Introduction to Pythagorean-hodograph curves

Rida T. Farouki

Department of Mechanical & Aeronautical Engineering,
University of California, Davis

— synopsis —

- impossibility of rational arc-length parameterizations
- "simple" parametric speed — Pythagorean-hodograph (PH) curves
- rational offsets and polynomial arc-length functions for PH curves
- planar PH curves — complex variable representation
- spatial PH curves — quaternion and Hopf map models
- extensions and generalizations — rational PH curves, Minkowski PH curves, Minkowski iosperimetric-hodograph curves
- special classes of spatial PH curves — helical polynomial curves, double PH curves, rational rotation-minimizing frame curves
- applications of PH curves to motion control problems

By virtue of their special algebraic structures, Pythagorean-hodograph (PH) curves offer unique advantages for computer-aided design and manufacturing, robotics, motion control, path planning, computer graphics, animation, and related fields. This book offers a comprehensive and self-contained treatment of the mathematical theory of PH curves, including algorithms for their construction and examples of their practical applications. Special features include an emphasis on the interplay of ideas from algebra and geometry and their historical origins, detailed algorithm descriptions, and many figures and worked examples. The book may appeal, in whole or in part, to mathematicians, computer scientists, and engineers.

ISBN 978-3-540-73397-3

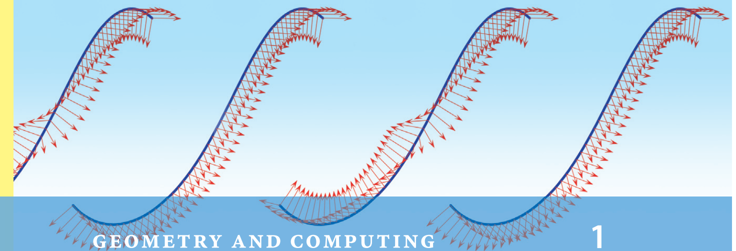
9 783540 733973

› springer.com

Farouki

GEOMETRY AND COMPUTING 1

Pythagorean-Hodograph Curves



GEOMETRY AND COMPUTING 1

Rida T. Farouki

Pythagorean-Hodograph Curves

Algebra and Geometry Inseparable

Springer

ISBN 978-3-540-73397-3 (2008) 728 pp. + 204 illustrations

curve representations — terminology

polynomial curve — $x(t) = \sum_{k=0}^{n} a_k t^k, \quad y(t) = \sum_{k=0}^{n} b_k t^k$

"simplest" non-trivial curves $\rightarrow$ piecewise-polynomial (spline) curves

rational curve — $x(t) = \dfrac{\sum_{k=0}^{n} a_k t^k}{\sum_{k=0}^{n} c_k t^k}, \quad y(t) = \dfrac{\sum_{k=0}^{n} b_k t^k}{\sum_{k=0}^{n} c_k t^k}$

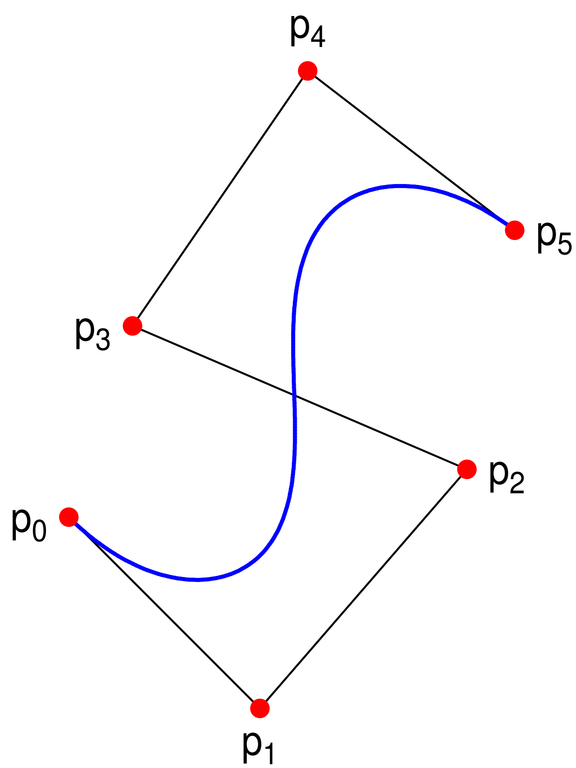
exact representation of conics, closure under projective transformations

algebraic curve — $f(x,y) = \sum_{j+k=n} c_{jk}\, x^j y^k = 0$

constitute a superset of the polynomial and rational (genus 0) curves

Bezier curve

P. de Casteljau (Citroen) – P. Bezier (Renault)



convex hull

subdivision

variation diminishing

numerical stability

$$r(t) = \sum_{k=0}^{n} p_k \binom{n}{k} (1-t)^{n-k} t^k$$

Bernstein basis on [0,1] :

$$[\,(1-t) + t\,]^n = (1-t)^n + n(1-t)^{n-1}t + \ldots + t^n$$

impossibility of rational arc-length parameterization

Theorem. It is impossible to parameterize any plane curve, other than a straight line, by rational functions of its arc length.

rational parameterization $\mathbf{r}(t) = (x(t), y(t)) \implies$ curve points can be exactly computed by *a finite sequence of arithmetic operations*

arc length parameterization $\mathbf{r}(t) = (x(t), y(t)) \implies$ equal parameter increments Δt generate *equidistantly spaced points along the curve*

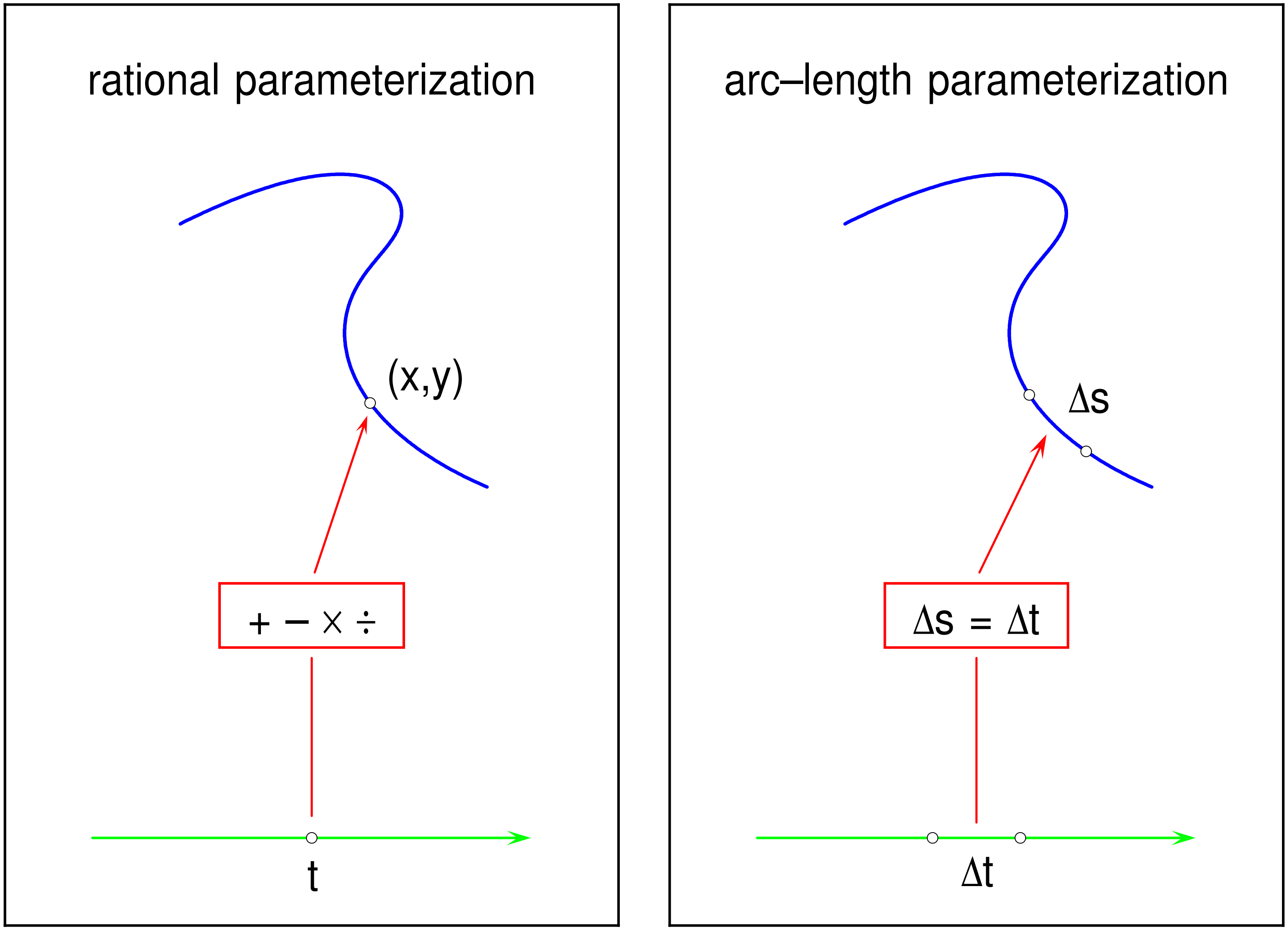
simple result but subtle proof — Pythagorean triples of polynomials, integration of rational functions, and calculus of residues

R. T. Farouki and T. Sakkalis (1991), Real rational curves are not "unit speed," *Comput. Aided Geom. Design* **8**, 151–157

R. T. Farouki and T. Sakkalis (2007), Rational space curves are not "unit speed," *Comput. Aided Geom. Design* **24**, 238–240

T. Sakkalis, R. T. Farouki, and L. Vaserstein (2009), Non–existence of rational arc length parameterizations for curves in $\mathbb{R}^n$, *J. Comp. Appl. Math.* **228**, 494–497

arc length parameterization by rational functions?



rational arc-length parameterization?

$$x(t) = \frac{X(t)}{W(t)}\,,\;\; y(t) = \frac{Y(t)}{W(t)} \quad \text{with } \gcd(W,X,Y) = 1\,,\; W(t) \neq \text{constant}$$

$$x'^2(t) + y'^2(t) \;\equiv\; 1 \quad \Rightarrow \quad (WX' - W'X)^2 + (WY' - W'Y)^2 \;\equiv\; W^4$$

Pythagorean triple $\Rightarrow \; (x', y') \;=\; \left(\dfrac{u^2 - v^2}{u^2 + v^2}\,, \dfrac{2uv}{u^2 + v^2} \right)$

are $\;\; x(t) \;=\; \displaystyle\int \frac{u^2 - v^2}{u^2 + v^2}\, \mathrm{d}t\,, \quad y(t) \;=\; \int \frac{2uv}{u^2 + v^2}\, \mathrm{d}t$ both rational?

$$\frac{f(t)}{g(t)} \;=\; \sum_{i=1}^{N} \sum_{j=1}^{m_i} \frac{C_{ij}}{(t - z_i)^j} \;+\; \frac{\bar{C}_{ij}}{(t - \bar{z}_i)^j}$$

$C_{i1} \;=\; \underset{t=z_i}{\text{residue}} \dfrac{f(t)}{g(t)}\,, \qquad \displaystyle\int \frac{f(t)}{g(t)}\, \mathrm{d}t$ is rational $\iff \; C_{i1} = \bar{C}_{i1} = 0$

$$\int_{-\infty}^{+\infty} \frac{f(t)}{g(t)}\,\mathrm{d}t \;=\; 2\pi\mathrm{i} \sum_{\mathrm{Im}(z_i)>0} \text{residue}\; \frac{f(t)}{g(t)}\bigg|_{t=z_i}$$

rational indefinite integral $\iff$ zero definite integral

proof by contradiction: $x(t) = \int \frac{u^2 - v^2}{u^2+v^2}\,\mathrm{d}t\,,\quad y(t) = \int \frac{2uv}{u^2+v^2}\,\mathrm{d}t$

assume both rational with $u(t), v(t) \not\equiv 0$ and $\gcd(u,v)=1$

choose α, β so that $\deg(\alpha u + \beta v)^2 \;<\; \deg(u^2+v^2)$

$\int \frac{(\alpha u+\beta v)^2}{u^2+v^2}\,\mathrm{d}t \;=\; \frac{1}{2}(\alpha^2-\beta^2)\,x(t) \;+\; \alpha\beta\, y(t) \;+\; \frac{1}{2}(\alpha^2+\beta^2)\,t$ is rational

$$\Rightarrow \quad \int_{-\infty}^{+\infty} \frac{(\alpha u+\beta v)^2}{u^2+v^2}\,\mathrm{d}t \;=\; 0 \quad \Rightarrow \quad \frac{[\,\alpha\, u(t) + \beta\, v(t)\,]^2}{u^2(t)+v^2(t)} \;\equiv\; 0$$

contradicts $u(t), v(t) \not\equiv 0$ or $\gcd(u,v)=1$

parametric speed of curve $\mathbf{r}(\xi)$

$$\sigma(\xi) = |\mathbf{r}'(\xi)| = \frac{\mathrm{d}s}{\mathrm{d}\xi} = \text{derivative of arc length } s \text{ w.r.t. parameter } \xi$$

$$= \sqrt{x'^2(\xi) + y'^2(\xi)} \quad \text{for plane curve}$$

$$= \sqrt{x'^2(\xi) + y'^2(\xi) + z'^2(\xi)} \quad \text{for space curve}$$

$\sigma(\xi) \equiv 1$ — i.e., $s \equiv \xi$ — for arc-length or “natural” parameterization, but impossible for any polynomial or rational curve except a straight line

irrational nature of $\sigma(\xi)$ has unfortunate computational implications:

- arc length must be computed approximately by numerical quadrature
- unit tangent $\mathbf{t}$, normal $\mathbf{n}$, curvature κ, etc, not rational functions of ξ
- offset curve $\mathbf{r}_d(\xi) = \mathbf{r}(\xi) + d\,\mathbf{n}(\xi)$ at distance d must be approximated
- requires approximate real-time CNC interpolator algorithms, for motion along $\mathbf{r}(\xi)$ with given speed (feedrate) $V = \mathrm{d}s/\mathrm{d}t$

curves with “simple” parametric speed

Although $\sigma(\xi) = 1$ is impossible, we can gain significant advantages by considering curves for which the argument of $\sqrt{x'^2(\xi) + y'^2(\xi)}$ or $\sqrt{x'^2(\xi) + y'^2(\xi) + z'^2(\xi)}$ is a perfect square — i.e., polynomial curves whose hodograph components satisfy the Pythagorean conditions

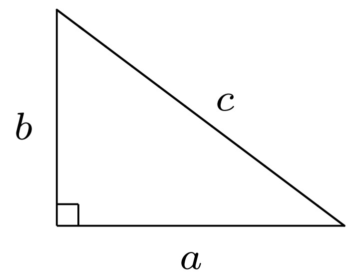
$$x'^2(\xi) + y'^2(\xi) \,=\, \sigma^2(\xi) \qquad \text{or} \qquad x'^2(\xi) + y'^2(\xi) + z'^2(\xi) \,=\, \sigma^2(\xi)$$

for some polynomial $\sigma(\xi)$. To achieve this, the Pythagorean structure must be built into the hodograph a priori, by a suitable algebraic model.

planar PH curves — Pythagorean structure of $\mathbf{r}'(t)$ achieved through complex variable model

spatial PH curves — Pythagorean structure of $\mathbf{r}'(t)$ achieved through quaternion or Hopf map models

higher dimenions or Minkowski metric — Clifford algebra formulation



a, b, c = **real numbers**

choose any $a, b \;\longrightarrow\; c = \sqrt{a^2 + b^2}$

a, b, c = **integers**

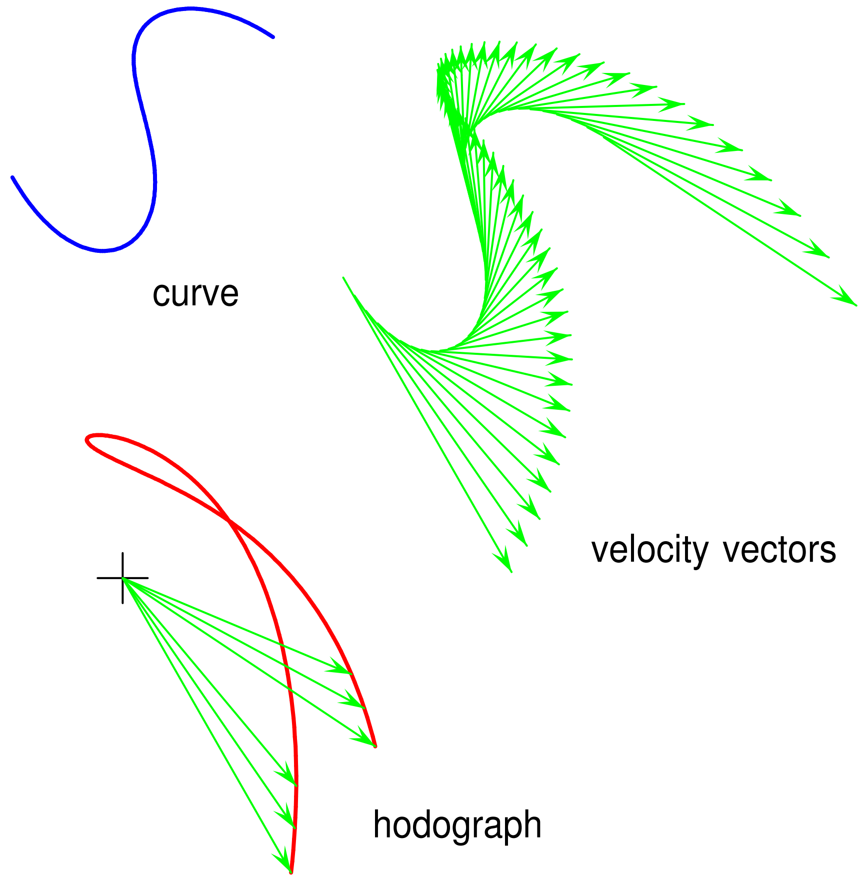
$$a^2 + b^2 = c^2 \iff \begin{cases} a = (u^2 - v^2)w \\ b = 2uvw \\ c = (u^2 + v^2)w \end{cases}$$

$a(t), b(t), c(t)$ = **polynomials**

$$a^2(t) + b^2(t) \equiv c^2(t) \iff \begin{cases} a(t) = [\,u^2(t) - v^2(t)\,]\,w(t) \\ b(t) = 2\,u(t)v(t)w(t) \\ c(t) = [\,u^2(t) + v^2(t)\,]\,w(t) \end{cases}$$

K. K. Kubota, *Amer. Math. Monthly* **79**, 503 (1972)

hodograph of curve $\mathbf{r}(t)$ = derivative $\mathbf{r}'(t)$



Pythagorean structure: $x'^2(t) + y'^2(t) = \sigma^2(t)$ for some polynomial $\sigma(t)$

Pythagorean-hodograph (PH) curves

$\mathbf{r}(t)$ is a PH curve in $\mathbb{R}^n \iff$ coordinate components of $\mathbf{r}'(t)$ are elements of a "Pythagorean $(n+1)$-tuple of polynomials"

PH curves exhibit special algebraic structures in their hodographs

- rational offset curves $\mathbf{r}_d(t) = \mathbf{r}(t) + d\,\mathbf{n}(t)$
- polynomial arc-length function $s(t) = \int_0^t |\mathbf{r}'(\tau)|\,\mathrm{d}\tau$
- closed-form evaluation of energy integral $E = \int_0^1 \kappa^2\,\mathrm{d}s$
- real–time CNC interpolators, rotation-minimizing frames, etc.

generalize PH curves to non-Euclidean metrics & other functional forms

planar offset curves

plane curve $\mathbf{r}(t) = (x(t), y(t))$ with unit normal $\mathbf{n}(t) = \dfrac{(y'(t), -x'(t))}{\sqrt{x'^2(t) + y'^2(t)}}$

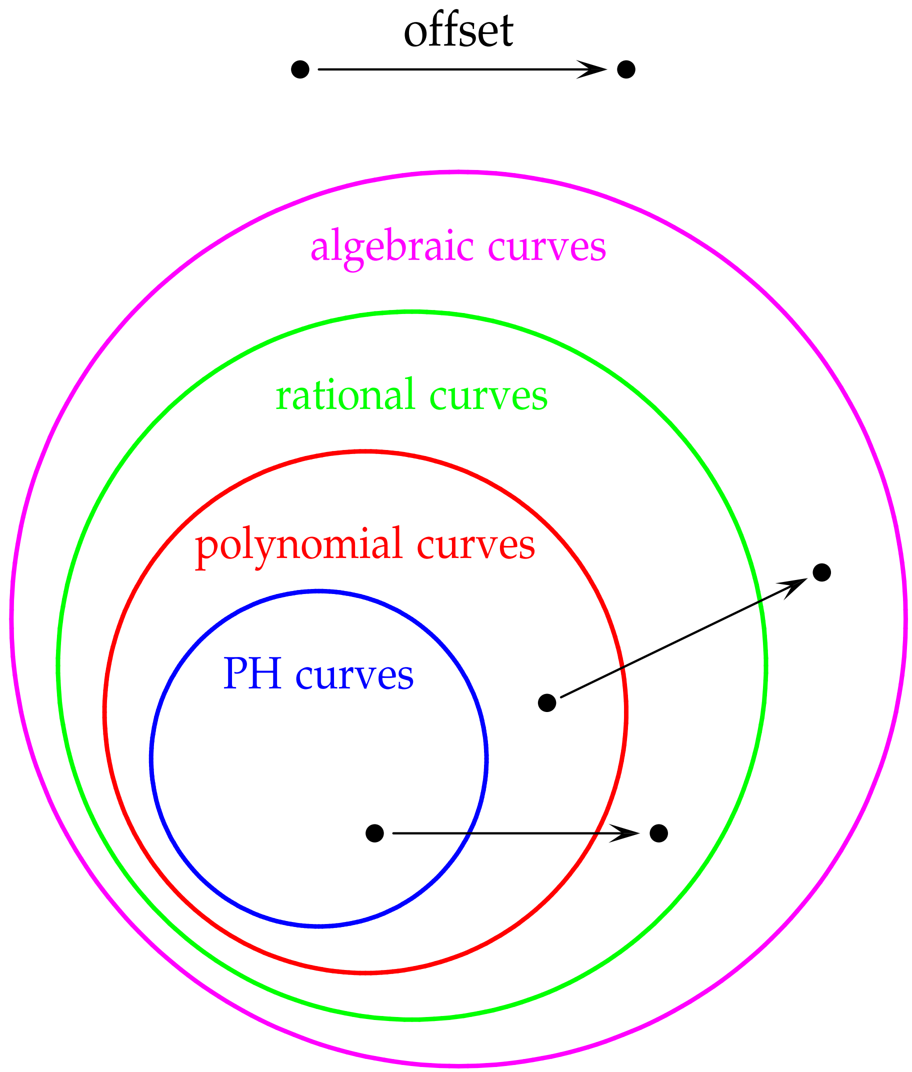
offset at distance d defined by $\mathbf{r}_d(t) = \mathbf{r}(t) + d\,\mathbf{n}(t)$

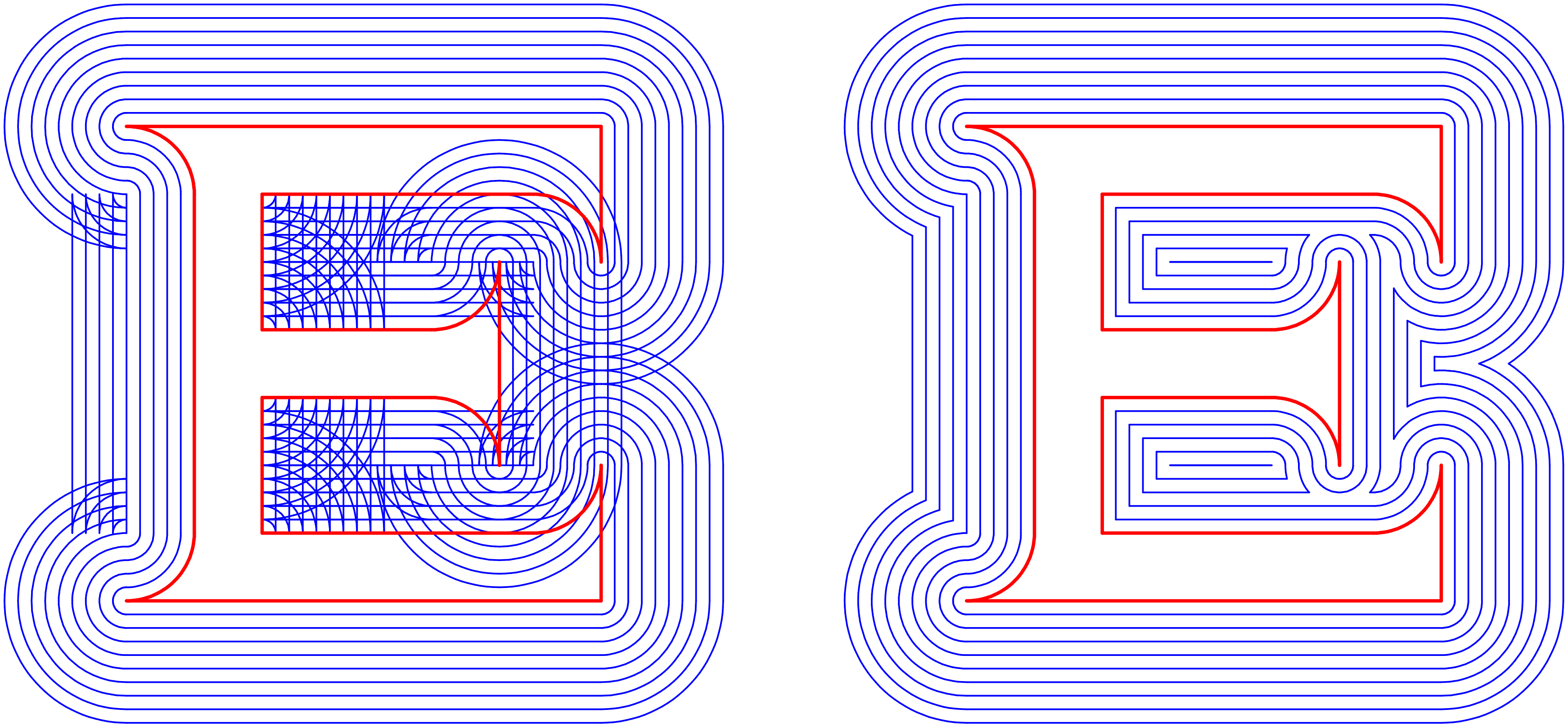
- defines center–line tool path, in order to cut a desired profile
- defines tolerance zone characterizing allowed variations in part shape
- defines erosion & dilation operators in mathematical morphology, image processing, geometrical smoothing procedures, etc.
- offset curves typically approximated in CAD systems
- PH curves have exact rational offset curve representations

taxonomy of offset curves

- offsets to $\{$ lines, circles $\} = \{$ lines, circles $\}$
- (2-sided) offset to parabola $=$ rational curve of degree 6
  (requires doubly-traced parameterization of parabola)
- (2-sided) offset to ellipse / hyperbola $=$ algebraic curve of degree 8
- (2-sided) offset to degree n Bézier curve
  $=$ algebraic curve of degree $4n-2$ in general
- (1-sided) offset to polynomial PH curve of degree n
  $=$ rational curve of degree $2n-1$
- (1-sided) offset to rational PH curve of degree n
  $=$ rational curve of degree n

offsets to Pythagorean–hodograph (PH) curves

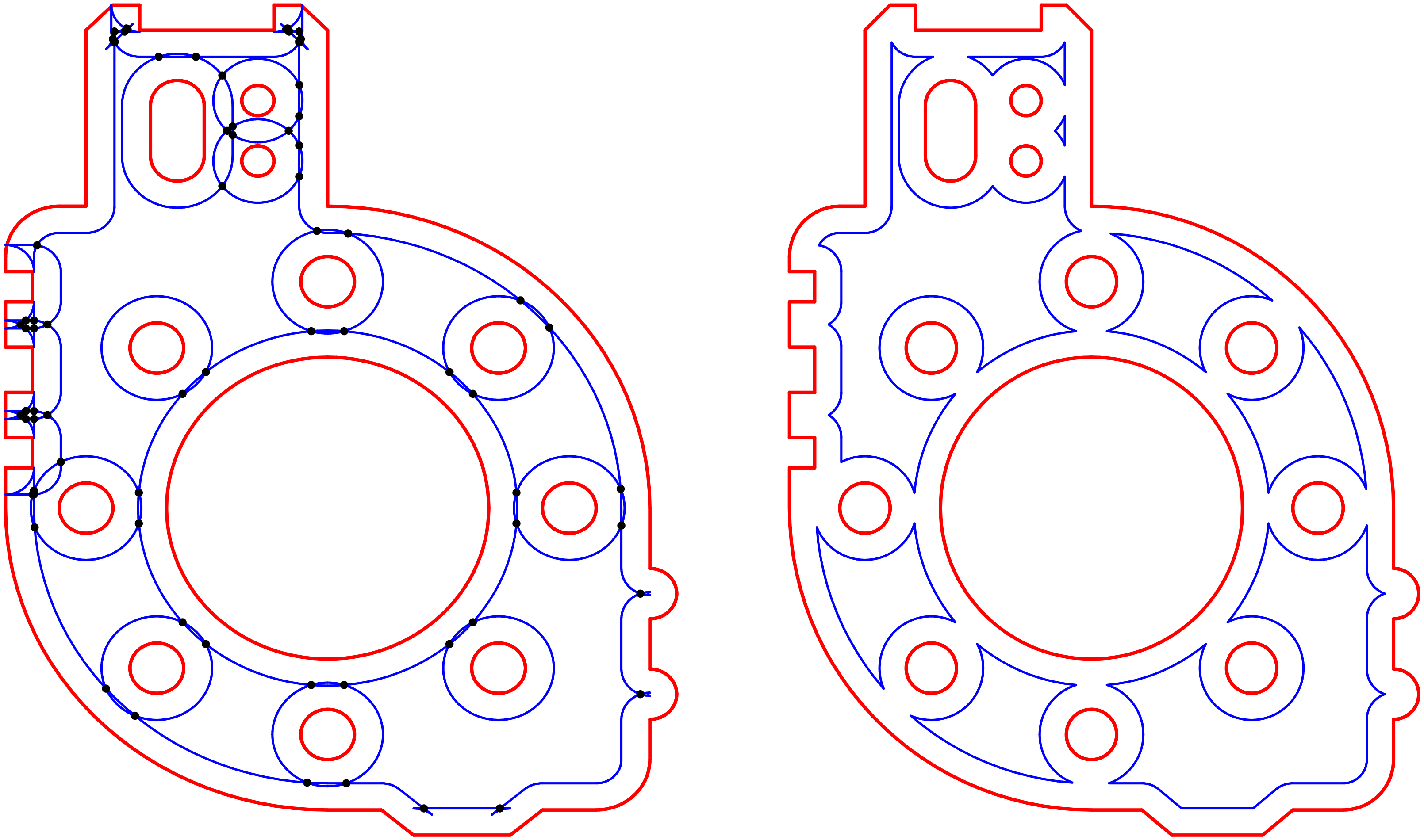




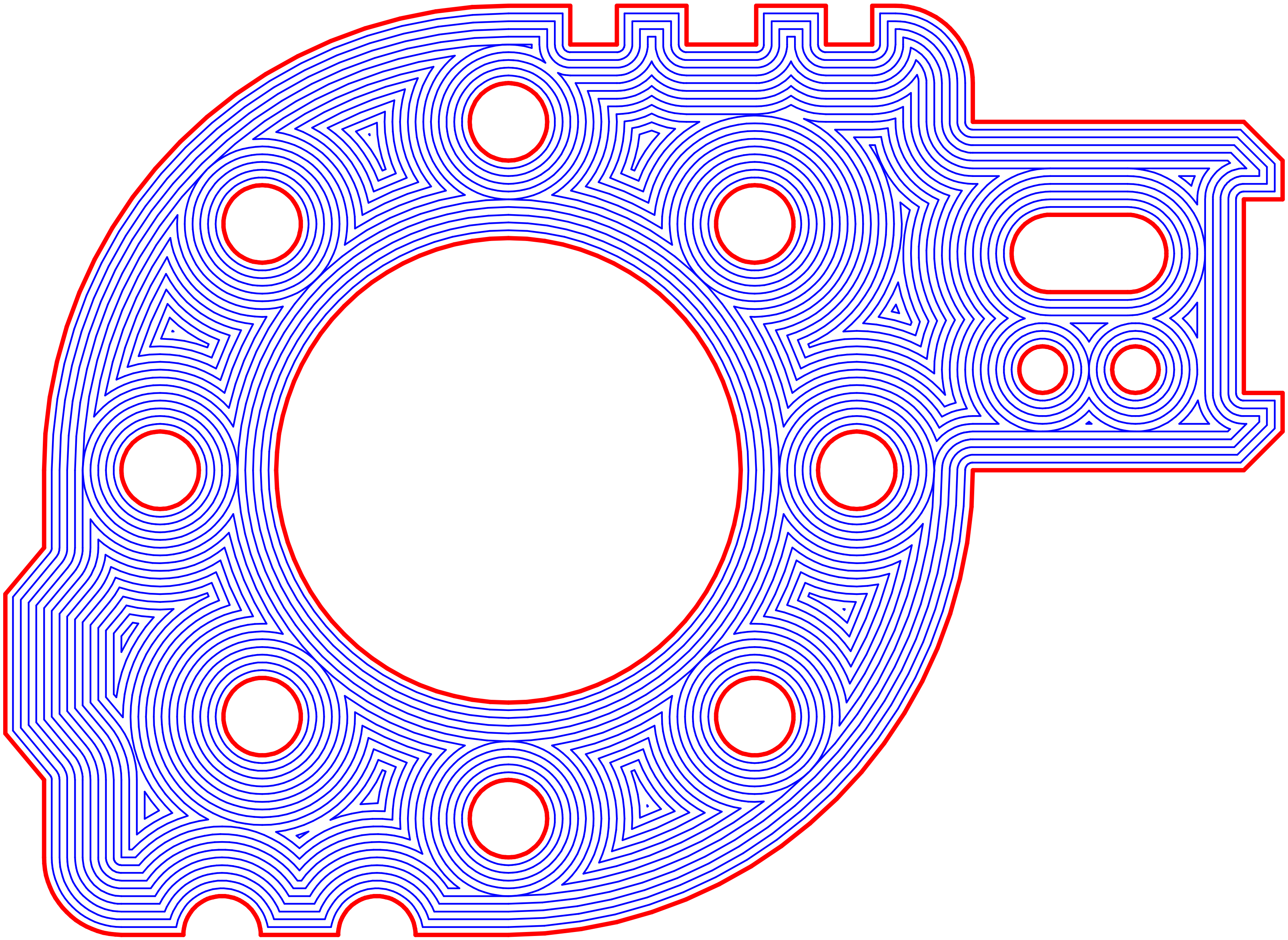
Left: untrimmed offsets obtained by seeeping a normal vector of length d around the original curve (including approrpiate rotations at vertices).

Right: trimmed offsets, obtained by deleting certain segments of the untrimmed offsets, that are not globally distance d from the given curve.

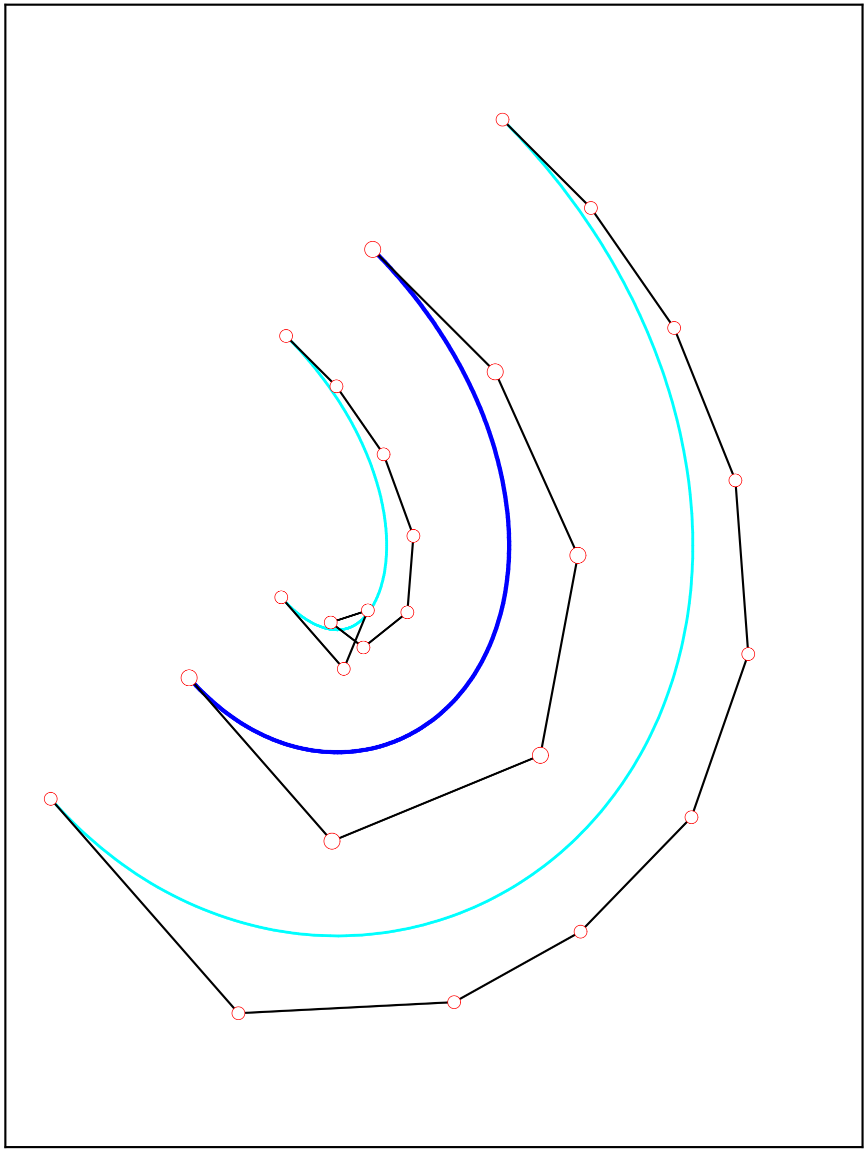
offset curve trimming procedure



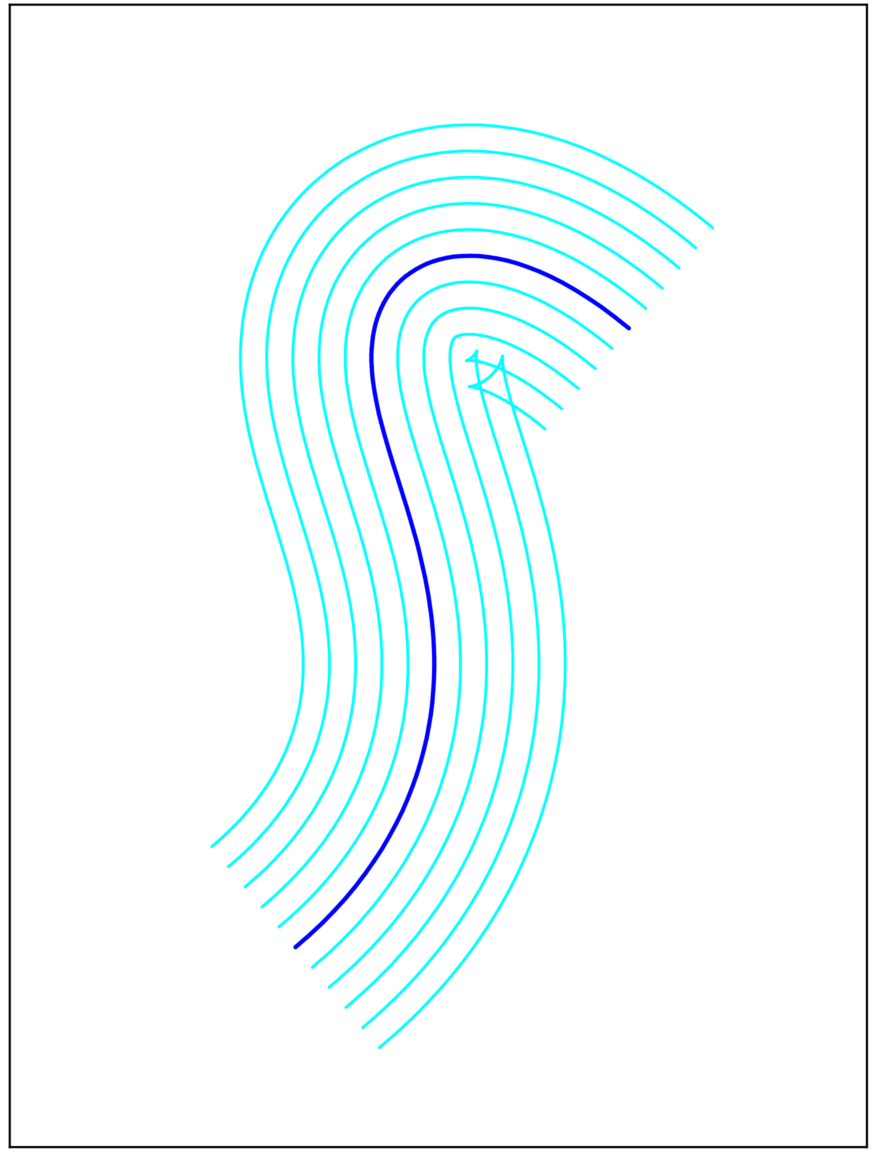
Left: self-intersections of the untrimmed offset. Right: trimmed offset, after discarding segments between these points that fail the distance test.



medial axis apparent as locus of tangent-discontinuities on offsets

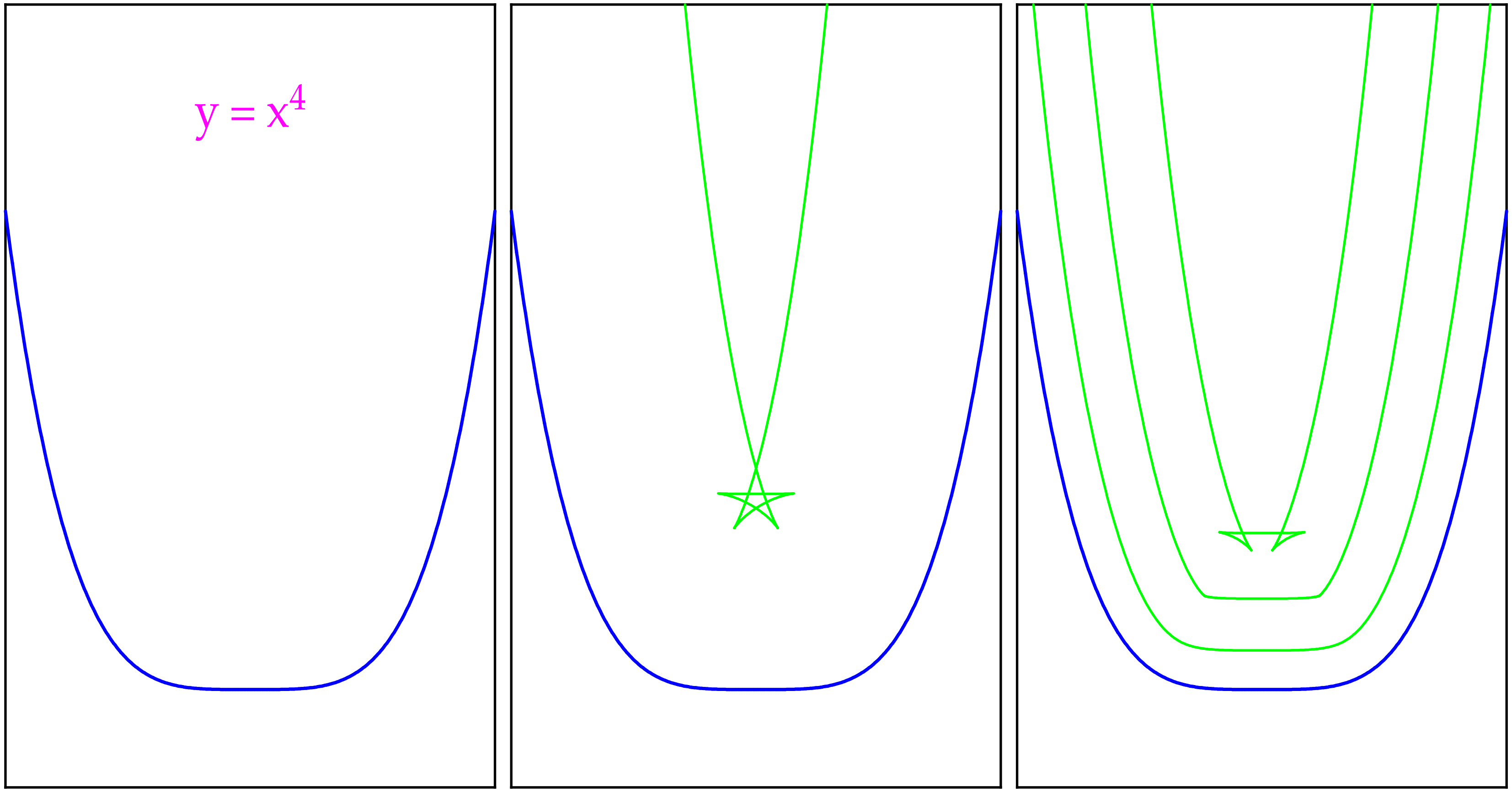


Bezier control polygons of rational offsets



offsets exact at any distance

intricate topology of parallel (offset) curves



"innocuous" curve

offset distance = 1

4 cusps, 6 self–intersections

offset distance

$< d_{crit}$ $= d_{crit}$ $> d_{crit}$

offset curve geometry governed by Huygens' principle (geometrical optics)

polynomial arc length function $s(\xi)$

for a planar PH curve of degree $n = 2m+1$ specified by

$\mathbf{r}'(\xi) = (x'(\xi), y'(\xi)) = (u^2(\xi) - v^2(\xi), 2u(\xi)v(\xi))$ where

$$u(\xi) \,=\, \sum_{k=0}^{m} u_k \binom{m}{k} (1-\xi)^{m-k}\xi^k \,, \quad v(\xi) \,=\, \sum_{k=0}^{m} v_k \binom{m}{k} (1-\xi)^{m-k}\xi^k \,,$$

the parametric speed can be expressed in Bernstein form as

$$\sigma(\xi) \,=\, |\mathbf{r}'(\xi)| \,=\, u^2(\xi) + v^2(\xi) \,=\, \sum_{k=0}^{2m} \sigma_k \binom{2m}{k} (1-\xi)^{2m-k}\xi^k \,,$$

where $\sigma_k \,=\, \displaystyle\sum_{j=\max(0,k-m)}^{\min(m,k)} \frac{\binom{m}{j}\binom{m}{k-j}}{\binom{n-1}{k}} (u_j u_{k-j} + v_j v_{k-j}) \,.$

The cumulative arc length $s(\xi)$ is then the polynomial function

$$s(\xi) \,=\, \int_0^\xi \sigma(\tau)\,\mathrm{d}\tau \,=\, \sum_{k=0}^{n} s_k \binom{n}{k} (1-\xi)^{n-k}\xi^k \,,$$

of the curve parameter ξ, with Bernstein coefficients given by

$$s_0 \,=\, 0 \qquad \text{and} \qquad s_k \,=\, \frac{1}{n}\sum_{j=0}^{k-1}\sigma_j \,, \quad k = 1,\ldots,n \,.$$

Hence, the total arc length S of the curve is simply

$$S \,=\, s(1) - s(0) \,=\, \frac{\sigma_0 + \sigma_1 + \cdots + \sigma_{n-1}}{n} \,,$$

and the arc length of any segment $\xi \in [\,a,b\,]$ is $s(b) - s(a)$. The result is *exact*, as compared to the approximate numerical quadrature required for “ordinary” polynomial curves.

inversion of arc length function — find parameter value ξ_* at which arc length has a given value s_* — i.e., solve equation

$$s(\xi_*) \,=\, s_*$$

note that s is *monotone-increasing* with ξ (since $\sigma = ds/dt \geq 0$) and hence this polynomial equation has just one (simple) real root — easily computed to machine precision by Newton-Raphson iteration

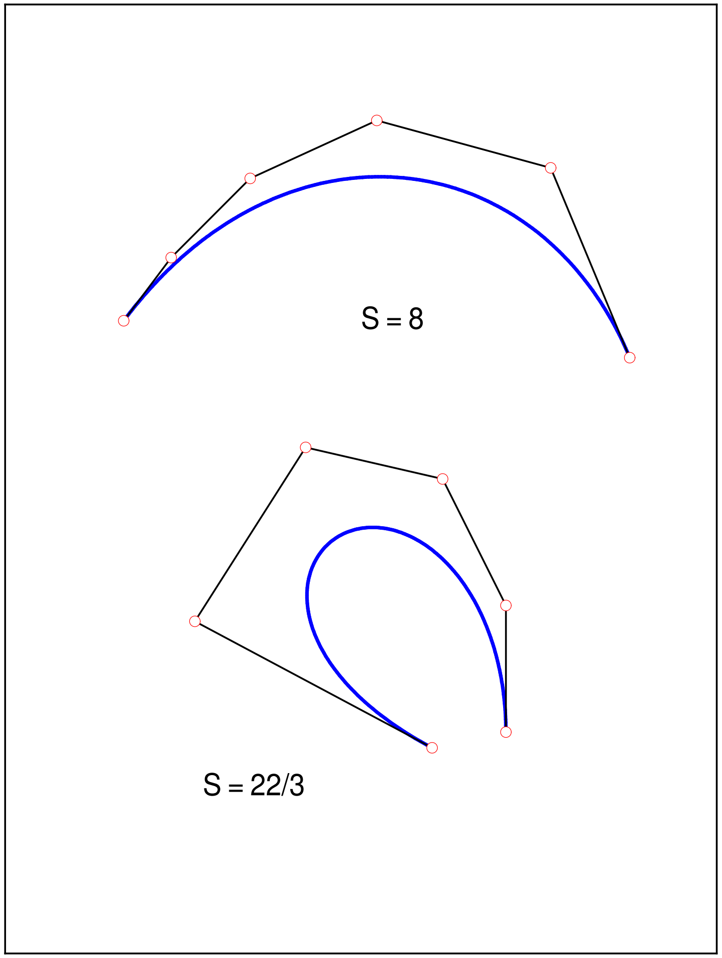
Example: *uniform rendering* of a PH curve — for given arc-length increment Δs,find parameter values $\xi_1,\ldots,\xi_N$ such that

$$s(\xi_k) \,=\, k\,\Delta s\,, \qquad k = 1,\ldots,N\,.$$

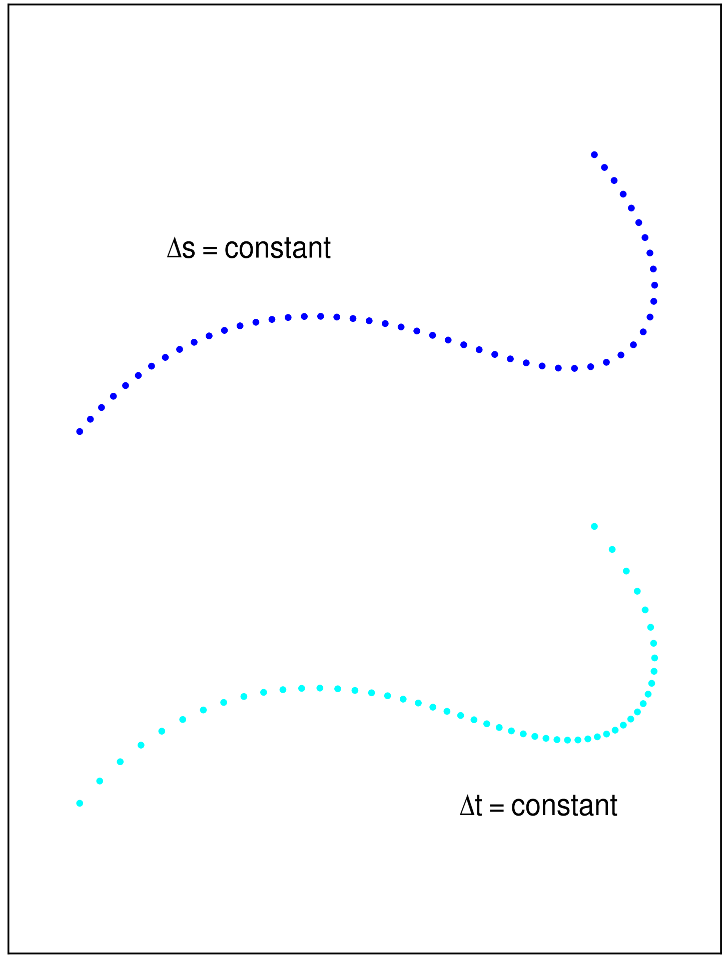
With initial approximation $\xi_k^{(0)} = \xi_{k-1} + \Delta s/\sigma(\xi_{k-1})$, use Newton iteration

$$\xi_k^{(r+1)} \,=\, \xi_k^{(r)} \,-\, \frac{s(\xi_k^{(r)})}{\sigma(\xi_k^{(r)})}\,, \qquad r = 0,1,\ldots$$

Values $\xi_1,\ldots,\xi_N$ define motion at *uniform speed* along a curve — simplest case of a broader class of problems addressed by real–time interpolator algorithms for digital motion controllers.



exact arc lengths



uniform arc–length rendering

planar PH curves — complex variable model

$$x'^2(t) + y'^2(t) = \sigma^2(t) \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} x'(t) = h(t)\,[\,u^2(t) - v^2(t)\,] \\ y'(t) = 2\,h(t)u(t)v(t) \\ \sigma(t) = h(t)\,[\,u^2(t) + v^2(t)\,] \end{array} \right.$$

usually choose $h(t) = 1$ to define a primitive hodograph

$$\gcd(u(t), v(t)) = 1 \iff \gcd(x'(t), y'(t)) = 1$$

if $\deg(u(t), v(t)) = m$, defines planar PH curve of odd degree $n = 2m + 1$

planar PH condition automatically satisfied using complex polynomials

$\mathbf{w}(t) = u(t) + \mathrm{i}\,v(t)$ maps to $\mathbf{r}'(t) = \mathbf{w}^2(t) = u^2(t) - v^2(t) + \mathrm{i}\,2\,u(t)v(t)$

$\rightarrow$ formulation of efficient complex arithmetic algorithms
for the construction and analysis of planar PH curves

summary of planar PH curve properties

- planar PH cubics are scaled, rotated, reparameterized segments of a unique curve, Tschinhausen's cubic (caustic for reflection by parabola)
- planar PH cubics characterized by intuitive geometrical constraints on Bézier control polygon, but not sufficiently flexible for free-form design
- planar PH quintics are excellent design tools — can inflect, and match first–order Hermite data by solving system of three quadratic equations
- select "good" interpolant from multiple solutions using shape measure — arc length, absolute rotation index, elastic bending energy
- generalizes to C^2 PH quintic splines smoothly interpolating sequence of points $\mathbf{p}_0, \ldots, \mathbf{p}_N$ — efficient complex arithmetic algorithms
- theory & algorithms for planar PH curves have attained a mature stage of development

curves with two-sided rational offsets

W. Lü (1995), Offset-rational parametric plane curves, *Comput. Aided Geom. Design* **12**, 601–616

parabola $\mathbf{r}(t) = (t, t^2)$ is simplest example

$$t \;=\; \frac{s^2 - 16}{16\,s} \;:\quad \left.\begin{array}{l} s \in (-\infty, 0) \\ s \in (0, +\infty) \end{array}\right\} \;\to\; t \in (-\infty, +\infty)$$

defines a “doubly-traced” rational re-parameterization

two-sided offset $\mathbf{r}_d(s) \;=\; \mathbf{r}(s) \pm d\,\mathbf{n}(s) \;=\; \left(\dfrac{X_d(s)}{W_d(s)}, \dfrac{Y_d(s)}{W_d(s)}\right)$ is rational :

$$\begin{aligned} X_d(s) &= 16\,(s^4 + 16d\,s^3 - 256d\,s - 256)\,s\,, \\ Y_d(s) &= s^6 - 16s^4 - 2048d\,s^3 - 256s^2 + 4096\,, \\ W_d(s) &= 256\,(s^2 + 16)\,s^2\,. \end{aligned}$$

offset is algebraic curve of degree 6 with implicit equation

$$
\begin{aligned}
f_d(x,y) \;=\;& 16\,(x^2+y^2)\,x^4 \;-\; 8\,(5x^2+4y^2)\,x^2y \\
-\;& (48d^2-1)\,x^4 - 32\,(d^2-1)\,x^2y^2 + 16\,y^4 \\
+\;& [\,2\,(4d^2-1)\,x^2 - 8\,(4d^2+1)\,y^2\,]\,y \\
+\;& 4\,d^2(12d^2-5)\,x^2 + (4d^2-1)^2y^2 \\
+\;& 8\,d^2(4d^2+1)\,y \;-\; d^2(4d^2+1)^2 \;=\; 0
\end{aligned}
$$

genus = 0 $\Rightarrow$ $\frac{1}{2}(n-1)(n-2)=10$ double points

one affine node $+$ six affine cusps

“non–ordinary” double point at infinity with
double points in first & second neighborhoods

generalized complex form (Lü 1995)

$h(t)$ = real polynomial, $\mathbf{w}(t) = u(t) + \mathrm{i}\, v(t)$ = complex polynomial

polynomial PH curve $\mathbf{r}(t) = \int h(t)\, \mathbf{w}^2(t)\, \mathrm{d}t$

two-sided rational offset curve $\mathbf{r}(t) = \int (\, \mathbf{k}\, t + 1\,)\, h(t)\, \mathbf{w}^2(t)\, \mathrm{d}t$

$h(t) = 1$ and $\mathbf{w}(t) = 1 \;\rightarrow$ parabola

$h(t)$ linear and $\mathbf{w}(t) = 1 \;\rightarrow$ cuspidal cubic

$\mathbf{k} = 0$ and $h(t) = 1 \;\rightarrow$ regular PH curve

describes all polynomial curves with rational offsets

characterization of spatial Pythagorean hodographs

R. T. Farouki and T. Sakkalis, Pythagorean–hodograph space curves, *Advances in Computational Mathematics* **2**, 41–66 (1994)

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \quad \Longleftarrow \quad \left\{ \begin{array}{l} x'(t) = u^2(t) - v^2(t) - w^2(t) \\ y'(t) = 2\,u(t)v(t) \\ z'(t) = 2\,u(t)w(t) \\ \sigma(t) = u^2(t) + v^2(t) + w^2(t) \end{array} \right.$$

only a sufficient condition — not invariant with respect to rotations in $\mathbb{R}^3$

R. Dietz, J. Hoschek, and B. Jüttler, An algebraic approach to curves and surfaces on the sphere and on other quadrics, *Computer Aided Geometric Design* **10**, 211–229 (1993)

H. I. Choi, D. S. Lee, and H. P. Moon, Clifford algebra, spin representation, and rational parameterization of curves and surfaces, *Advances in Computational Mathematics* **17**, 5-48 (2002)

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) \\ y'(t) = 2\,[\,u(t)q(t) + v(t)p(t)\,] \\ z'(t) = 2\,[\,v(t)q(t) - u(t)p(t)\,] \\ \sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t) \end{array} \right.$$

spatial PH curves — quaternion & Hopf map models

quaternion model $(\mathbb{H} \to \mathbb{R}^3)$ $\quad \mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$

$$\to \; \mathbf{r}'(t) \; = \; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) \; = \; [\,u^2(t) + v^2(t) - p^2(t) - q^2(t)\,]\,\mathbf{i}$$
$$+ \, 2\,[\,u(t)q(t) + v(t)p(t)\,]\,\mathbf{j} \, + \, 2\,[\,v(t)q(t) - u(t)p(t)\,]\,\mathbf{k}$$

Hopf map model $(\mathbb{C}^2 \to \mathbb{R}^3)$ $\quad \boldsymbol{\alpha}(t) = u(t) + \mathrm{i}\,v(t), \; \boldsymbol{\beta}(t) = q(t) + \mathrm{i}\,p(t)$

$$\to \; (x'(t), y'(t), z'(t)) \; = \; (|\boldsymbol{\alpha}(t)|^2 - |\boldsymbol{\beta}(t)|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)), 2\,\mathrm{Im}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)))$$

equivalence — identify “i” with “$\mathbf{i}$” and set $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t)$

both forms invariant under general spatial rotation by θ about axis $\mathbf{n}$

summary of spatial PH curve properties

- all spatial PH cubics are helical curves — satisfy $\mathbf{a} \cdot \mathbf{t} = \cos \alpha$ (where $\mathbf{a} =$ axis of helix, $\alpha =$ pitch angle) and $\kappa/\tau =$ constant

- spatial PH cubics characterized by intuitive geometrical constraints on Bézier control polygons

- spatial PH quintics well-suited to free-from design applications — two-parameter family of interpolants to first-order Hermite data

- optimal choice of free parameters is a rather subtle problem — one parameter controls curve shape, the other total arc length

- generalization to spatial C^2 PH quintic splines is problematic — too many free parameters!

- many interesting subspecies — helical polynomial curves, "double" PH curves, rational rotation-minimizing frame curves, etc.

rational Pythagorean-hodograph curves

J. C. Fiorot and T. Gensane (1994), Characterizations of the set of rational parametric curves with rational offsets, in *Curves and Surfaces in Geometric Design* AK Peters, 153–160

H. Pottmann (1995), Rational curves and surfaces with rational offsets, *Comput. Aided Geom. Design* **12**, 175–192

- employs dual representation — plane curve regarded as envelope of tangent lines, rather than point locus
- offsets to a rational PH curve are of the same degree as that curve
- admist natural generalization to rational surfaces with rational offsets
- parametric speed, but not arc length, is a rational function of curve parameter (rational functions do not, in general, have rational integrals)
- geometrical optics interpretation — rational PH curves are caustics for reflection of parallel light rays by rational plane curves
- Laguerre geometry model — oriented contact of lines & circles

rational unit normal to planar curve $\mathbf{r}(t) = \left(\dfrac{X(t)}{W(t)}, \dfrac{Y(t)}{W(t)}\right)$

$$n_x(t) \;=\; \frac{2a(t)b(t)}{a^2(t)+b^2(t)}\,, \qquad n_y(t) \;=\; \frac{a^2(t)-b^2(t)}{a^2(t)+b^2(t)}$$

equation of tangent line at point (x, y) on rational curve

$$\ell(x,y,t) \;=\; n_x(t)\,x \;+\; n_y(t)\,y \;-\; \frac{f(t)}{g(t)} \;=\; 0$$

envelope of tangent lines — solve $\ell(x,y,t) = \dfrac{\partial \ell}{\partial t}(x,y,t) = 0$

for (x, y) and set $x = X(t)/W(t)$, $y = Y(t)/W(t)$ to obtain

$$\begin{aligned}
W &\;=\; (a^2+b^2)(a'b-ab')g^2\,,\\
X &\;=\; 2ab(a'b-ab')fg \;-\; \tfrac{1}{2}(a^4-b^4)(f'g-fg')\,,\\
Y &\;=\; (a^2-b^2)(a'b-ab')fg \;+\; ab(a^2+b^2)(f'g-fg')\,.
\end{aligned}$$

dual representation in line coordinates $K(t)$, $L(t)$, $M(t)$ is simpler

define set of all tangent lines to rational PH curve by

$$K(t)\,W \,+\, L(t)\,X \,+\, M(t)\,Y \,=\, 0$$

line coordinates are given in terms $a(t)$, $b(t)$ and $f(t)$, $g(t)$ by

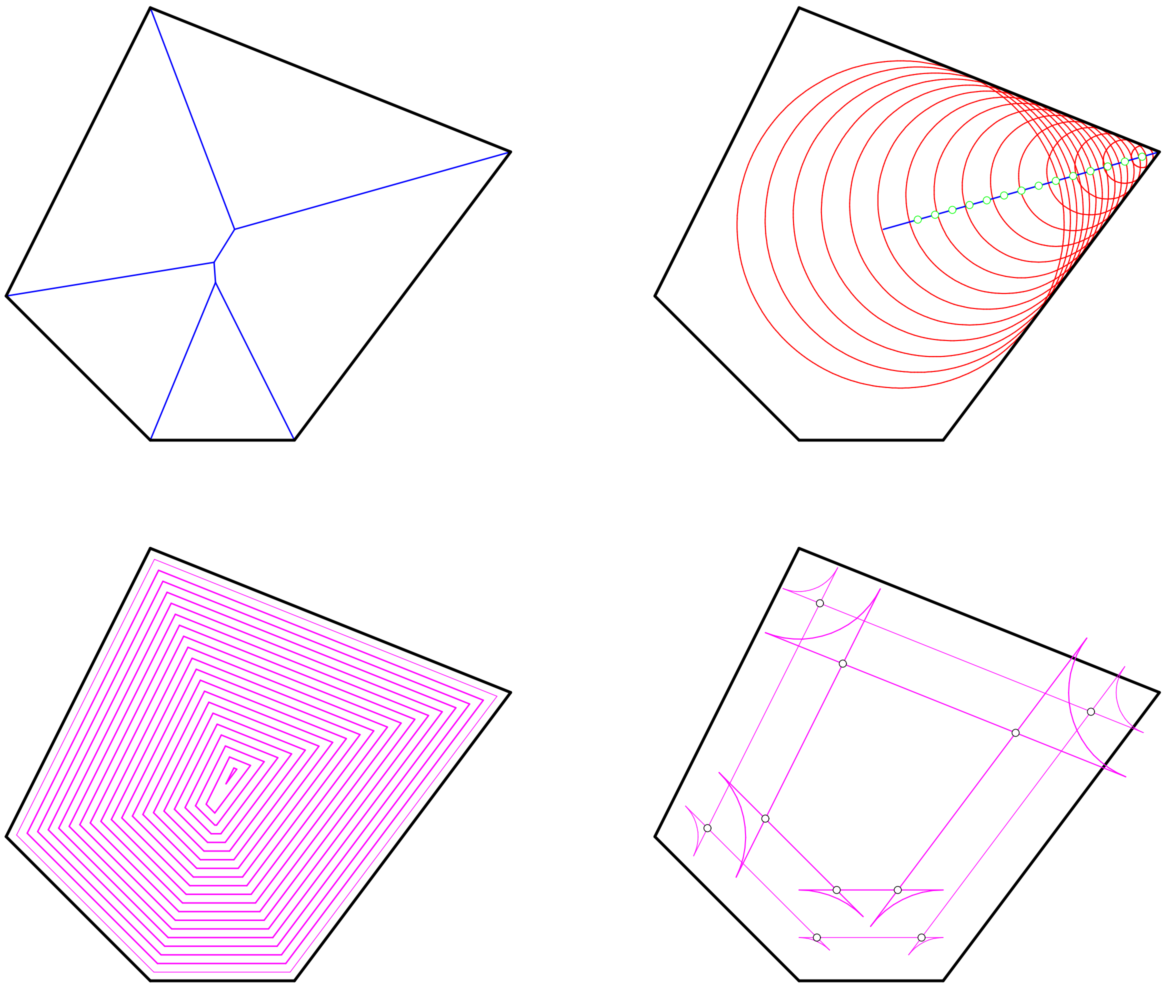
$$K \,:\, L \,:\, M \,=\, -\,(a^2+b^2)f \,:\, 2abg \,:\, (a^2-b^2)g$$

for rational PH curves, class (= degree of line representation)
is less than order (= degree of point representation)

dual Bézier representation — control points replaced by control lines

rational offsets constructed by parallel displacement of control lines

medial axis transform of planar domain



medial axis $=$ locus of centers of maximal inscribed disks, touching domain boundary in at least two points; medial axis transform (MAT) $=$ medial axis $+$ superposed function specifying radii of maximal disks

Minkowski Pythagorean-hodograph (MPH) curves

H. P. Moon (1999), Minkowski Pythagorean hodographs, *Comput. Aided Geom. Design* **16**, 739–753

$(x(t),y(t),r(t))$ = medial axis transform (MAT) of planar domain $\mathcal{D}$

characterizes domain $\mathcal{D}$ as union of one-parameter family of circular disks $\mathcal{C}(t)$ with centers $(x(t),y(t))$ and radii $r(t)$

recovery of domain boundary $\partial\mathcal{D}$ as envelope of one–parameter family of circular disks specified by the MAT $(x(t),y(t),r(t))$

$$x_e(t) \;=\; x(t) \,-\, r(t)\,\frac{r'(t)x'(t) \pm \sqrt{x'^2(t)+y'^2(t)-r'^2(t)}\,y'(t)}{x'^2(t)+y'^2(t)}\,,$$

$$y_e(t) \;=\; y(t) \,-\, r(t)\,\frac{r'(t)y'(t) \mp \sqrt{x'^2(t)+y'^2(t)-r'^2(t)}\,x'(t)}{x'^2(t)+y'^2(t)}\,.$$

for parameterization to be rational, MAT hodograph must satisfy

$$x'^2(t) \,+\, y'^2(t) \,-\, r'^2(t) \,=\, \sigma^2(t)$$

— this is a Pythagorean condition in the Minkowski space $\mathbb{R}^{(2,1)}$

metric of Minkowski space $\mathbb{R}^{(2,1)}$ has signature $+ + -$ rather than usual signature $+ + +$ for metric of Euclidean space $\mathbb{R}^3$

Moon (1999): sufficient–and–necessary characterization of Minkowski Pythagorean hodographs in terms of four polynomials $u(t)$, $v(t)$, $p(t)$, $q(t)$

$$\begin{aligned}
x'(t) &= u^2(t) + v^2(t) - p^2(t) - q^2(t)\,, \\
y'(t) &= 2\,[\,u(t)p(t) - v(t)q(t)\,]\,, \\
r'(t) &= 2\,[\,u(t)v(t) - p(t)q(t)\,]\,, \\
\sigma(t) &= u^2(t) - v^2(t) + p^2(t) - q^2(t)\,.
\end{aligned}$$

interpretation of Minkowski metric

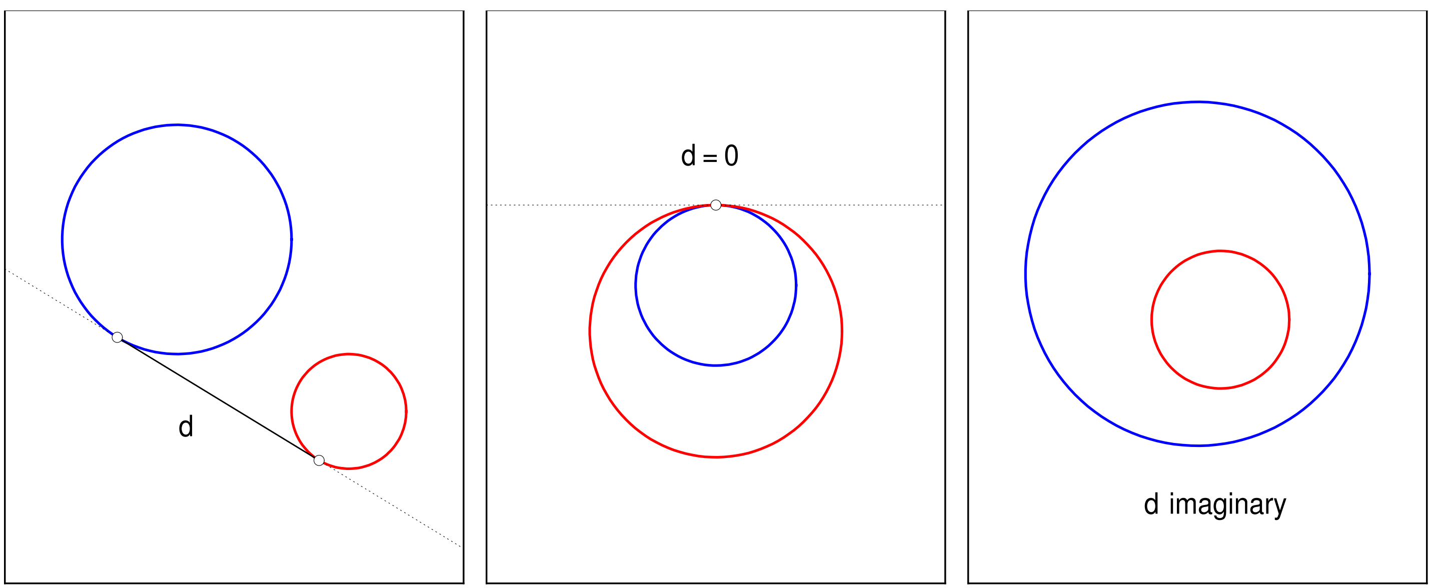
originates in special relativity: distance d between events with space–time coordinates (x_1, y_1, t_1) and (x_2, y_2, t_2) is defined by

$$d^2 \;=\; (x_2 - x_1)^2 + (y_2 - y_1)^2 - c^2(t_2 - t_1)^2$$

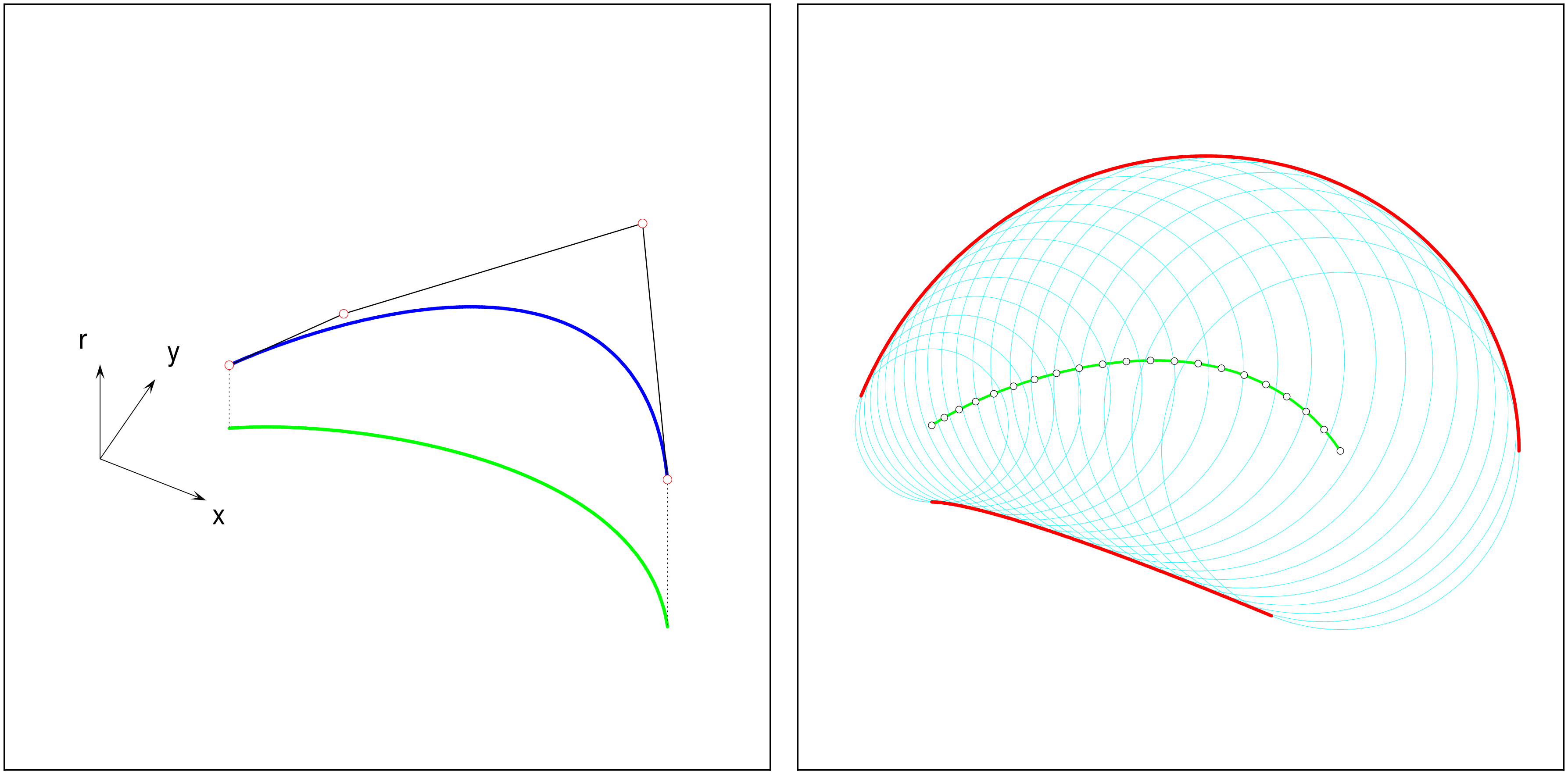
space-like if d real, light-like if $d = 0$, time-like if d imaginary

distance between circles (x_1, y_1, r_1) and (x_2, y_2, r_2) as points in $\mathbb{R}^{(2,1)}$

$$d^2 \;=\; (x_2 - x_1)^2 + (y_2 - y_1)^2 - (r_2 - r_1)^2$$



rational boundary reconstructed from MPH curve

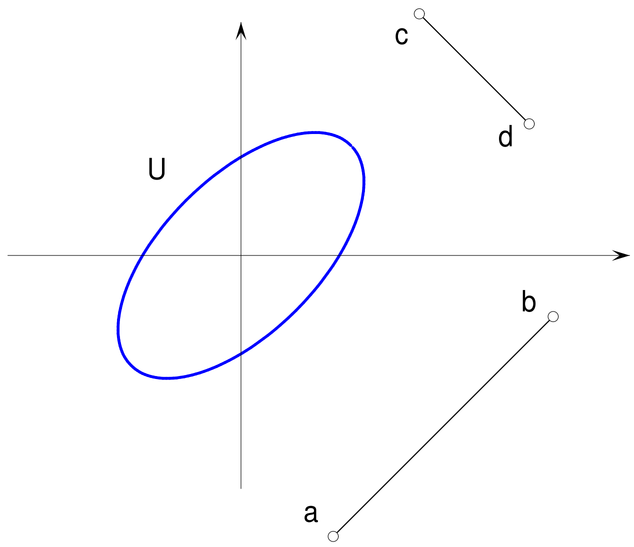


Minkowski isoperimetric-hodograph curves

R. Ait-Haddou, L. Biard, and M. Slawinski (2000), Minkowski isoperimetric-hodograph curves,
Comput. Aided Geom. Design **17**, 835–861

(two–sided) offset at distance $\pm d$ from a plane curve $=$ boundary of *Minkowski sum* or convolution of curve with a circle of radius d

replace circle with convex, centrally–symmetric curve $\mathcal{U}$, the indicatrix



line segments ab and cd have same length under metric defined by $\mathcal{U}$

If $\mathcal{U}$ is defined in polar coordinates by a π–periodic function $r(\theta)$, the Minkowski distance between points $\mathbf{p}_1 = (x_1, y_1)$ and $\mathbf{p}_2 = (x_2, y_2)$ is

$$d(\mathbf{p}_1, \mathbf{p}_2) = \frac{|\,\mathbf{p}_2 - \mathbf{p}_1|}{r(\theta)}$$

where $|\mathbf{p}_2 - \mathbf{p}_1| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$ is the Euclidean distance, and θ is the angle that the vector $\mathbf{p}_2 - \mathbf{p}_1$ makes with the x–axis.

indicatrix $\mathcal{U}$ defines "anisotropic unit circle" in the Minkowski plane (should not be confused with metric for "pseudo–Euclidean" Minkowski space, that differs only in signature from the Euclidean metric)

- differential geometry of plane curves under the Minkowski metric
- conditions for rational offsets under this metric (Minkowski IH curves)
- point and line Bézier control structures for Minkowski IH curves

special classes of spatial PH curves

helical polynomial space curves

satisfy $\mathbf{a} \cdot \mathbf{t} = \cos\alpha$ ($\mathbf{a} =$ axis, $\alpha =$ pitch angle) and $\kappa/\tau = \tan\alpha$

all helical polynomial curves are PH curves (implied by $\mathbf{a} \cdot \mathbf{t} = \cos\alpha$)
all spatial PH cubics are helical, but not all PH curves of degree ≥ 5

"double" Pythagorean–hodograph (DPH) curves

$\mathbf{r}'(t)$ and $\mathbf{r}'(t) \times \mathbf{r}''(t)$ both have Pythagorean structures
— have rational Frenet frames $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ and curvatures κ

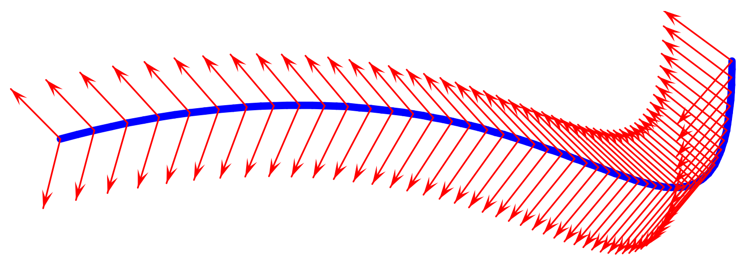
all helical polynomial curves are DPH — not just PH — curves
all DPH quintics are helical, but not all DPH curves of degree ≥ 7

rational rotation–minimizing frame (RRMF) curves

rational frames $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ with angular velocity satisfying $\boldsymbol{\omega} \cdot \mathbf{t} \equiv 0$

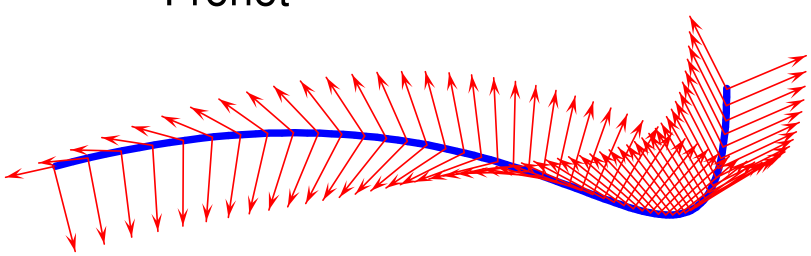
RRMF curves are of minimum degree 5 (proper subset of PH quintics) identifiable by quadratic (vector) constraint on quaternion coefficients

useful in spatial motion planning and rigid–body orientation control



RMF

Frenet



real-time CNC interpolators

computer numerical control (CNC) machine has digital controller

- in each sampling interval ($\Delta t \sim 10^{-3}$ sec) of servo system, compare actual position (measured by encoders on each machine axis) with reference position computed by real–time interpolator algorithm

- real-time CNC interpolator: for parametric curve $\mathbf{r}(\xi)$ and speed (feedrate) function v, compute reference-point parameter values $\xi_1, \xi_2, \ldots$ in real time:

$$\int_0^{\xi_k} \frac{|\mathbf{r}'(\xi)|\,\mathrm{d}\xi}{v} = k\Delta t\,, \quad k = 1, 2, \ldots$$

- Pythagorean-hodograph (PH) curves — analytic reduction of "interpolation integral"
  $\Longrightarrow$ accurate & efficient real-time interpolator

advantages of PH curves in motion control

- PH curves admit analytic reduction of interpolation integral, rather than truncated Taylor series expansion

- using analytic curve description (instead of piecewise linear/circular G code approximations) eliminates "aliasing" effects, yields smoother motions, and allows greater acceleration rates

- flexible repertoire of variable feedrate functions — dependent on time, arc length, curvature, etc.

- solve inverse dynamics problem to compensate for contour errors due to machine inertia, friction, etc.

- applications of rotation–minimizing frames to 5-axis machining

closure

- advantages of PH curves: rational offset curves, exact arc–length computation, real-time CNC interpolators, exact rotation–minimizing frames, bending energies, etc.

- applications of PH curves in digital motion control, path planning, robotics, animation, computer graphics, etc.

- investigation of PH curves involves a wealth of concepts from algebra and geometry with a long and fascinating history

- many open problems remain: optimal choice of degrees of freedom, C^2 spline formulations, control polygons for design of PH splines, deeper geometrical insight into quaternion representation, etc.