Helical polynomial curves and "double" Pythagorean-hodograph curves

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— synopsis —

- introduction: properties of Pythagorean-hodograph curves
- computing rotation-minimizing frames on spatial PH curves
- helical polynomial space curves — are always PH curves
- standard quaternion representation for spatial PH curves
- "double" Pythagorean hodograph structure — requires both $|\mathbf{r}'(t)|$ and $|\mathbf{r}'(t) \times \mathbf{r}''(t)|$ to be polynomials in t
- Hermite interpolation problem: selection of free parameters

Pythagorean-hodograph (PH) curves

$\mathbf{r}(\xi) =$ PH curve $\iff$ coordinate components of $\mathbf{r}'(\xi)$

comprise a "Pythagorean n-tuple of polynomials" in $\mathbb{R}^n$

PH curves incorporate special algebraic structures in their hodographs

(complex number & quaternion models for planar & spatial PH curves)

- rational offset curves $\mathbf{r}_d(\xi) = \mathbf{r}(\xi) + d\,\mathbf{n}(\xi)$
- polynomial arc-length function $s(\xi) = \int_0^\xi |\mathbf{r}'(\xi)| \, \mathrm{d}\xi$
- closed-form evaluation of energy integral $E = \int_0^1 \kappa^2 \, \mathrm{d}s$
- real-time CNC interpolators, rotation-minimizing frames, etc.

helical polynomial space curves

several equivalent characterizations of helical curves

- tangent $\mathbf{t}$ maintains constant inclination ψ with fixed vector $\mathbf{a}$
- $\mathbf{a} \cdot \mathbf{t} = \cos\psi$, where ψ = pitch angle and $\mathbf{a}$ = axis vector of helix
- fixed curvature/torsion ratio, $\kappa/\tau = \tan\psi$ (Theorem of Lancret)
- curve has a circular tangent indicatrix on the unit sphere
  (small circle for space curve, great circle for planar curve)
- $(\mathbf{r}^{(2)} \times \mathbf{r}^{(3)}) \cdot \mathbf{r}^{(4)} \equiv 0$ — where $\mathbf{r}^{(k)}$ = $k^{\rm th}$ arc–length derivative
- circular helix (κ & τ *individually* constant) is transcendental curve

all helical polynomial space curves are PH curves

constant inclination $\Rightarrow$ $\mathbf{a} \cdot \mathbf{r}'(t) \equiv \cos\psi \, |\mathbf{r}'(t)|$

$\mathbf{a} \cdot \mathbf{r}'(t) =$ polynomial in t for any polynomial curve $\mathbf{r}(t)$

$\cos\psi \, |\mathbf{r}'(t)| =$ polynomial in t only if $\mathbf{r}(t)$ is a PH curve

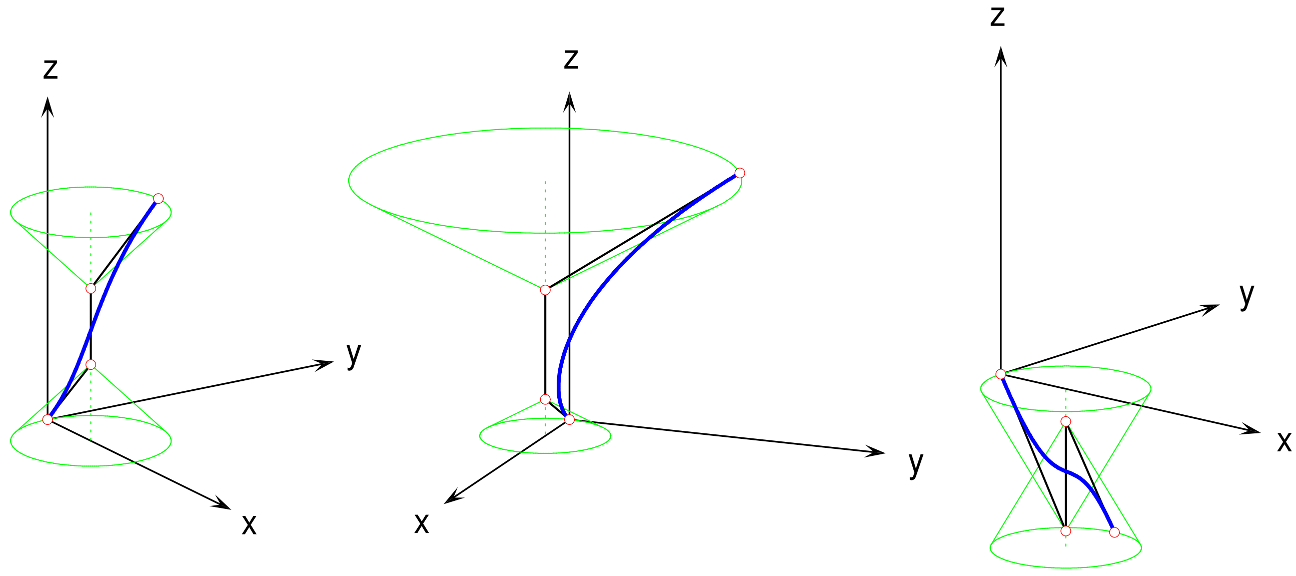
all spatial PH cubics, but not all spatial PH quintics, are helical

problem : distinguish between helical & non–helical spatial PH curves

characterization of spatial PH cubics

A cubic with Bézier control–polygon legs $\mathbf{L}_0, \mathbf{L}_1, \mathbf{L}_2$ has a Pythagorean hodograph if and only if $\mathbf{L}_0$ and $\mathbf{L}_2$ lie on a right–circular cone of some half–angle ϑ about $\mathbf{L}_1$ as axis, and their azimuthal separation φ on this cone is given in terms of the lengths L_0, L_1, L_2 by $\cos\varphi = 1 - 2L_1^2/L_0L_2$

(generalizes constraints for Tschirnhaus cubic in planar case)



can specify pitch angle and helix axis in terms of $L_0, L_1, L_2, \vartheta, \varphi$

Pythagorean quartuples of polynomials

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \quad \Longleftrightarrow \quad \begin{cases} x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) \\ y'(t) = 2\,[\,u(t)q(t) + v(t)p(t)\,] \\ z'(t) = 2\,[\,v(t)q(t) - u(t)p(t)\,] \\ \sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t) \end{cases}$$

R. Dietz, J. Hoschek, and B. Jüttler, An algebraic approach to curves and surfaces on the sphere and on other quadrics, *Computer Aided Geometric Design* **10**, 211–229 (1993)

H. I. Choi, D. S. Lee, and H. P. Moon, Clifford algebra, spin representation, and rational parameterization of curves and surfaces, *Advances in Computational Mathematics* **17**, 5-48 (2002)

choose quaternion polynomial $\mathcal{A}(t) \;=\; u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$

$\rightarrow$ spatial Pythagorean hodograph $\mathbf{r}'(t) \;=\; (x'(t), y'(t), z'(t)) \;=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$

fundamentals of quaternion algebra

quaternions are four-dimensional numbers of the form

$$\mathcal{A} = a + a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k} \qquad \text{and} \qquad \mathcal{B} = b + b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}$$

that obey the sum and (non-commutative) product rules

$$\mathcal{A} + \mathcal{B} = (a+b) + (a_x + b_x)\,\mathbf{i} + (a_y + b_y)\,\mathbf{j} + (a_z + b_z)\,\mathbf{k}$$

$$\begin{aligned}\mathcal{A}\,\mathcal{B} &= (ab - a_x b_x - a_y b_y - a_z b_z) \\ &+ (a b_x + b a_x + a_y b_z - a_z b_y)\,\mathbf{i} \\ &+ (a b_y + b a_y + a_z b_x - a_x b_z)\,\mathbf{j} \\ &+ (a b_z + b a_z + a_x b_y - a_y b_x)\,\mathbf{k}\end{aligned}$$

basis elements 1, $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1$

equivalently, $\mathbf{i}\,\mathbf{j} = -\mathbf{j}\,\mathbf{i} = \mathbf{k}\,,\quad \mathbf{j}\,\mathbf{k} = -\mathbf{k}\,\mathbf{j} = \mathbf{i}\,,\quad \mathbf{k}\,\mathbf{i} = -\mathbf{i}\,\mathbf{k} = \mathbf{j}$

scalar-vector form of quaternions

set $\mathcal{A} = (a, \mathbf{a})$ and $\mathcal{B} = (b, \mathbf{b})$ — a, b and $\mathbf{a}$, $\mathbf{b}$ are scalar and vector parts

(a, b and $\mathbf{a}, \mathbf{b}$ also called the real and imaginary parts of $\mathcal{A}, \mathcal{B}$)

$$\mathcal{A} + \mathcal{B} = (a + b, \mathbf{a} + \mathbf{b})$$

$$\mathcal{A}\mathcal{B} = (ab - \mathbf{a} \cdot \mathbf{b}, a\,\mathbf{b} + b\,\mathbf{a} + \mathbf{a} \times \mathbf{b})$$

(historical note: Hamilton's quaternions preceded, but were eventually supplanted by, the 3-dimensional vector analysis of Gibbs and Heaviside)

$\mathcal{A}^* = (a, -\mathbf{a})$ is the conjugate of $\mathcal{A}$

modulus : $|\mathcal{A}|^2 = \mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = a^2 + |\mathbf{a}|^2$

note that $|\mathcal{A}\,\mathcal{B}| = |\mathcal{A}|\,|\mathcal{B}|$ and $(\mathcal{A}\,\mathcal{B})^* = \mathcal{B}^*\mathcal{A}^*$

unit quaternions & spatial rotations

any unit quaternion has the form $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$

describes a spatial rotation by angle θ about unit vector $\mathbf{n}$

for any vector $\mathbf{v}$ the quaternion product

$$\boxed{\mathbf{v}' = \mathcal{U}\,\mathbf{v}\,\mathcal{U}^*}$$

yields the vector $\mathbf{v}'$ corresponding to a rotation of $\mathbf{v}$ by θ about $\mathbf{n}$

here $\mathbf{v}$ is short-hand for a “pure vector” quaternion $\mathcal{V} = (0, \mathbf{v})$

unit quaternions $\mathcal{U}$ form a (non-commutative) group under multiplication

concatenation of spatial rotations

rotate θ_1 about $\mathbf{n}_1$ then θ_2 about $\mathbf{n}_2$ $\longrightarrow$ equivalent rotation θ about $\mathbf{n}$

$$\theta \;=\; \pm 2\, \cos^{-1}(\cos\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 - \mathbf{n}_1\cdot\mathbf{n}_2 \sin\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2)$$

$$\mathbf{n} \;=\; \pm\, \frac{\sin\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2\, \mathbf{n}_1 \;+\; \cos\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2\, \mathbf{n}_2 \;-\; \sin\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2\, \mathbf{n}_1\times\mathbf{n}_2}{\sqrt{1 \;-\; (\cos\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 \;-\; \mathbf{n}_1\cdot\mathbf{n}_2 \sin\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2)^2}}$$

sign ambiguity: equivalence of $-\theta$ about $-\mathbf{n}$ and θ about $\mathbf{n}$

formulae discovered by Olinde Rodrigues (1794-1851)

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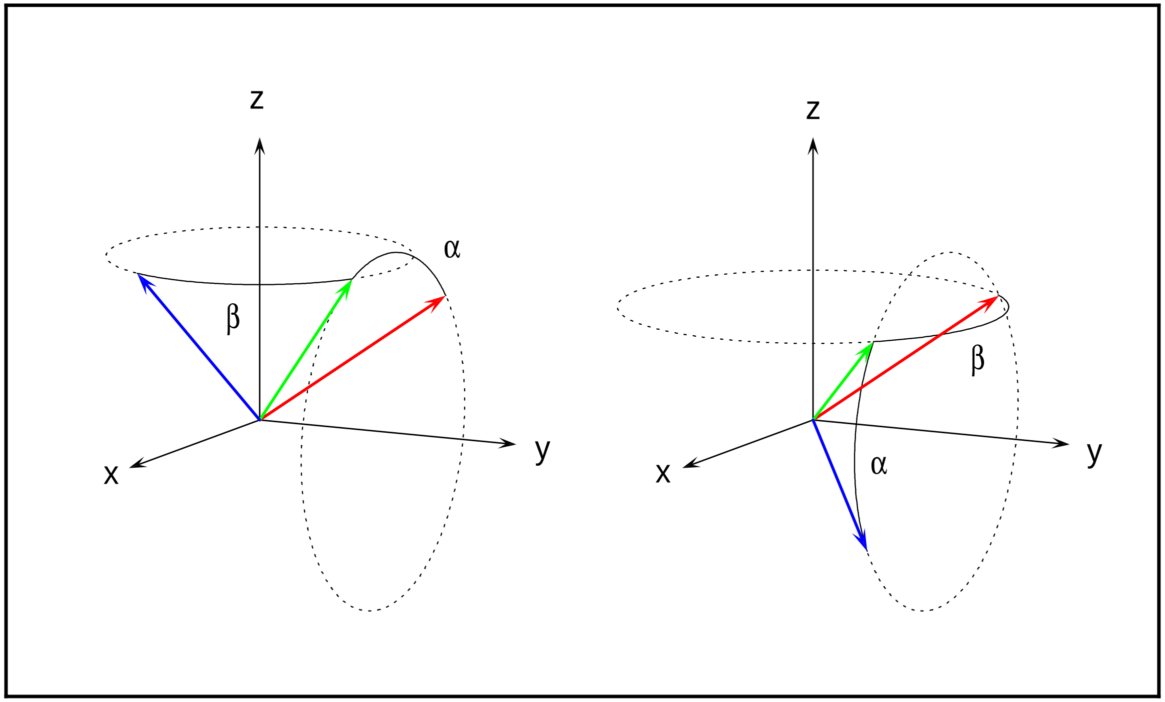
set $\mathcal{U}_1 = (\cos\tfrac{1}{2}\theta_1, \sin\tfrac{1}{2}\theta_1\mathbf{n}_1)$ and $\mathcal{U}_2 = (\cos\tfrac{1}{2}\theta_2, \sin\tfrac{1}{2}\theta_2\mathbf{n}_2)$

$\mathcal{U} = \mathcal{U}_2\mathcal{U}_1 = (\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta\mathbf{n})$ defines angle, axis of compound rotation

spatial rotations do not commute



blue vector is obtained from red vector by the concatenation of two spatial rotations — left: $R_y(\alpha)\,R_z(\beta)$, right: $R_z(\beta)\,R_y(\alpha)$ — the end results differ

define $\mathcal{U}_1 = (\cos\frac{1}{2}\alpha, \sin\frac{1}{2}\alpha\,\mathbf{j})$, $\mathcal{U}_2 = (\cos\frac{1}{2}\beta, \sin\frac{1}{2}\beta\,\mathbf{k})$ — $\mathcal{U}_1\mathcal{U}_2 \neq \mathcal{U}_2\mathcal{U}_1$

quaternion model for spatial PH curves

quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$

maps to

$$\mathbf{r}'(t) \;=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) \;=\; [\,u^2(t) + v^2(t) - p^2(t) - q^2(t)\,]\,\mathbf{i} + 2\,[\,u(t)q(t) + v(t)p(t)\,]\,\mathbf{j} \;+\; 2\,[\,v(t)q(t) - u(t)p(t)\,]\,\mathbf{k}$$

rotation invariance of spatial PH form: rotate by θ about $\mathbf{n} = (n_x, n_y, n_z)$

define $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$ — then $\mathbf{r}'(t) \;\to\; \tilde{\mathbf{r}}'(t) = \tilde{\mathcal{A}}(t)\,\mathbf{i}\,\tilde{\mathcal{A}}^*(t)$

where $\tilde{\mathcal{A}}(t) \;=\; \mathcal{U}\,\mathcal{A}(t)$ (can interpret as rotation in $\mathbb{R}^4$)

matrix form of $\tilde{\mathcal{A}}(t) = \mathcal{U}\,\mathcal{A}(t)$

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{p} \\ \tilde{q} \end{bmatrix} = \begin{bmatrix} \cos\frac{1}{2}\theta & -n_x\sin\frac{1}{2}\theta & -n_y\sin\frac{1}{2}\theta & -n_z\sin\frac{1}{2}\theta \\ n_x\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta & -n_z\sin\frac{1}{2}\theta & n_y\sin\frac{1}{2}\theta \\ n_y\sin\frac{1}{2}\theta & n_z\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta & -n_x\sin\frac{1}{2}\theta \\ n_z\sin\frac{1}{2}\theta & -n_y\sin\frac{1}{2}\theta & n_x\sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ p \\ q \end{bmatrix}$$

matrix $\in \mathrm{SO}(4)$

in general, points have non-closed orbits under rotations in $\mathbb{R}^4$

degenerate forms of spatial PH curves

Lemma 1. *For any quaternion $\mathcal{A} \neq 0$, the quaternions $\mathcal{A}$, $\mathcal{A}\,\mathbf{i}$, $\mathcal{A}\,\mathbf{j}$, $\mathcal{A}\,\mathbf{k}$ — interpreted as vectors in $\mathbb{R}^4$ — define an orthogonal basis, in terms of which any quaternion can be represented by four real values α, β, γ, δ as the linear combination*

$$\alpha\,\mathcal{A} + \beta\,\mathcal{A}\,\mathbf{i} + \gamma\,\mathcal{A}\,\mathbf{j} + \delta\,\mathcal{A}\,\mathbf{k} = \mathcal{A}\,(\alpha + \beta\,\mathbf{i} + \gamma\,\mathbf{j} + \delta\,\mathbf{k})\,.$$

if $\mathcal{A} = u + v\,\mathbf{i} + p\,\mathbf{j} + q\,\mathbf{k}$, components of $\mathcal{A}$, $\mathcal{A}\,\mathbf{i}$, $\mathcal{A}\,\mathbf{j}$, $\mathcal{A}\,\mathbf{k}$ define columns of an orthogonal 4×4 matrix

$$\begin{bmatrix} u & -v & -p & -q \\ v & u & -q & p \\ p & q & u & -v \\ q & -p & v & u \end{bmatrix}$$

if $|\mathcal{A}| = 1$, specifies a rotation $(1, \mathbf{i}, \mathbf{j}, \mathbf{k}) \to (\mathcal{A}, \mathcal{A}\,\mathbf{i}, \mathcal{A}\,\mathbf{j}, \mathcal{A}\,\mathbf{k})$ in $\mathbb{R}^4$

degenerate spatial PH cubics

spatial PH cubics : $\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$ with $\mathcal{A}(t) = \mathcal{A}_0(1-t) + \mathcal{A}_1 t$

writing $\mathcal{A}_1 = \mathcal{A}_0\,(\alpha_1 + \beta_1\,\mathbf{i} + \gamma_1\,\mathbf{j} + \delta_1\,\mathbf{k})$ we have:

$\mathbf{r}(t)$ = straight line $\iff$ $(\gamma_1, \delta_1) = (0, 0)$

$\mathbf{r}(t)$ = plane curve $\iff$ $\beta_1 = 0$ and $(\gamma_1, \delta_1) \neq (0, 0)$

NOTE: all spatial PH cubics are helical curves

degenerate spatial PH quintics

spatial PH quintics : use $\mathcal{A}(t) = \mathcal{A}_0(1-t)^2 + \mathcal{A}_1 2(1-t)t + \mathcal{A}_2 t^2$

writing $\mathcal{A}_r = \mathcal{A}_0\,(\alpha_r + \beta_r\,\mathbf{i} + \gamma_r\,\mathbf{j} + \delta_r\,\mathbf{k})$ for $r = 1, 2$ we have:

$\mathbf{r}(t)$ = straight line $\iff$ $(\gamma_1, \delta_1) = (\gamma_2, \delta_2) = (0, 0)$

$\mathbf{r}(t)$ = plane curve $\iff$ $\beta_1 = \beta_2 = 0$ and $\gamma_1\delta_2 - \gamma_2\delta_1 = 0$

with $(\gamma_1, \delta_1) \neq (0, 0)$ and $(\gamma_2, \delta_2) \neq (0, 0)$

conditions for plane curve equivalent to linear dependence of $\mathcal{A}_0, \mathcal{A}_1, \mathcal{A}_2$

morphology of helical PH quintics

$$\gcd(u,v,p,q) = \text{constant} \quad \not\Rightarrow \quad \gcd(x',y',z') = \text{constant}$$

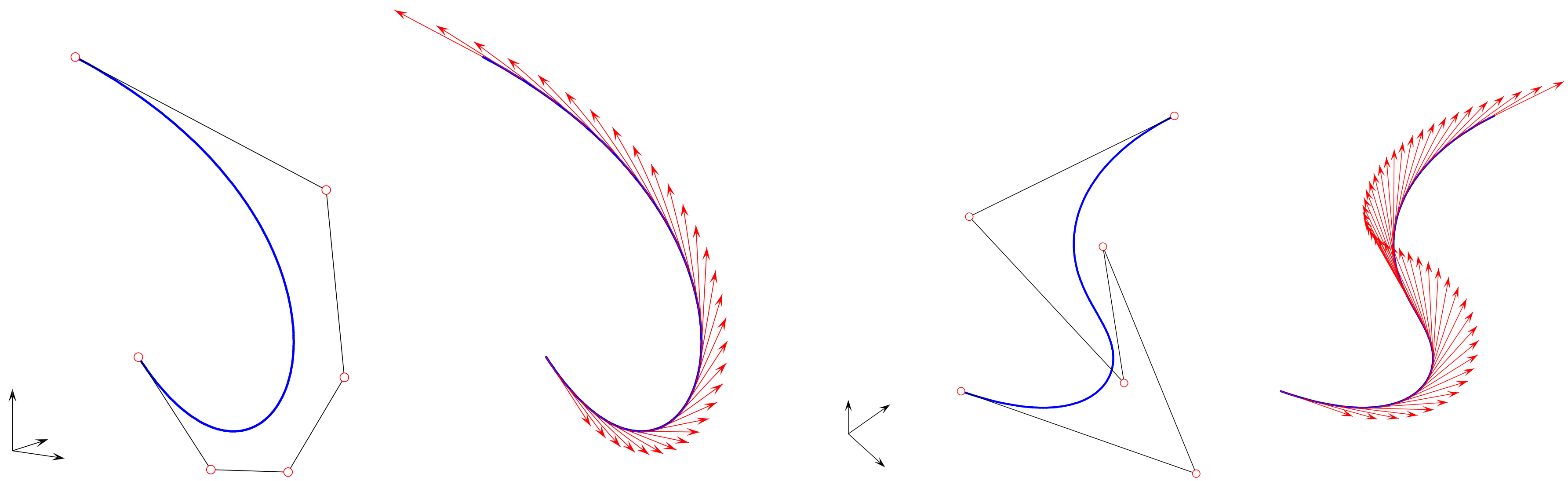
specifically, $\gcd(x',y',z') \,=\, \gcd(u+\mathrm{i}\,v, p-\mathrm{i}\,q)\cdot\gcd(u-\mathrm{i}\,v, p+\mathrm{i}\,q)$

- monotone-helical PH quintics — $\gcd(x',y',z')$ is quadratic

  tangent indicatrix is singly-traced circle (curve tangent maintains a consistent sense of rotation about helix axis)

- general helical PH quintics — $\gcd(x',y',z')$ is a constant

  tangent indicatrix is doubly-traced circle (curve tangent may exhibit reversals in sense of rotation about helix axis)



examples of monotone-helical (left) and general helical (right) PH quintics

$$\mathbf{r}'(t) = \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) \quad \text{with} \quad \mathcal{A}(t) = \mathcal{A}_0(1-t)^2 + \mathcal{A}_1 2(1-t)t + \mathcal{A}_2 t^2$$

$$\text{set} \quad \mathcal{A}_r = \mathcal{A}_0\,(\alpha_r + \beta_r\,\mathbf{i} + \gamma_r\,\mathbf{j} + \delta_r\,\mathbf{k}) \quad \text{for} \quad r = 1, 2$$

$\mathbf{r}(t) \; = \;$ general helical PH quintic

$$\iff \quad \gamma_1 : \gamma_2 = \delta_1 : \delta_2 \quad \text{and} \quad \beta_1 : \beta_2 = (\gamma_1^2 + \delta_1^2) : (\gamma_1\gamma_2 + \delta_1\delta_2)$$

$\mathbf{r}(t) \; = \;$ monotone-helical PH quintic

$$\iff \quad \alpha_2 = \frac{r\alpha_1 + s\beta_1}{\gamma_1^2 + \delta_1^2} + \frac{s^2 - r^2}{4(\gamma_1^2 + \delta_1^2)^2}\,, \quad \beta_2 = \frac{r\beta_1 - s\alpha_1}{\gamma_1^2 + \delta_1^2} + \frac{2rs}{4(\gamma_1^2 + \delta_1^2)^2}$$

where $r = \gamma_1\gamma_2 + \delta_1\delta_2$ and $s = \gamma_1\delta_2 - \gamma_2\delta_1$

for a helical PH space curve with $\sigma(t) = |\mathbf{r}'(t)|$ we have

$$\frac{\kappa}{\tau} = \tan\psi \quad \Rightarrow \quad |\mathbf{r}' \times \mathbf{r}''|^3 = \tan\psi\,\sigma^3\,(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''$$

using also the property $|\mathbf{r}' \times \mathbf{r}''|^2 = \sigma^2\rho$ of all PH space curves gives

$$\rho^{3/2} = \tan\psi\,(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''$$

Lemma. *A necessary condition for a spatial PH curve to be helical is that the polynomial $\rho(t)$ be a perfect square — i.e., the curve must be a double PH curve.*

trivially satisfied for all PH cubics, since $(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''' =$ constant

$\deg((\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}''') = 6$ for PH quintics, so we must have $\rho(t) = \omega^2(t)$ for a quadratic polynomial $\omega(t)$ if $\mathbf{r}(t)$ is a helical PH quintic

“double” Pythagorean-hodograph structure

$|\mathbf{r}'(t)|$ and $|\mathbf{r}'(t) \times \mathbf{r}''(t)|$ are both polynomials in curve parameter t

$$x'^2 + y'^2 + z'^2 \equiv \sigma^2 \,,$$

$$(y'z'' - y''z')^2 + (z'x'' - z''x')^2 + (x'y'' - x''y')^2 \equiv (\sigma\omega)^2 \,.$$

Frenet frame, curvature, torsion are all rational functions of t

$$\mathbf{t} = \frac{\mathbf{r}'}{|\mathbf{r}'|} \,, \qquad \mathbf{n} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \times \mathbf{t} \,, \qquad \mathbf{b} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \,,$$

$$\kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3} \,, \qquad \tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$$

Beltran & Monterde (2007) have called them “2-PH curves” every spatial PH curve satisfies $|\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,|^2 \,=\, \sigma^2(t)\rho(t)$

the polynomial $\rho(t)$ can be defined in terms of $u(t), v(t), p(t), q(t)$ and $u'(t), v'(t), p'(t), q'(t)$ in several different ways :

$$\rho \;=\; 4\,[\;(up' - u'p)^2 \,+\, (uq' - u'q)^2 \,+\, (vp' - v'p)^2 \,+\, (vq' - v'q)^2 \\ +\, 2(uv' - u'v)(pq' - p'q)\;] \qquad (1)$$

$$\rho \;=\; 4\,[\;(uv' - u'v + pq' - p'q)^2 \,+\, (up' - u'p - vq' + v'q)^2 \\ +\, (uq' - u'q + vp' - v'p)^2 \,-\, (uv' - u'v - pq' + p'q)^2\;] \qquad (2)$$

$$\rho \;=\; 4\,[\;(up' - u'p + vq' - v'q)^2 \,+\, (uq' - u'q - vp' + v'p)^2\;] \qquad (3)$$

“double” PH structure — triples and quartuples

for a double PH curve, $\rho(t) = \omega^2(t)$ for some polynomial $\omega(t)$

form (3) of $\rho(t) \;\Rightarrow\; 2(up' - u'p + vq' - v'q)$, $2(uq' - u'q - vp' + v'p)$, ω
must comprise a Pythagorean triple of polynomials

$$
\begin{aligned}
2(up' - u'p + vq' - v'q) &= k(a^2 - b^2) \\
2(uq' - u'q - vp' + v'p) &= 2\,k\,a\,b \\
\omega &= k(a^2 + b^2)
\end{aligned}
$$

for polynomials $k(t)$, $a(t)$, $b(t)$ with $\gcd(a(t), b(t)) =$ constant

hence, double PH curves involve both Pythagorean triples
and Pythagorean quartuples of polynomials !

helical PH quintics as "double" PH curves

$2(up' - u'p + vq' - v'q)$, $2(uq' - u'q - vp' + v'p)$, ω are quadratic

to satisfy second Pythagorean condition, we must have either

(1) $\deg(a(t), b(t)) = 1$ and $k(t) =$ constant

(2) $a(t), b(t) =$ constants and $\deg(k(t)) = 2$

cases (1) & (2) identify monotone-helical and general helical PH quintics

$\Rightarrow$ all double PH quintics are helical curves

there exist non-helical double PH curves

Beltran & Monterde (2007): all double PH cubics and quintics are helical — but there exist double PH curves of degree 7 that are not helical

$$x(t) \,=\, \frac{1}{21}\,t^7 + \frac{1}{5}\,t^5 + t^3 - 3\,t\,, \quad y(t) \,=\, -\frac{1}{2}\,t^4 + 3\,t^2\,, \quad z(t) \,=\, -\,2\,t^3$$

$$|\mathbf{r}'(t)| \,=\, \frac{t^6 + 3\,t^4 + 9\,t^2 + 9}{3}\,, \quad |\mathbf{r}'(t) \times \mathbf{r}''(t)| \,=\, 2(t^2+1)(t^6+3\,t^4+9\,t^2+9)$$

$$\frac{\kappa(t)}{\tau(t)} \,=\, -\,\frac{9\,(t^2+1)^2}{2\,t^6 + 9\,t^4 - 9} \,\neq\, \text{constant}$$

In general, the curvature/torsion ratio for a double PH curve is

$$\frac{\kappa(t)}{\tau(t)} \,=\, \frac{\omega^3(t)}{[\,\mathbf{r}'(t) \times \mathbf{r}''(t)\,] \cdot \mathbf{r}'''(t)}$$

Hopf map model for spatial PH curves

Choi et al. (2002) — alternative to the quaternion representation

Hopf map $\mathbb{C} \times \mathbb{C} = \mathbb{R}^4 \to \mathbb{R}^3$ generates Pythagorean hodographs in $\mathbb{R}^3$

from two complex polynomials $\boldsymbol{\alpha}(t) = u(t) + \mathrm{i}\,v(t)$, $\boldsymbol{\beta}(t) = q(t) + \mathrm{i}\,p(t)$:

$$\begin{aligned}\mathbf{r}'(t) \,=\, H(\boldsymbol{\alpha}(t),\boldsymbol{\beta}(t)) \,&=\, (|\boldsymbol{\alpha}(t)|^2 - |\boldsymbol{\beta}(t)|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)), 2\,\mathrm{Im}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t))) \\ &=\, (\,u^2(t) + v^2(t) - p^2(t) - q^2(t)\,, \\ &\qquad 2(u(t)q(t) + v(t)p(t))\,,\, 2(v(t)q(t) - u(t)p(t))\,)\end{aligned}$$

identify imaginary unit i with quaternion basis element $\mathbf{i}$ — quaternion polynomial $\mathcal{A}(t)$ is related to the complex polynomials $\boldsymbol{\alpha}(t)$ and $\boldsymbol{\beta}(t)$ by

$$\mathcal{A}(t) \,=\, u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k} \,=\, \boldsymbol{\alpha}(t) \,+\, \mathbf{k}\,\boldsymbol{\beta}(t)$$

polynomial $\rho(t)$ has simpler formulation in Hopf map model

$$\boldsymbol{\alpha}\,\boldsymbol{\beta}' - \boldsymbol{\alpha}'\boldsymbol{\beta} \;=\; (uq' - u'q - vp' + v'p) \;+\; \mathrm{i}\,(up' - u'p + vq' - v'q)$$

$$\Rightarrow \quad \rho(t) \;=\; 4\,|\,\boldsymbol{\alpha}(t)\boldsymbol{\beta}'(t) - \boldsymbol{\alpha}'(t)\boldsymbol{\beta}(t)\,|^2$$

restricting $H(\boldsymbol{\alpha},\boldsymbol{\beta})$ to complex numbers satisfying $|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2 = 1$, it defines a map between the "3–sphere" $S^3 \,:\, u^2 + v^2 + p^2 + q^2 = 1$ in the space $\mathbb{R}^4$ spanned by coordinates (u,v,p,q) and the familiar "2–sphere" $S^2 \,:\, x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$ with coordinates (x,y,z)

great circles of S^3 are mapped to points of S^2 by $H(\boldsymbol{\alpha},\boldsymbol{\beta})$

first known map between higher and lower dimension spheres that is not null homotopic (applications to quantum computing)

spatial PH quintic Hermite interpolants

spatial PH quintic interpolating end points $\mathbf{p}_i$, $\mathbf{p}_f$ & derivatives $\mathbf{d}_i$, $\mathbf{d}_f$

$$\mathbf{r}'(t) \;=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$$

where $\quad \mathcal{A}(t) \;=\; \mathcal{A}_0(1-t)^2 \,+\, \mathcal{A}_1 2(1-t)t \,+\, \mathcal{A}_2 t^2$

three equations in three quaternion unknowns $\mathcal{A}_0$, $\mathcal{A}_1$, $\mathcal{A}_2$

$$\mathbf{r}'(0) \;=\; \mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^* \;=\; \mathbf{d}_i \qquad \text{and} \qquad \mathbf{r}'(1) \;=\; \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^* \;=\; \mathbf{d}_f$$

$$\begin{aligned} \int_0^1 \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)\,\mathrm{d}t \;&=\; \tfrac{1}{5}\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_0^* \,+\, \tfrac{1}{10}(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_1^* + \mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_0^*) \\ &+\; \tfrac{1}{30}(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* + 4\,\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*) \\ &+\; \tfrac{1}{10}(\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_1^*) \,+\, \tfrac{1}{5}\mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_2^* \;=\; \mathbf{p}_f - \mathbf{p}_i \end{aligned}$$

solution of fundamental equation

given vector $\mathbf{c} = |\mathbf{c}|(\lambda, \mu, \nu)$ find quaternion $\mathcal{A}$ such that

$$\mathcal{A}\,\mathbf{i}\,\mathcal{A}^* \;=\; \mathbf{c}$$

one–parameter family of solutions

$$\mathcal{A}(\phi) \;=\; \sqrt{\tfrac{1}{2}(1+\lambda)|\mathbf{c}|}\,\Big(-\sin\phi \;+\; \cos\phi\,\mathbf{i} \\ +\; \frac{\mu\cos\phi + \nu\sin\phi}{1+\lambda}\mathbf{j} \;+\; \frac{\nu\cos\phi - \mu\sin\phi}{1+\lambda}\mathbf{k}\Big)$$

in $\mathbb{R}^3$ there is a continuous family of rotations
mapping the vector $\mathbf{i}$ into a given vector (λ, μ, ν)

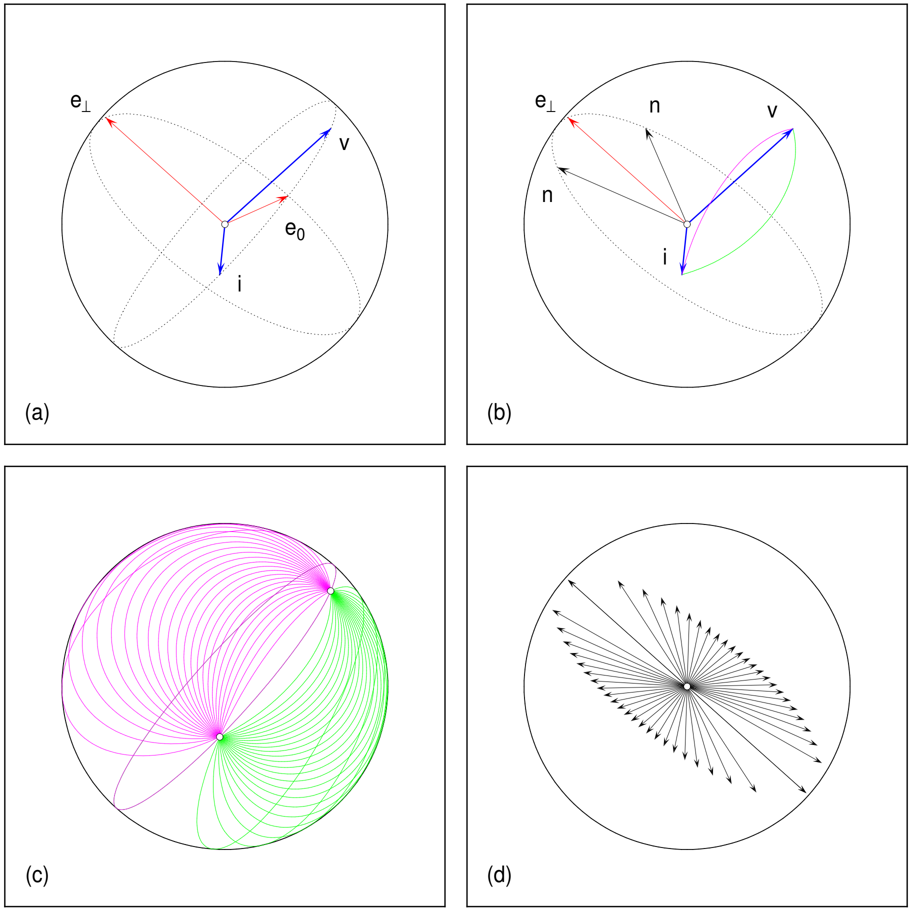
families of spatial rotations

find $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$ that rotates $\mathbf{i} = (1,0,0) \longrightarrow \mathbf{v} = (\lambda, \mu, \nu)$

$$\begin{aligned} n_x^2(1-\cos\theta) + \cos\theta &= \lambda\,, \\ n_x n_y(1-\cos\theta) + n_z\sin\theta &= \mu\,, \\ n_z n_x(1-\cos\theta) - n_y\sin\theta &= \nu\,. \end{aligned}$$

$$\begin{aligned} n_x &= \frac{\pm\sqrt{\cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\theta}}{\sin\frac{1}{2}\theta}\,, \\ n_y &= \frac{\pm\,\mu\sqrt{\cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\theta}\; - \;\nu\cos\frac{1}{2}\theta}{(1+\lambda)\sin\frac{1}{2}\theta}\,, \\ n_z &= \frac{\pm\,\nu\sqrt{\cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\theta}\; + \;\mu\cos\frac{1}{2}\theta}{(1+\lambda)\sin\frac{1}{2}\theta}\,. \end{aligned}$$

general solution, where $\alpha = \cos^{-1}\lambda$ and $\alpha \le \theta \le 2\pi - \alpha$



Spatial rotations of unit vectors $\mathbf{i} \to \mathbf{v}$. (a) Vectors $\mathbf{e}_\perp$ (orthogonal to $\mathbf{i}$, $\mathbf{v}$) and $\mathbf{e}_0$ (bisector of $\mathbf{i}, \mathbf{v}$) — the plane Π of $\mathbf{e}_\perp$ and $\mathbf{e}_0$ is orthogonal to that of $\mathbf{i}$ and $\mathbf{v}$. (b) For any rotation angle $\theta \in (\alpha, 2\pi - \alpha)$, where $\alpha = \cos^{-1}(\mathbf{i} \cdot \mathbf{v})$, there are two possible rotations, with axes $\mathbf{n}$ inclined equally to $\mathbf{e}_\perp$ in the plane Π. (c) Sampling of the family of spatial rotations $\mathbf{i} \to \mathbf{v}$, shown as loci on the unit sphere. (d) Axes $\mathbf{n}$ for these rotations, lying in the plane Π.

construction of Hermite interpolants

derivative conditions have form of fundamental equation

— can be solved directly for $\mathcal{A}_0$ and $\mathcal{A}_2$

end-point condition can then be cast in fundamental form as

$$(3\mathcal{A}_0 + 4\mathcal{A}_1 + 3\mathcal{A}_2)\,\mathbf{i}\,(3\mathcal{A}_0 + 4\mathcal{A}_1 + 3\mathcal{A}_2)^*$$

$$= 120(\mathbf{p}_f - \mathbf{p}_i) - 15(\mathbf{d}_i + \mathbf{d}_f) + 5(\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* + \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^*)$$

— solve for $\mathcal{A}_1$, since $\mathcal{A}_0$ and $\mathcal{A}_2$ known

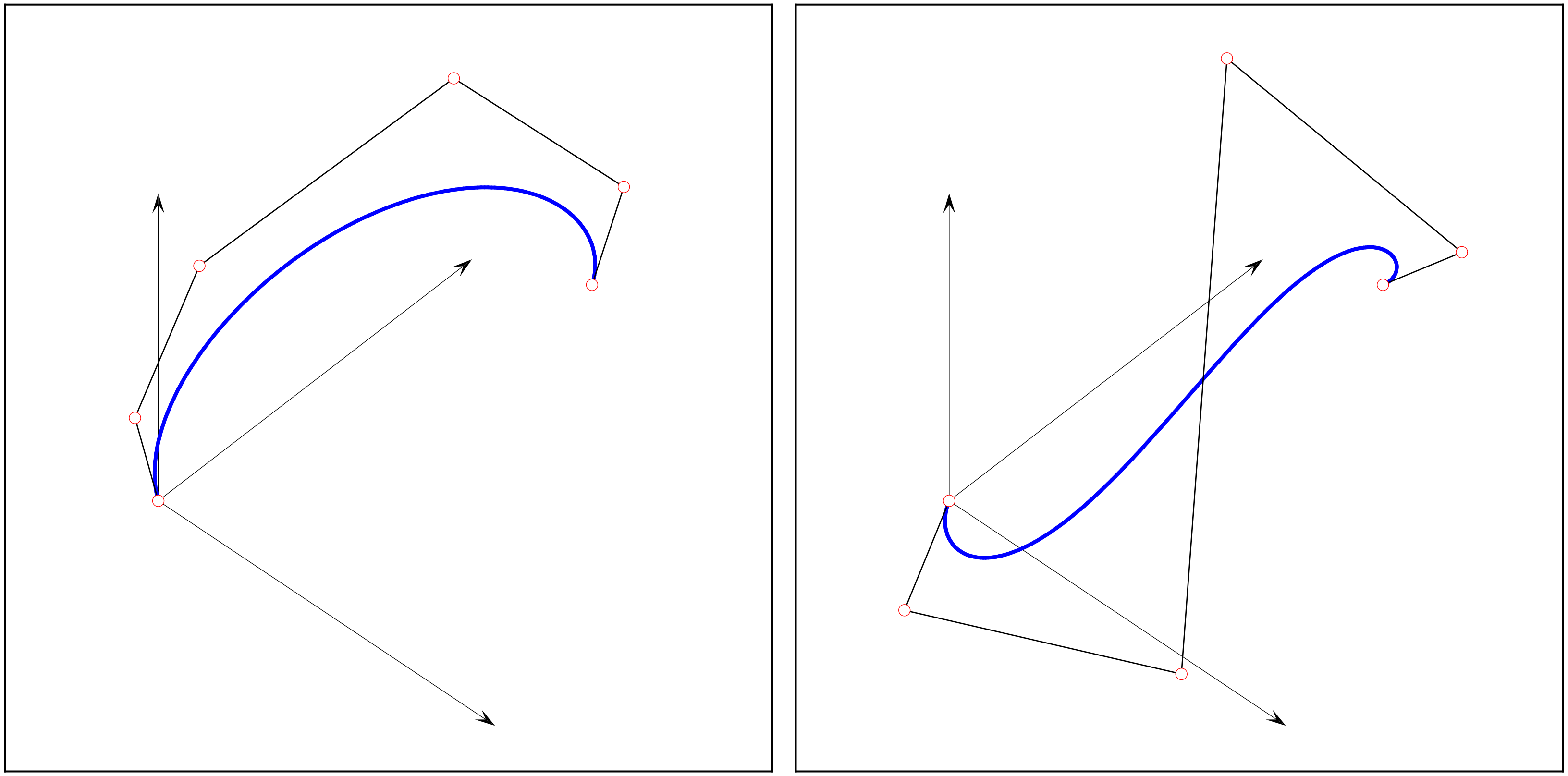
solution contains three free parameters ϕ_0, ϕ_1, ϕ_2

but shape of interpolants depends only on their differences

$\Longrightarrow$ $\exists$ two-parameter family of spatial PH quintic interpolants

to given Hermite data $\mathbf{p}_i$, $\mathbf{d}_i$ and $\mathbf{p}_f$, $\mathbf{d}_f$

spatial PH quintic Hermite interpolants



$\mathbf{p}_i = (0,0,0)$ and $\mathbf{p}_f = (1,1,1)$ for both curves

$\mathbf{d}_i = (-0.8, 0.3, 1.2)$ and $\mathbf{d}_f = (0.5, -1.3, -1.0)$ for curve on left,

$\mathbf{d}_i = (0.4, -1.5, -1.2)$ and $\mathbf{d}_f = (-1.2, -0.6, -1.2)$ for curve on right

open problem: find "optimal" ϕ_0, ϕ_2 values

shape of interpolants depends strongly on free parameters

- minimize a shape-measure integral, e.g., $E = \int \kappa^2 \, \mathrm{d}s$
  (but highly non-linear in the free parameters)

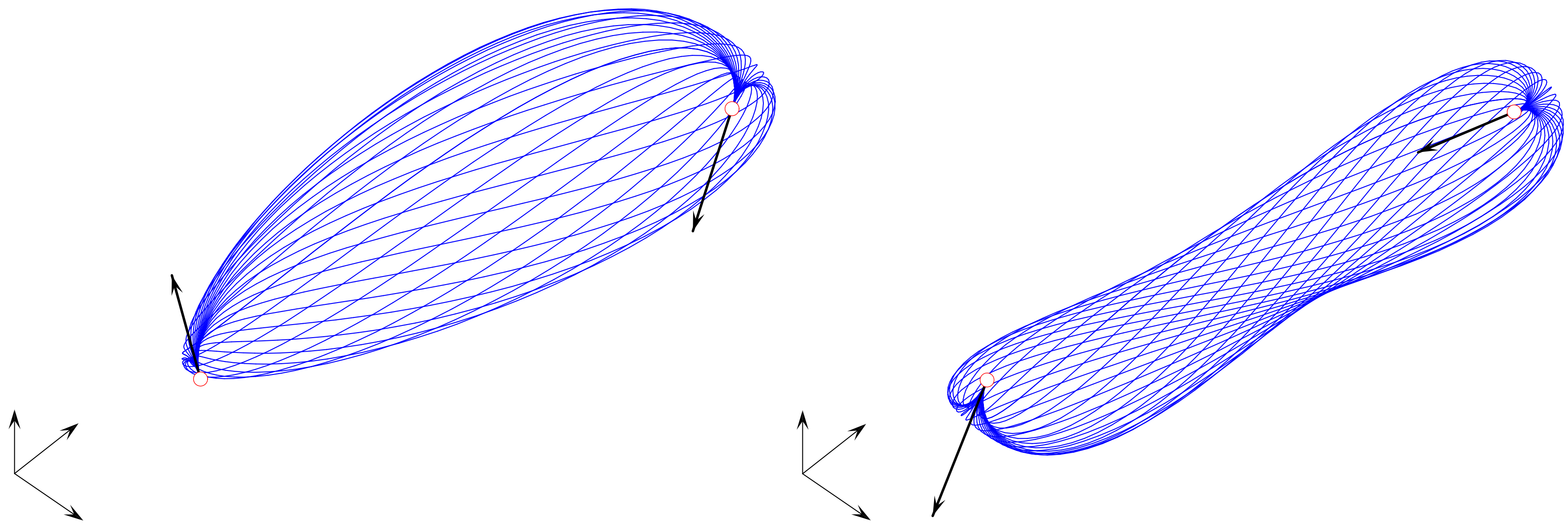
- impose additional conditions (restrict solution space)
  — e.g., helicity condition $\kappa / \tau = \mathrm{constant}$

- study geometry of quaternion curve $\mathcal{A}(t)$
  — need better insight on geometry of quaternion space $\mathbb{H}$

- extension to spatial C^2 PH quintic splines

two-parameter family of Hermite interpolants

nominal parameters: ϕ_0, ϕ_2 — arc length of interpolants depends only on difference $\phi_2 - \phi_0$, shape of interpolants depends only on mean $\frac{1}{2}(\phi_0 + \phi_2)$



sampling of the one-parameter families of spatial PH quintic interpolants, of identical arc length, to given first-order Hermite data — defined by holding $\phi_2 - \phi_0$ constant, and varying only $\frac{1}{2}(\phi_0 + \phi_2)$

recent results on Hermite interpolants

(Farouki, Giannelli, Manni, Sestini, 2007)

- dependence of total arc length S exhibits a single minimum and a single maximum with respect to the variable $\phi_2 - \phi_0$
- these extremal arc length interpolants correspond to helical PH quintics
- $\Rightarrow$ helical PH quintic interpolants exist for any first-order Hermite data
- three "practical" criteria for identifying interpolants with near-optimal shape properties (all reproduce cubic PH interpolants when they exist)
- give values of the energy integral close to the absolute minimum, at modest computational cost

closure

- spatial PH curves ideally suited to computing rotation-minimizing frames (symbolic integration or rational approximation)

- helical polynomial space curves are always PH curves
  — two quintic types (monotone and general helical PH quintics)

- double PH curves have rational Frenet frames, curvature, torsion
  — all helical PH curves are necessarily double PH curves

- properties of solutions to first-order Hermite interpolation problem

- don’t believe a Russian who tells you he has stopped drinking