Helical polynomial curves and “double” Pythagorean-hodograph curves

Rida T. Farouki

Department of Mechanical & Aeronautical Engineering,
University of California, Davis
— synopsis —

- introduction: properties of Pythagorean-hodograph curves
- computing rotation-minimizing frames on spatial PH curves
- helical polynomial space curves — are always PH curves
- standard quaternion representation for spatial PH curves
- “double” Pythagorean hodograph structure — requires both $|r'(t)|$ and $|r'(t) \times r''(t)|$ to be polynomials in $t$
- Hermite interpolation problem: selection of free parameters
Pythagorean-hodograph (PH) curves

\[ r(\xi) = \text{PH curve} \iff \text{coordinate components of } r'(\xi) \]
comprise a “Pythagorean \( n \)-tuple of polynomials” in \( \mathbb{R}^n \)

PH curves incorporate special algebraic structures in their hodographs
(complex number & quaternion models for planar & spatial PH curves)

- rational offset curves \( r_d(\xi) = r(\xi) + d\mathbf{n}(\xi) \)
- polynomial arc-length function \( s(\xi) = \int_0^\xi |r'(\xi)| \, d\xi \)
- closed-form evaluation of energy integral \( E = \int_0^1 \kappa^2 \, ds \)
- real-time CNC interpolators, rotation-minimizing frames, etc.
helical polynomial space curves

several equivalent characterizations of helical curves

- tangent $t$ maintains constant inclination $\psi$ with fixed vector $a$
- $a \cdot t = \cos \psi$, where $\psi =$ pitch angle and $a =$ axis vector of helix
- fixed curvature/torsion ratio, $\kappa/\tau = \tan \psi$ (Theorem of Lancret)
- curve has a circular tangent indicatrix on the unit sphere (small circle for space curve, great circle for planar curve)
- $(r^{(2)} \times r^{(3)}) \cdot r^{(4)} \equiv 0$ — where $r^{(k)} = k^{th}$ arc–length derivative
- circular helix ($\kappa$ & $\tau$ individually constant) is transcendental curve
all helical polynomial space curves are PH curves

constant inclination \Rightarrow a \cdot r'(t) \equiv \cos \psi |r'(t)|

a \cdot r'(t) = \text{polynomial in } t \text{ for any polynomial curve } r(t)

\cos \psi |r'(t)| = \text{polynomial in } t \text{ only if } r(t) \text{ is a PH curve}

all spatial PH cubics, but not all spatial PH quintics, are helical

problem: distinguish between helical & non–helical spatial PH curves
characterization of spatial PH cubics

A cubic with Bézier control–polygon legs \( \mathbf{L}_0, \mathbf{L}_1, \mathbf{L}_2 \) has a Pythagorean hodograph if and only if \( \mathbf{L}_0 \) and \( \mathbf{L}_2 \) lie on a right–circular cone of some half–angle \( \vartheta \) about \( \mathbf{L}_1 \) as axis, and their azimuthal separation \( \varphi \) on this cone is given in terms of the lengths \( L_0, L_1, L_2 \) by

\[
\cos \varphi = 1 - \frac{2L_1^2}{L_0L_2}
\]

(generalizes constraints for Tschirnhaus cubic in planar case)

can specify pitch angle and helix axis in terms of \( L_0, L_1, L_2, \vartheta, \varphi \)
Pythagorean quartuples of polynomials

\[ x'(t)^2 + y'(t)^2 + z'(t)^2 = \sigma^2(t) \iff \begin{cases} 
    x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) \\
    y'(t) = 2 [ u(t)q(t) + v(t)p(t) ] \\
    z'(t) = 2 [ v(t)q(t) - u(t)p(t) ] \\
    \sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t) 
\end{cases} \]


choose quaternion polynomial \( \mathcal{A}(t) = u(t) + v(t) \mathbf{i} + p(t) \mathbf{j} + q(t) \mathbf{k} \)

→ spatial Pythagorean hodograph \( \mathbf{r}'(t) = (x'(t), y'(t), z'(t)) = \mathcal{A}(t) \mathbf{i} \mathcal{A}^*(t) \)
fundamentals of quaternion algebra

quaternions are four-dimensional numbers of the form

\[ \mathcal{A} = a + a_x i + a_y j + a_z k \quad \text{and} \quad \mathcal{B} = b + b_x i + b_y j + b_z k \]

that obey the sum and (non-commutative) product rules

\[ \mathcal{A} + \mathcal{B} = (a + b) + (a_x + b_x) i + (a_y + b_y) j + (a_z + b_z) k \]

\[ \mathcal{A} \mathcal{B} = (ab - a_x b_x - a_y b_y - a_z b_z) + (ab_x + ba_x + a_y b_z - a_z b_y) i + (ab_y + ba_y + a_z b_x - a_x b_z) j + (ab_z + ba_z + a_x b_y - a_y b_x) k \]

basis elements \( 1, i, j, k \) satisfy \( i^2 = j^2 = k^2 = i j k = -1 \)
equivalently, \( i j = -j i = k, \quad j k = -k j = i, \quad k i = -i k = j \)
scalar-vector form of quaternions

set $\mathcal{A} = (a, a)$ and $\mathcal{B} = (b, b) — a, b$ and $a, b$ are scalar and vector parts

$(a, b$ and $a, b$ also called the real and imaginary parts of $\mathcal{A}, \mathcal{B})$

$$\mathcal{A} + \mathcal{B} = (a + b, a + b)$$

$$\mathcal{A} \mathcal{B} = (ab - a \cdot b, a b + b a + a \times b)$$

(historical note: Hamilton’s quaternions preceded, but were eventually supplanted by, the 3-dimensional vector analysis of Gibbs and Heaviside)

$\mathcal{A}^* = (a, -a)$ is the conjugate of $\mathcal{A}$

modulus: $|\mathcal{A}|^2 = \mathcal{A}^* \mathcal{A} = \mathcal{A} \mathcal{A}^* = a^2 + |a|^2$

note that $|\mathcal{A} \mathcal{B}| = |\mathcal{A}| |\mathcal{B}|$ and $(\mathcal{A} \mathcal{B})^* = \mathcal{B}^* \mathcal{A}^*$
any unit quaternion has the form $U = (\cos \frac{1}{2} \theta, \sin \frac{1}{2} \theta \mathbf{n})$

describes a spatial rotation by angle $\theta$ about unit vector $\mathbf{n}$

for any vector $\mathbf{v}$ the quaternion product $v' = U \mathbf{v} U^*$

yields the vector $v'$ corresponding to a rotation of $\mathbf{v}$ by $\theta$ about $\mathbf{n}$

here $\mathbf{v}$ is short-hand for a “pure vector” quaternion $\mathcal{V} = (0, \mathbf{v})$

unit quaternions $U$ form a (non-commutative) group under multiplication
concatenation of spatial rotations

rotate $\theta_1$ about $n_1$ then $\theta_2$ about $n_2$ → equivalent rotation $\theta$ about $n$

\[
\theta = \pm 2 \cos^{-1}(\cos \frac{1}{2} \theta_1 \cos \frac{1}{2} \theta_2 - n_1 \cdot n_2 \sin \frac{1}{2} \theta_1 \sin \frac{1}{2} \theta_2)
\]

\[
n = \pm \frac{\sin \frac{1}{2} \theta_1 \cos \frac{1}{2} \theta_2 n_1 + \cos \frac{1}{2} \theta_1 \sin \frac{1}{2} \theta_2 n_2 - \sin \frac{1}{2} \theta_1 \sin \frac{1}{2} \theta_2 n_1 \times n_2}{\sqrt{1 - (\cos \frac{1}{2} \theta_1 \cos \frac{1}{2} \theta_2 - n_1 \cdot n_2 \sin \frac{1}{2} \theta_1 \sin \frac{1}{2} \theta_2)^2}}
\]

sign ambiguity: equivalence of $-\theta$ about $-n$ and $\theta$ about $n$

formulae discovered by Olinde Rodrigues (1794-1851)

---

set $U_1 = (\cos \frac{1}{2} \theta_1, \sin \frac{1}{2} \theta_1 n_1)$ and $U_2 = (\cos \frac{1}{2} \theta_2, \sin \frac{1}{2} \theta_2 n_2)$

$U = U_2 U_1 = (\cos \frac{1}{2} \theta, \sin \frac{1}{2} \theta n)$ defines angle, axis of compound rotation
spatial rotations do not commute

blue vector is obtained from red vector by the concatenation of two spatial rotations — left: $R_y(\alpha) R_z(\beta)$, right: $R_z(\beta) R_y(\alpha)$ — the end results differ

define $\mathcal{U}_1 = (\cos \frac{1}{2}\alpha, \sin \frac{1}{2}\alpha \mathbf{j})$, $\mathcal{U}_2 = (\cos \frac{1}{2}\beta, \sin \frac{1}{2}\beta \mathbf{k})$ — $\mathcal{U}_1 \mathcal{U}_2 \neq \mathcal{U}_2 \mathcal{U}_1$
quaternion model for spatial PH curves

quaternion polynomial \( \mathcal{A}(t) = u(t) + v(t) \mathbf{i} + p(t) \mathbf{j} + q(t) \mathbf{k} \)

maps to \( \mathbf{r}'(t) = \mathcal{A}(t) \mathbf{i} \mathcal{A}^*(t) = \left[ u^2(t) + v^2(t) - p^2(t) - q^2(t) \right] \mathbf{i} + 2 \left[ u(t)q(t) + v(t)p(t) \right] \mathbf{j} + 2 \left[ v(t)q(t) - u(t)p(t) \right] \mathbf{k} \)

rotation invariance of spatial PH form: rotate by \( \theta \) about \( \mathbf{n} = (n_x, n_y, n_z) \)

define \( \mathcal{U} = (\cos \frac{1}{2} \theta, \sin \frac{1}{2} \theta \mathbf{n}) \) — then \( \mathbf{r}'(t) \rightarrow \tilde{\mathbf{r}}'(t) = \tilde{\mathcal{A}}(t) \mathbf{i} \tilde{\mathcal{A}}^*(t) \)

where \( \tilde{\mathcal{A}}(t) = \mathcal{U} \mathcal{A}(t) \) (can interpret as rotation in \( \mathbb{R}^4 \))
matrix form of \( \tilde{A}(t) = U A(t) \)

\[
\begin{bmatrix}
\tilde{u} \\
\tilde{v} \\
\tilde{p} \\
\tilde{q}
\end{bmatrix} =
\begin{bmatrix}
\cos \frac{1}{2}\theta & -n_x \sin \frac{1}{2}\theta & -n_y \sin \frac{1}{2}\theta & -n_z \sin \frac{1}{2}\theta \\
n_x \sin \frac{1}{2}\theta & \cos \frac{1}{2}\theta & -n_z \sin \frac{1}{2}\theta & n_y \sin \frac{1}{2}\theta \\
n_y \sin \frac{1}{2}\theta & n_z \sin \frac{1}{2}\theta & \cos \frac{1}{2}\theta & -n_x \sin \frac{1}{2}\theta \\
n_z \sin \frac{1}{2}\theta & -n_y \sin \frac{1}{2}\theta & n_x \sin \frac{1}{2}\theta & \cos \frac{1}{2}\theta
\end{bmatrix}
\begin{bmatrix}
u \\
v \\
p \\
q
\end{bmatrix}
\]

matrix \( \in \text{SO}(4) \)

in general, points have non-closed orbits under rotations in \( \mathbb{R}^4 \)
Lemma 1. For any quaternion \( A \neq 0 \), the quaternions \( A, Ai, Aj, Ak \) — interpreted as vectors in \( \mathbb{R}^4 \) — define an orthogonal basis, in terms of which any quaternion can be represented by four real values \( \alpha, \beta, \gamma, \delta \) as the linear combination

\[
\alpha A + \beta Ai + \gamma Aj + \delta Ak = A(\alpha + \beta i + \gamma j + \delta k).
\]

If \( A = u + vi + pj + qk \), components of \( A, Ai, Aj, Ak \) define columns of an orthogonal \( 4 \times 4 \) matrix

\[
\begin{bmatrix}
  u & -v & -p & -q \\
  v & u & -q & -p \\
  p & q & u & -v \\
  q & -p & v & u
\end{bmatrix}
\]

If \( |A| = 1 \), specifies a rotation \((1, i, j, k) \rightarrow (A, Ai, Aj, Ak)\) in \( \mathbb{R}^4 \).
degenerate spatial PH cubics

spatial PH cubics: \( \mathbf{r}'(t) = \mathcal{A}(t) \mathbf{i} \mathcal{A}^{*}(t) \) with \( \mathcal{A}(t) = \mathcal{A}_0(1 - t) + \mathcal{A}_1 t \)

writing \( \mathcal{A}_1 = \mathcal{A}_0 (\alpha_1 + \beta_1 \mathbf{i} + \gamma_1 \mathbf{j} + \delta_1 \mathbf{k}) \) we have:

\[
\begin{align*}
\mathbf{r}(t) &= \text{straight line} \iff (\gamma_1, \delta_1) = (0, 0) \\
\mathbf{r}(t) &= \text{plane curve} \iff \beta_1 = 0 \text{ and } (\gamma_1, \delta_1) \neq (0, 0)
\end{align*}
\]

NOTE: all spatial PH cubics are helical curves
degenerate spatial PH quintics

spatial PH quintics: use \( \mathcal{A}(t) = \mathcal{A}_0(1 - t)^2 + \mathcal{A}_1 2(1 - t)t + \mathcal{A}_2 t^2 \)

writing \( \mathcal{A}_r = \mathcal{A}_0 (\alpha_r + \beta_r \mathbf{i} + \gamma_r \mathbf{j} + \delta_r \mathbf{k}) \) for \( r = 1, 2 \) we have:

\[ \mathbf{r}(t) = \text{straight line} \iff (\gamma_1, \delta_1) = (\gamma_2, \delta_2) = (0, 0) \]

\[ \mathbf{r}(t) = \text{plane curve} \iff \beta_1 = \beta_2 = 0 \text{ and } \gamma_1 \delta_2 - \gamma_2 \delta_1 = 0 \]

with \( (\gamma_1, \delta_1) \neq (0, 0) \) and \( (\gamma_2, \delta_2) \neq (0, 0) \)

conditions for plane curve equivalent to linear dependence of \( \mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2 \)
morphology of helical PH quintics

\[
gcd(u, v, p, q) = \text{constant} \not\Rightarrow gcd(x', y', z') = \text{constant}
\]

specifically,
\[
gcd(x', y', z') = gcd(u + i v, p - i q) \cdot gcd(u - i v, p + i q)
\]

• monotone-helical PH quintics — $gcd(x', y', z')$ is quadratic
  
  tangent indicatrix is singly-traced circle (curve tangent maintains a consistent sense of rotation about helix axis)

• general helical PH quintics — $gcd(x', y', z')$ is a constant
  
  tangent indicatrix is doubly-traced circle (curve tangent may exhibit reversals in sense of rotation about helix axis)
examples of monotone-helical (left) and general helical (right) PH quintics
\[ r'(t) = A(t) \mathbf{i} A^*(t) \quad \text{with} \quad A(t) = A_0(1-t)^2 + A_12(1-t)t + A_2t^2 \]

\[ \text{set} \quad A_r = A_0 (\alpha_r + \beta_r \mathbf{i} + \gamma_r \mathbf{j} + \delta_r \mathbf{k}) \quad \text{for} \quad r = 1, 2 \]

\[ r(t) = \text{general helical PH quintic} \quad \iff \quad \gamma_1 : \gamma_2 = \delta_1 : \delta_2 \quad \text{and} \quad \beta_1 : \beta_2 = (\gamma_1^2 + \delta_1^2) : (\gamma_1 \gamma_2 + \delta_1 \delta_2) \]

\[ r(t) = \text{monotone-helical PH quintic} \quad \iff \quad \alpha_2 = \frac{r \alpha_1 + s \beta_1}{\gamma_1^2 + \delta_1^2} + \frac{s^2 - r^2}{4(\gamma_1^2 + \delta_1^2)^2}, \quad \beta_2 = \frac{r \beta_1 - s \alpha_1}{\gamma_1^2 + \delta_1^2} + \frac{2rs}{4(\gamma_1^2 + \delta_1^2)^2} \]

where \( r = \gamma_1 \gamma_2 + \delta_1 \delta_2 \) and \( s = \gamma_1 \delta_2 - \gamma_2 \delta_1 \).
for a helical PH space curve with $\sigma(t) = |r'(t)|$ we have

$$\frac{\kappa}{\tau} = \tan \psi \quad \Rightarrow \quad |r' \times r''|^3 = \tan \psi \sigma^3 (r' \times r'') \cdot r'''$$

using also the property $|r' \times r''|^2 = \sigma^2 \rho$ of all PH space curves gives

$$\rho^{3/2} = \tan \psi (r' \times r'') \cdot r'''$$

**Lemma.** A necessary condition for a spatial PH curve to be helical is that the polynomial $\rho(t)$ be a perfect square — i.e., the curve must be a double PH curve.

trivially satisfied for all PH cubics, since $(r' \times r'') \cdot r''' = \text{constant}$

$$\deg((r' \times r'') \cdot r''') = 6$$ for PH quintics, so we must have $\rho(t) = \omega^2(t)$ for a quadratic polynomial $\omega(t)$ if $r(t)$ is a helical PH quintic
“double” Pythagorean-hodograph structure

\[ |r'(t)| \text{ and } |r'(t) \times r''(t)| \text{ are both polynomials in curve parameter } t \]

\[ x'^2 + y'^2 + z'^2 \equiv \sigma^2, \]

\[ (y'z'' - y''z')^2 + (z'x'' - z''x')^2 + (x'y'' - x''y')^2 \equiv (\sigma\omega)^2. \]

Frenet frame, curvature, torsion are all rational functions of \( t \)

\[ t = \frac{r'}{|r'|}, \quad n = \frac{r' \times r''}{|r' \times r''|} \times t, \quad b = \frac{r' \times r''}{|r' \times r''|}, \]

\[ \kappa = \frac{|r' \times r''|}{|r'|^3}, \quad \tau = \frac{(r' \times r'') \cdot r'''}{|r' \times r''|^2}. \]

Beltran & Monterde (2007) have called them “2-PH curves”
every spatial PH curve satisfies \[ | \mathbf{r}'(t) \times \mathbf{r}''(t) |^2 = \sigma^2(t) \rho(t) \]

the polynomial \( \rho(t) \) can be defined in terms of \( u(t), v(t), p(t), q(t) \) and \( u'(t), v'(t), p'(t), q'(t) \) in several different ways:

\[ \rho = 4 \left[ (up' - u'p)^2 + (uq' - u'q)^2 + (vp' - v'p)^2 + (vq' - v'q)^2 
+ 2(uv' - u'v)(pq' - p'q) \right] \tag{1} \]

\[ \rho = 4 \left[ (uv' - u'v + pq' - p'q)^2 + (up' - u'p - vq' + v'q)^2 
+ (uq' - u'q + vp' - v'p)^2 - (uv' - u'v - pq' + p'q)^2 \right] \tag{2} \]

\[ \rho = 4 \left[ (up' - u'p + vq' - v'q)^2 + (uq' - u'q - vp' + v'p)^2 \right] \tag{3} \]
“double” PH structure — triples and quartuples

for a double PH curve, \( \rho(t) = \omega^2(t) \) for some polynomial \( \omega(t) \)

form (3) of \( \rho(t) \) \( \Rightarrow \) \( 2(up' - u'p + vq' - v'q), 2(uq' - u'q - vp' + v'p) \), \( \omega \) must comprise a Pythagorean triple of polynomials

\[
2(up' - u'p + vq' - v'q) = k(a^2 - b^2) \\
2(uq' - u'q - vp' + v'p) = 2kab \\
\omega = k(a^2 + b^2)
\]

for polynomials \( k(t), a(t), b(t) \) with \( \gcd(a(t), b(t)) = \text{constant} \)

hence, double PH curves involve both Pythagorean triples and Pythagorean quartuples of polynomials!
helical PH quintics as “double” PH curves

\[2(u'p - u'p + v'q - v'q), \quad 2(u'q - u'q - v'p + v'p), \quad \omega \text{ are quadratic}\]

to satisfy second Pythagorean condition, we must have either

(1) \(\deg(a(t), b(t)) = 1\) and \(k(t) = \text{constant}\)

(2) \(a(t), b(t) = \text{constants}\) and \(\deg(k(t)) = 2\)

cases (1) & (2) identify monotone-helical and general helical PH quintics

\[\Rightarrow \text{all double PH quintics are helical curves}\]
there exist non-helical double PH curves

Beltran & Monterde (2007): all double PH cubics and quintics are helical — but there exist double PH curves of degree 7 that are not helical

\[
x(t) = \frac{1}{21} t^7 + \frac{1}{5} t^5 + t^3 - 3t, \quad y(t) = -\frac{1}{2} t^4 + 3 t^2, \quad z(t) = -2 t^3
\]

\[
|r'(t)| = \frac{t^6 + 3t^4 + 9t^2 + 9}{3}, \quad |r'(t) \times r''(t)| = 2(t^2 + 1)(t^6 + 3t^4 + 9t^2 + 9)
\]

\[
\frac{\kappa(t)}{\tau(t)} = -\frac{9(t^2 + 1)^2}{2t^6 + 9t^4 - 9} \neq \text{constant}
\]

In general, the curvature/torsion ratio for a double PH curve is

\[
\frac{\kappa(t)}{\tau(t)} = \frac{\omega^3(t)}{[r'(t) \times r''(t)] \cdot r'''(t)}
\]
Hopf map model for spatial PH curves

Choi et al. (2002) — alternative to the quaternion representation

Hopf map $\mathbb{C} \times \mathbb{C} = \mathbb{R}^4 \rightarrow \mathbb{R}^3$ generates Pythagorean hodographs in $\mathbb{R}^3$

from two complex polynomials $\alpha(t) = u(t) + i v(t)$, $\beta(t) = q(t) + i p(t)$:

$$r'(t) = H(\alpha(t), \beta(t)) = (|\alpha(t)|^2 - |\beta(t)|^2, 2 \Re(\alpha(t)\overline{\beta}(t)), 2 \Im(\alpha(t)\overline{\beta}(t)))$$

$$= (u^2(t) + v^2(t) - p^2(t) - q^2(t),$$

$$2(u(t)q(t) + v(t)p(t)), 2(v(t)q(t) - u(t)p(t)))$$

identify imaginary unit $i$ with quaternion basis element $i$ — quaternion polynomial $A(t)$ is related to the complex polynomials $\alpha(t)$ and $\beta(t)$ by

$$A(t) = u(t) + v(t) i + p(t) j + q(t) k = \alpha(t) + k \beta(t)$$
polynomial $\rho(t)$ has simpler formulation in Hopf map model

$$\alpha \beta' - \alpha' \beta = (uq' - u'q - vp' + v'p) + i (up' - u'p + vq' - v'q)$$

$$\Rightarrow \rho(t) = 4 |\alpha(t)\beta'(t) - \alpha'(t)\beta(t)|^2$$

restricting $H(\alpha, \beta)$ to complex numbers satisfying $|\alpha|^2 + |\beta|^2 = 1$, it defines a map between the “3–sphere” $S^3 : u^2 + v^2 + p^2 + q^2 = 1$ in the space $\mathbb{R}^4$ spanned by coordinates $(u, v, p, q)$ and the familiar “2–sphere” $S^2 : x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ with coordinates $(x, y, z)$

great circles of $S^3$ are mapped to points of $S^2$ by $H(\alpha, \beta)$

first known map between higher and lower dimension spheres that is not null homotopic (applications to quantum computing)
spatial PH quintic Hermite interpolants

spatial PH quintic interpolating end points $p_i, p_f$ & derivatives $d_i, d_f$

$$r'(t) = A(t) i A^*(t)$$

where $A(t) = A_0(1-t)^2 + A_1 2(1-t)t + A_2 t^2$

three equations in three quaternion unknowns $A_0, A_1, A_2$

$$r'(0) = A_0 i A_0^* = d_i \quad \text{and} \quad r'(1) = A_2 i A_2^* = d_f$$

$$\int_0^1 A(t) i A^*(t) \, dt = \frac{1}{5} A_0 i A_0^* + \frac{1}{10} (A_0 i A_1^* + A_1 i A_0^*)$$

$$+ \frac{1}{30} (A_0 i A_2^* + 4 A_1 i A_1^* + A_2 i A_0^*)$$

$$+ \frac{1}{10} (A_1 i A_2^* + A_2 i A_1^*) + \frac{1}{5} A_2 i A_2^* = p_f - p_i$$
solution of fundamental equation

given vector \( c = |c| (\lambda, \mu, \nu) \) find quaternion \( \mathcal{A} \) such that

\[
\mathcal{A} i \mathcal{A}^* = c
\]

one–parameter family of solutions

\[
\mathcal{A}(\phi) = \sqrt{\frac{1}{2}(1 + \lambda)}|c| \left( -\sin \phi + \cos \phi \, i \\
+ \frac{\mu \cos \phi + \nu \sin \phi}{1 + \lambda} \, j + \frac{\nu \cos \phi - \mu \sin \phi}{1 + \lambda} \, k \right)
\]

in \( \mathbb{R}^3 \) there is a continuous family of rotations
mapping the vector \( i \) into a given vector \( (\lambda, \mu, \nu) \)
families of spatial rotations

find \( U = (\cos \frac{1}{2}\theta, \sin \frac{1}{2}\theta n) \) that rotates \( i = (1, 0, 0) \rightarrow v = (\lambda, \mu, \nu) \)

\[
\begin{align*}
n_x^2(1 - \cos \theta) + \cos \theta &= \lambda, \\
n_x n_y (1 - \cos \theta) + n_z \sin \theta &= \mu, \\
n_z n_x (1 - \cos \theta) - n_y \sin \theta &= \nu.
\end{align*}
\]

\[
\begin{align*}
n_x &= \pm \frac{\sqrt{\cos^2 \frac{1}{2} \alpha - \cos^2 \frac{1}{2} \theta}}{\sin \frac{1}{2} \theta}, \\
n_y &= \pm \frac{\mu \sqrt{\cos^2 \frac{1}{2} \alpha - \cos^2 \frac{1}{2} \theta - \nu \cos \frac{1}{2} \theta}}{(1 + \lambda) \sin \frac{1}{2} \theta}, \\
n_z &= \pm \frac{\nu \sqrt{\cos^2 \frac{1}{2} \alpha - \cos^2 \frac{1}{2} \theta + \mu \cos \frac{1}{2} \theta}}{(1 + \lambda) \sin \frac{1}{2} \theta}.
\end{align*}
\]

general solution, where \( \alpha = \cos^{-1} \lambda \) and \( \alpha \leq \theta \leq 2\pi - \alpha \)
Spatial rotations of unit vectors $\mathbf{i} \rightarrow \mathbf{v}$. (a) Vectors $\mathbf{e}_\perp$ (orthogonal to $\mathbf{i}$, $\mathbf{v}$) and $\mathbf{e}_0$ (bisector of $\mathbf{i}$, $\mathbf{v}$) — the plane $\Pi$ of $\mathbf{e}_\perp$ and $\mathbf{e}_0$ is orthogonal to that of $\mathbf{i}$ and $\mathbf{v}$. (b) For any rotation angle $\theta \in (\alpha, 2\pi - \alpha)$, where $\alpha = \cos^{-1}(\mathbf{i} \cdot \mathbf{v})$, there are two possible rotations, with axes $\mathbf{n}$ inclined equally to $\mathbf{e}_\perp$ in the plane $\Pi$. (c) Sampling of the family of spatial rotations $\mathbf{i} \rightarrow \mathbf{v}$, shown as loci on the unit sphere. (d) Axes $\mathbf{n}$ for these rotations, lying in the plane $\Pi$. 
construction of Hermite interpolants

derivative conditions have form of fundamental equation — can be solved directly for $A_0$ and $A_2$

dend-point condition can then be cast in fundamental form as

$$(3A_0 + 4A_1 + 3A_2)i (3A_0 + 4A_1 + 3A_2)^* = 120(p_f - p_i) - 15(d_i + d_f) + 5(A_0iA_2^* + A_2iA_0^*)$$

— solve for $A_1$, since $A_0$ and $A_2$ known

solution contains three free parameters $\phi_0$, $\phi_1$, $\phi_2$ but shape of interpolants depends only on their differences

$\implies \exists$ two-parameter family of spatial PH quintic interpolants to given Hermite data $p_i$, $d_i$ and $p_f$, $d_f$
spatial PH quintic Hermite interpolants

\[ p_i = (0, 0, 0) \text{ and } p_f = (1, 1, 1) \text{ for both curves} \]
\[ d_i = (-0.8, 0.3, 1.2) \text{ and } d_f = (0.5, -1.3, -1.0) \text{ for curve on left,} \]
\[ d_i = (0.4, -1.5, -1.2) \text{ and } d_f = (-1.2, -0.6, -1.2) \text{ for curve on right} \]
open problem: find “optimal” $\phi_0, \phi_2$ values

shape of interpolants depends strongly on free parameters

- minimize a shape-measure integral, e.g., $E = \int \kappa^2 \, ds$
  (but highly non-linear in the free parameters)

- impose additional conditions (restrict solution space)
  — e.g., helicity condition $\kappa/\tau = \text{constant}$

- study geometry of quaternion curve $A(t)$
  — need better insight on geometry of quaternion space $\mathbb{H}$

- extension to spatial $C^2$ PH quintic splines
two-parameter family of Hermite interpolants

nominal parameters: $\phi_0, \phi_2$ — arc length of interpolants depends only on difference $\phi_2 - \phi_0$, shape of interpolants depends only on mean $\frac{1}{2}(\phi_0 + \phi_2)$

sampling of the one-parameter families of spatial PH quintic interpolants, of identical arc length, to given first-order Hermite data — defined by holding $\phi_2 - \phi_0$ constant, and varying only $\frac{1}{2}(\phi_0 + \phi_2)$
recent results on Hermite interpolants

(Farouki, Giannelli, Manni, Sestini, 2007)

• dependence of total arc length $S$ exhibits a **single minimum** and a **single maximum** with respect to the variable $\phi_2 - \phi_0$

• these extremal arc length interpolants correspond to **helical PH quintics**

$\Rightarrow$ helical PH quintic interpolants exist for any first-order Hermite data

• three “practical” criteria for identifying interpolants with **near-optimal shape properties** (all reproduce cubic PH interpolants when they exist)

• give values of the **energy integral** close to the absolute minimum, at modest computational cost
• spatial PH curves ideally suited to computing rotation-minimizing frames (symbolic integration or rational approximation)

• helical polynomial space curves are always PH curves
  — two quintic types (monotone and general helical PH quintics)

• double PH curves have rational Frenet frames, curvature, torsion
  — all helical PH curves are necessarily double PH curves

• properties of solutions to first-order Hermite interpolation problem

• don’t believe a Russian who tells you he has stopped drinking