Rotation-minimizing frames on space curves — theory, algorithms, applications

Rida T. Farouki

Department of Mechanical & Aerospace Engineering,
University of California, Davis

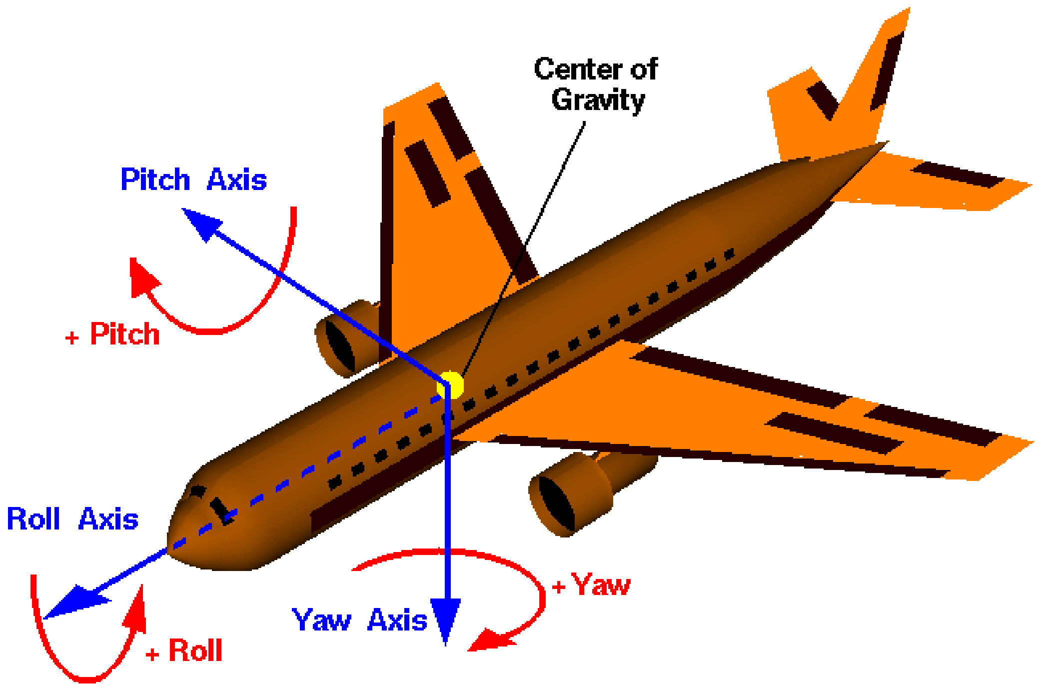
— in collaboration with —

Carlotta Giannelli, Carla Manni, Alessandra Sestini, Takis Sakkalis

— synopsis —

- rotation-minimizing frames (RMFs) on space curves
- "defects" of the Frenet frame — applications of RMFs
- RMFs for spatial Pythagorean–hodograph (PH) curves
- characterization of PH curves with exact rational RMFs
- design of rational rotation–minimizing rigid body motions
- directed frames — camera orientation control
  computation of rotation-minimizing directed frames
- polar differential geometry — anti-hodograph,
  Frenet directed frame, polar curvature and torsion

aircraft attitude — pitch, yaw, roll



"rotation–minimizing" motion $\Longrightarrow \ \omega_{\mathrm{roll}} \equiv 0$

rotation-minimizing frames on space curves

- an adapted frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ on a space curve $\mathbf{r}(\xi)$ is a system of three orthonormal vectors, such that $\mathbf{e}_1 = \mathbf{r}'/|\mathbf{r}'|$ is the curve tangent and $(\mathbf{e}_2, \mathbf{e}_3)$ span the curve normal plane at each point

- on any given space curve, there are infinitely many adapted frames — the Frenet frame is perhaps the most familiar

  R. L. Bishop (1975), There is more than one way to frame a curve, *Amer. Math. Monthly* **82**, 246–251

- for an adapted rotation–minimizing frame (RMF), the normal–plane vectors $(\mathbf{e}_2, \mathbf{e}_3)$ exhibit no instantaneous rotation about $\mathbf{e}_1$

  F. Klok (1986), Two moving coordinate frames for sweeping along a 3D trajectory, *Comput. Aided Geom. Design* **3**, 217–229

- angular orientation of RMF relative to Frenet frame $=$ integral of curve torsion w.r.t. arc length ($\Rightarrow$ one–parameter family of RMFs)

  H. Guggenheimer (1989), Computing frames along a trajectory, *Comput. Aided Geom. Design* **6**, 77–78

- spatial PH curves admit exact evaluation of torsion integral, but expression for RMF contains transcendental terms

  R. T. Farouki (2002), Exact rotation–minimizing frames for spatial Pythagorean–hodograph curves, *Graphical Models* **64**, 382–395

- piecewise–rational RMF approximation on polynomial & rational curves

  B. Jüttler and C. Mäurer (1999), Rational approximation of rotation minimizing frames using Pythagorean–hodograph cubics, *J. Geom. Graphics* **3**, 141–159

  R. T. Farouki and C. Y. Han (2003), Rational approximation schemes for rotation–minimizing frames on Pythagorean–hodograph curves, *Comput. Aided Geom. Design* **20**, 435–454

- Euler–Rodrigues frame (ERF) is better reference than Frenet frame for identifying curves with rational RMFs (RRMF curves)

  H. I. Choi and C. Y. Han (2002), Euler–Rodrigues frames on spatial Pythagorean–hodograph curves, *Comput. Aided Geom. Design* **19**, 603–620

  ERF $=$ rational adapted frame defined on spatial PH curves that is non–singular at inflection points

- “implicit” algebraic condition for rational RMFs on spatial PH curves — no rational RMFs for non–degenerate cubics

  C. Y. Han (2008), Nonexistence of rational rotation–minimizing frames on cubic curves, *Comput. Aided Geom. Design* **25**, 298–304

- sufficient–and–necessary conditions on Hopf map coefficients of spatial PH quintics for rational RMF

  R. T. Farouki, C. Giannelli, C. Manni, A. Sestini (2009), Quintic space curves with rational rotation–minimizing frames, *Comput. Aided Geom. Design* **26**, 580–592

- directed rotation–minimizing frames (camera orientation control)

  R. T. Farouki and C. Giannelli (2009), Spatial camera orientation control by rotation–minimizing directed frames, *Comput. Anim. Virtual Worlds* **20**, 457–472

- simplified (quadratic) RRMF conditions for quaternion & Hopf map representations of spatial PH quintics

  R. T. Farouki (2010), Quaternion and Hopf map characterizations for the existence of rational rotation–minimizing frames on quintic space curves, *Adv. Comp. Math.* **33**, 331–348

- general RRMF conditions for spatial PH curves of any degree

  R. T. Farouki and T. Sakkalis (2010), Rational rotation–minimizing frames on polynomial space curves of arbitrary degree, *J. Symb. Comp.* **45**, 844–856

- spatial motion design by RRMF quintic Hermite interpolation

  R. T. Farouki, C. Giannelli, C. Manni, A. Sestini (2011), Design of rational rotation–minimizing rigid body motions by Hermite interpolation, *Math. Comp.*, to appear

- design of interpolatory rotation–minimizing camera motions

  R. T. Farouki, C. Giannelli, A. Sestini (2011), An interpolation scheme for designing rational rotation–minimizing camera motions, *Adv. Comp. Math.*, to appear

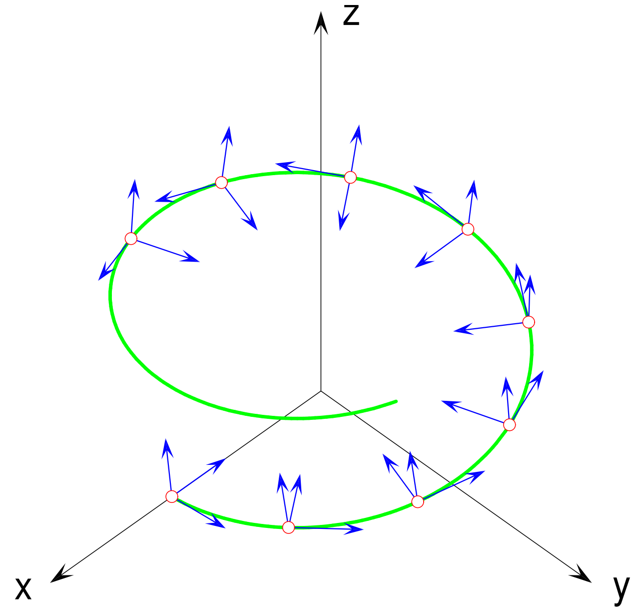
- several different classes of RRMF curves of given degree

  R. T. Farouki and T. Sakkalis (2011), A complete classification of quintic space curves with rational rotation–minimizing frames *J. Symb. Comp.*, to appear

differential geometry of space curves

Frenet frame $(\mathbf{t}(\xi), \mathbf{n}(\xi), \mathbf{b}(\xi))$ on space curve $\mathbf{r}(\xi)$ defined by

$$\mathbf{t} = \frac{\mathbf{r}'}{|\mathbf{r}'|}, \qquad \mathbf{n} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|} \times \mathbf{t}, \qquad \mathbf{b} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\mathbf{r}' \times \mathbf{r}''|}$$



$\mathbf{t}$ defines instantaneous direction of motion along curve;
$\mathbf{n}$ points toward center of curvature; $\mathbf{b} = \mathbf{t} \times \mathbf{n}$ completes frame

variation of frame $(\mathbf{t}(\xi), \mathbf{n}(\xi), \mathbf{b}(\xi))$ along curve $\mathbf{r}(\xi)$

specified in terms of parametric speed, curvature, torsion functions

$$\sigma = |\mathbf{r}'|, \qquad \kappa = \frac{|\mathbf{r}' \times \mathbf{r}''|}{|\mathbf{r}'|^3}, \qquad \tau = \frac{(\mathbf{r}' \times \mathbf{r}'') \cdot \mathbf{r}'''}{|\mathbf{r}' \times \mathbf{r}''|^2}$$

by Frenet–Serret equations

$$\begin{bmatrix} \mathbf{t}' \\ \mathbf{n}' \\ \mathbf{b}' \end{bmatrix} = \sigma \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

- $(\mathbf{t}, \mathbf{n})$ span osculating plane (second–order contact at each point)
- $(\mathbf{n}, \mathbf{b})$ span normal plane (cuts curve orthogonally at each point)
- $(\mathbf{b}, \mathbf{t})$ span rectifying plane (envelope of these planes defines *rectifying developable*, allows curve to be flattened onto a plane)

“defects” of Frenet frame on space curves

- $(\mathbf{t},\mathbf{n},\mathbf{b})$ do not depend rationally on curve parameter ξ
- normal–plane vectors $(\mathbf{n},\mathbf{b})$ become indeterminate and can suddenly “flip” at inflection points of curve, where $\kappa = 0$
- exhibits “unnecessary rotation” in the curve normal plane

$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \mathbf{d}\times\mathbf{t}\,, \qquad \frac{\mathrm{d}\mathbf{n}}{\mathrm{d}s} = \mathbf{d}\times\mathbf{n}\,, \qquad \frac{\mathrm{d}\mathbf{b}}{\mathrm{d}s} = \mathbf{d}\times\mathbf{b}$$

Darboux vector $\quad \mathbf{d} = \kappa\,\mathbf{b} + \tau\,\mathbf{t} =$ Frenet frame rotation rate

component $\tau\,\mathbf{t}$ describes instantaneous rotation in normal plane (unnecessary for “smoothly varying” adapted orthonormal frame) total curvature $|\mathbf{d}| = \sqrt{\kappa^2 + \tau^2} =$ angular velocity of Frenet frame

rotation–minimizing adapted frame $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ satisfying

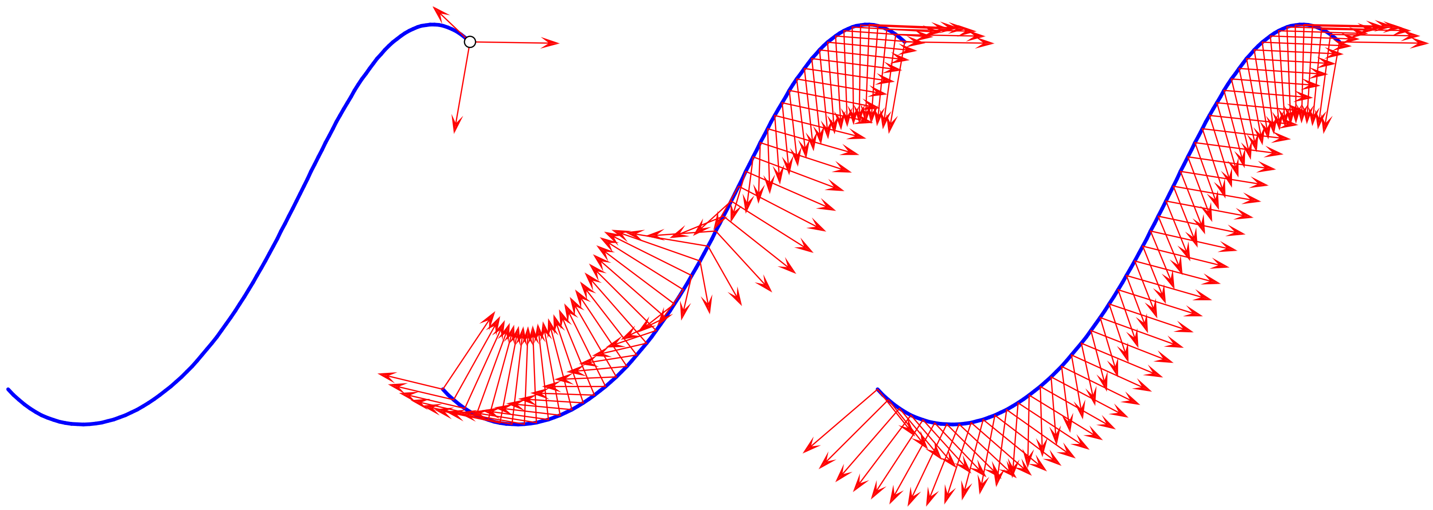
$$\frac{\mathrm{d}\mathbf{t}}{\mathrm{d}s} = \boldsymbol{\omega} \times \mathbf{t}\,, \qquad \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} = \boldsymbol{\omega} \times \mathbf{u}\,, \qquad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s} = \boldsymbol{\omega} \times \mathbf{v}$$

RMF characteristic property — angular velocity $\boldsymbol{\omega}$ satisfies $\boldsymbol{\omega} \cdot \mathbf{t} \equiv 0$

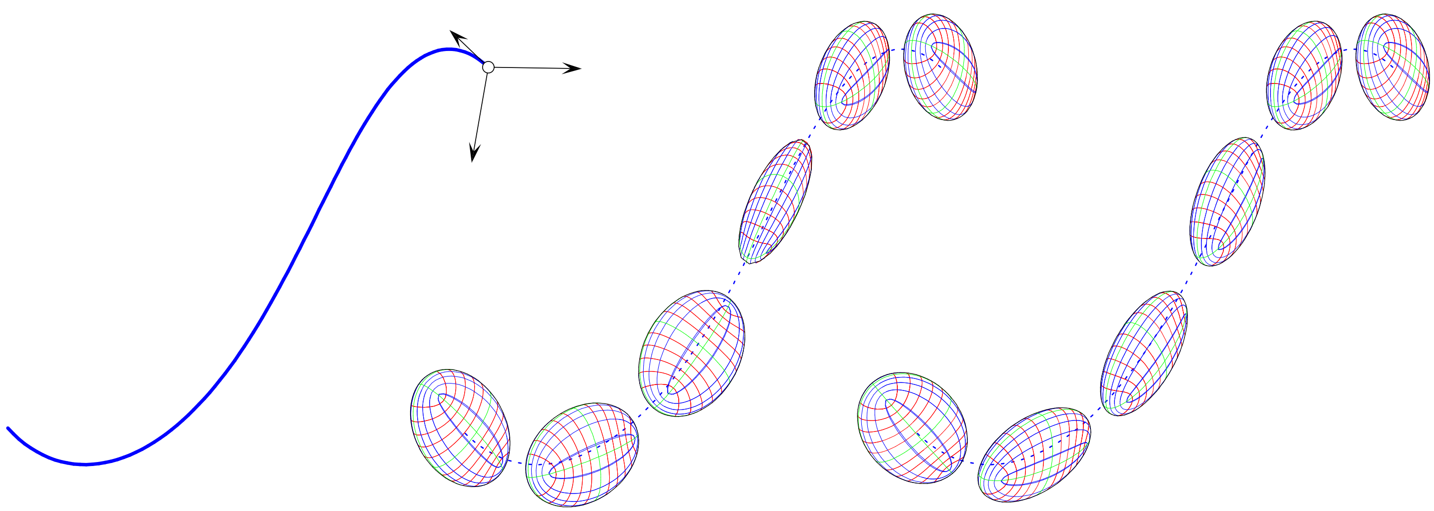
no instantaneous rotation of normal–plane vectors $(\mathbf{u}, \mathbf{v})$ about tangent $\mathbf{t}$

$\rightarrow$ rotation–minimizing frame much better than Frenet frame for applications in animation, path planning, swept surface constructions, etc.

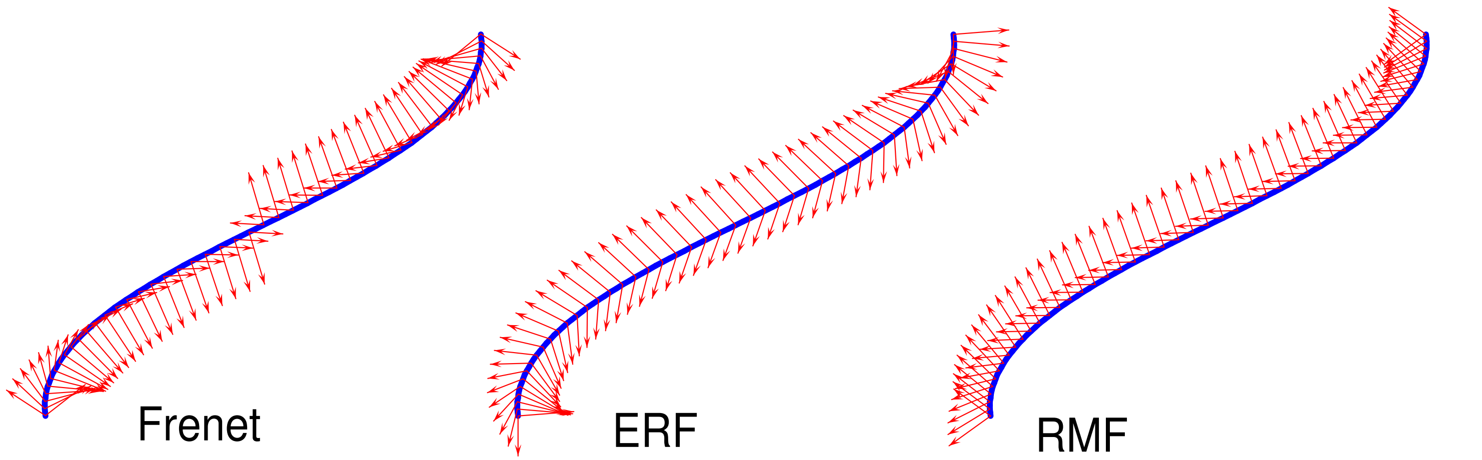
among all adapted frames on a space curve, the RMF identifies least elastic energy associated with twisting (as distinct from bending)



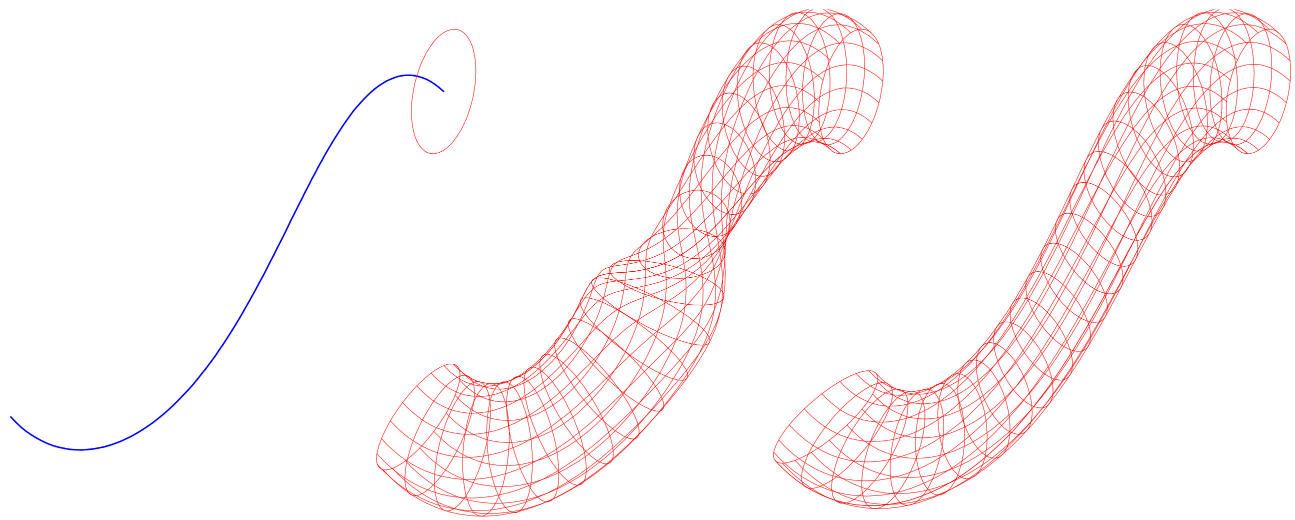
Frenet frame (center) & rotation-minimizing frame (right) on space curve



motion of an ellipsoid oriented by Frenet & rotation-minimizing frames



sudden reversal of Frenet frame through an inflection point



surface constructed by sweeping an ellipse along a space curve using Frenet frame (center) & rotation-minimizing frame (right)

Pythagorean-hodograph (PH) curves

$\mathbf{r}(\xi) =$ PH curve in $\mathbb{R}^n \iff$ coordinate components of $\mathbf{r}'(\xi)$

elements of "Pythagorean $(n+1)$-tuple of polynomials"

PH curves incorporate special algebraic structures in their hodographs

(complex number & quaternion models for planar & spatial PH curves)

- rational offset curves $\mathbf{r}_d(\xi) = \mathbf{r}(\xi) + d\,\mathbf{n}(\xi)$
- polynomial arc-length function $s(\xi) = \int_0^\xi |\mathbf{r}'(\xi)|\,\mathrm{d}\xi$
- closed-form evaluation of energy integral $E = \int_0^1 \kappa^2\,\mathrm{d}s$
- real-time CNC interpolators, rotation-minimizing frames, etc.

Pythagorean quartuples of polynomials

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) \\ y'(t) = 2\,[\,u(t)q(t) + v(t)p(t)\,] \\ z'(t) = 2\,[\,v(t)q(t) - u(t)p(t)\,] \\ \sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t) \end{array} \right.$$

H. I. Choi, D. S. Lee, and H. P. Moon, Clifford algebra, spin representation, and rational parameterization of curves and surfaces, *Advances in Computational Mathematics* **17**, 5-48 (2002)

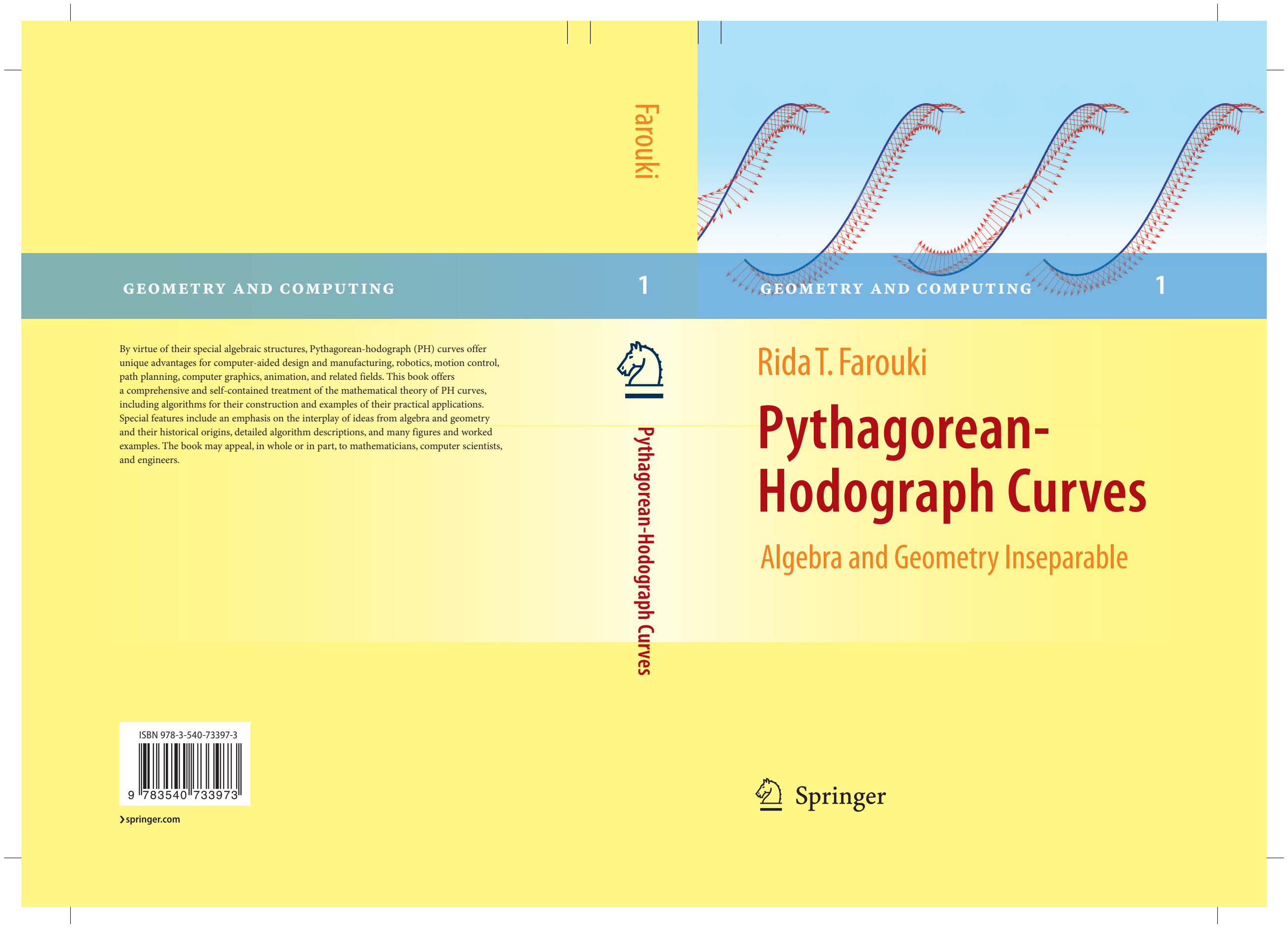
quaternion representation $\quad \mathcal{A}(t) \;=\; u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$

$\rightarrow$ spatial Pythagorean hodograph $\quad \mathbf{r}'(t) \;=\; (x'(t), y'(t), z'(t)) \;=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$

Hopf map representation $\quad \boldsymbol{\alpha}(t) = u(t) + \mathrm{i}\,v(t),\; \boldsymbol{\beta}(t) = q(t) + \mathrm{i}\,p(t)$

$$\rightarrow \; (x'(t), y'(t), z'(t)) \;=\; (|\boldsymbol{\alpha}(t)|^2 - |\boldsymbol{\beta}(t)|^2, 2\,\mathrm{Re}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)), 2\,\mathrm{Im}(\boldsymbol{\alpha}(t)\overline{\boldsymbol{\beta}}(t)))$$

equivalence — identify "i" with "$\mathbf{i}$" and set $\mathcal{A}(t) = \boldsymbol{\alpha}(t) + \mathbf{k}\,\boldsymbol{\beta}(t)$



As long as algebra and geometry were separated, their progress was slow and their uses limited; but once these sciences were united, they lent each other mutual support and advanced rapidly together towards perfection.

Joseph-Louis Lagrange (1736-1813)

rotation-minimizing frames on spatial PH curves

$$\text{new basis in normal plane} \quad \begin{bmatrix} \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

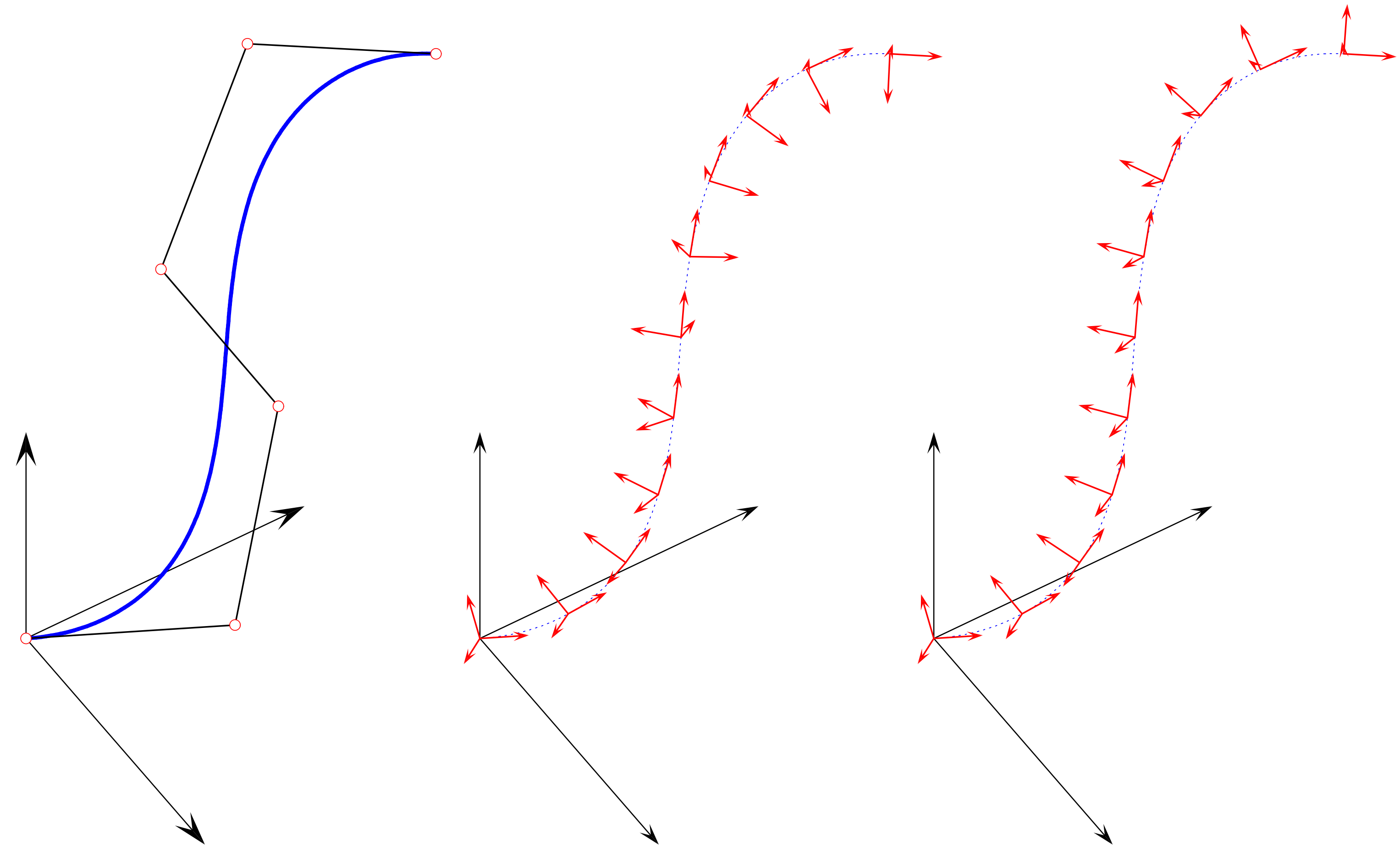
where $\theta = -\int \tau \, \mathrm{d}s$: cancels “unnecessary rotation” in normal plane

free integration constant $\Longrightarrow$ $\exists$ one–parameter family of RMFs

options for construction of RMF $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ on spatial PH quintics:

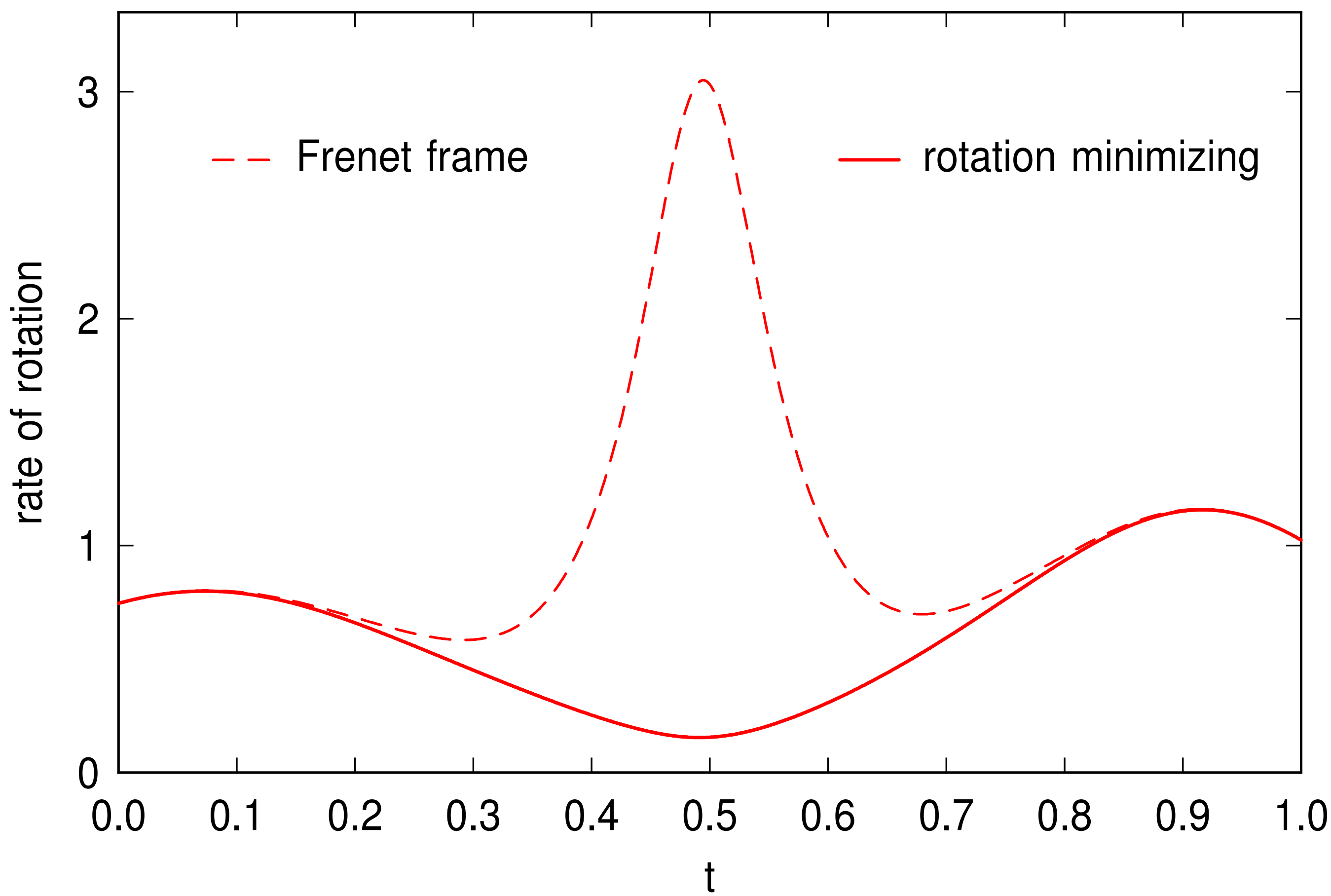
- analytic reduction — involves rational function integration, logarithmic dependence on curve parameter
- rational approximation — use Padé (rational Hermite) approach: simple algorithm & rapid convergence
- exact rational RMFs — identify sufficient and necessary conditions for rational RMFs on spatial PH curves

comparison of Frenet & rotation-minimizing frames



spatial PH quintic   Frenet frame   rotation–minimizing frame

rotation rates — RMF vs Frenet frame



compared with the rotation-minimizing frame $(\mathbf{t}, \mathbf{u}, \mathbf{v})$, the Frenet frame $(\mathbf{t}, \mathbf{n}, \mathbf{b})$ exhibits a lot of "unnecessary" rotation (in the curve normal plane)

rational RMFs on spatial PH curves

any space curve with a rational RMF must be a PH curve
(since only PH curves have rational unit tangents)

Choi & Han (2002): for PH curve with hodograph $\mathbf{r}'(\xi) = \mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)$

$$\mathbf{e}_1(\xi) = \frac{\mathcal{A}(\xi)\,\mathbf{i}\,\mathcal{A}^*(\xi)}{|\mathcal{A}(\xi)|^2}, \quad \mathbf{e}_2(\xi) = \frac{\mathcal{A}(\xi)\,\mathbf{j}\,\mathcal{A}^*(\xi)}{|\mathcal{A}(\xi)|^2}, \quad \mathbf{e}_3(\xi) = \frac{\mathcal{A}(\xi)\,\mathbf{k}\,\mathcal{A}^*(\xi)}{|\mathcal{A}(\xi)|^2}$$

defines Euler–Rodrigues frame (ERF) — $\mathbf{e}_1$ is curve tangent,
while $(\mathbf{e}_2, \mathbf{e}_3)$ span the normal plane at each curve point

ERF $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is a better "reference" than Frenet frame
$(\mathbf{t}, \mathbf{n}, \mathbf{b})$ for seeking rational RMFs $(\mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3)$ on spatial PH curves

ERF is not intrinsic: depends on chosen basis $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ for $\mathbb{R}^3$
— but is inherently rational and non–singular at inflection points

seek rational rotation ERF $\rightarrow$ RMF

Han (2008) — RMF vectors $(\mathbf{f}_2, \mathbf{f}_3)$ must be obtainable from ERF vectors $(\mathbf{e}_2, \mathbf{e}_3)$ by rational rotation in curve normal plane at each point of $\mathbf{r}(\xi)$, specified by two polynomials $a(\xi)$, $b(\xi)$:

$$\mathbf{f}_2(\xi) \;=\; \frac{a^2(\xi) - b^2(\xi)}{a^2(\xi) + b^2(\xi)}\,\mathbf{e}_2(\xi) \,-\, \frac{2\,a(\xi)b(\xi)}{a^2(\xi) + b^2(\xi)}\,\mathbf{e}_3(\xi)\,,$$

$$\mathbf{f}_3(\xi) \;=\; \frac{2\,a(\xi)b(\xi)}{a^2(\xi) + b^2(\xi)}\,\mathbf{e}_2(\xi) \,+\, \frac{a^2(\xi) - b^2(\xi)}{a^2(\xi) + b^2(\xi)}\,\mathbf{e}_3(\xi)\,.$$

if such polynomials $a(\xi)$, $b(\xi)$ exist, we have an RRMF curve — i.e., a PH curve with a rational rotation–minimizing frame

ERF angular velocity

$$\mathbf{e}_1' = \boldsymbol{\omega} \times \mathbf{e}_1 \,, \quad \mathbf{e}_2' = \boldsymbol{\omega} \times \mathbf{e}_2 \,, \quad \mathbf{e}_3' = \boldsymbol{\omega} \times \mathbf{e}_3$$

express ERF angular velocity $\boldsymbol{\omega}$ in basis $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ as

$$\boldsymbol{\omega} = \omega_1 \mathbf{e}_1 + \omega_2 \mathbf{e}_2 + \omega_3 \mathbf{e}_3$$

$$\omega_1 = \mathbf{e}_3 \cdot \mathbf{e}_2' = -\mathbf{e}_2 \cdot \mathbf{e}_3' = \frac{2(uv' - u'v - pq' + p'q)}{u^2 + v^2 + p^2 + q^2} \,,$$

$$\omega_2 = \mathbf{e}_1 \cdot \mathbf{e}_3' = -\mathbf{e}_3 \cdot \mathbf{e}_1' = \frac{2(up' - u'p + vq' - v'q)}{u^2 + v^2 + p^2 + q^2} \,,$$

$$\omega_3 = \mathbf{e}_2 \cdot \mathbf{e}_1' = -\mathbf{e}_1 \cdot \mathbf{e}_2' = \frac{2(uq' - u'q - vp' + v'p)}{u^2 + v^2 + p^2 + q^2} \,.$$

$\Longrightarrow$ find $a(\xi), b(\xi)$ so ω_1 is cancelled by $\omega = \dfrac{2(ab' - a'b)}{a^2 + b^2}$

“implicit” algebraic condition for RRMF curves

Han (2008): PH curve defined by $\mathcal{A}(\xi) = u(\xi) + v(\xi)\,\mathbf{i} + p(\xi)\,\mathbf{j} + q(\xi)\,\mathbf{k}$ is an RRMF curve if and only if polynomials $a(\xi)$, $b(\xi)$ exist such that

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{ab' - a'b}{a^2 + b^2}$$

Hopf map representation with $\boldsymbol{\alpha}(\xi) = u(\xi) + \mathrm{i}\,v(\xi)$, $\boldsymbol{\beta}(\xi) = q(\xi) + \mathrm{i}\,p(\xi)$ requires existence of complex polynomial $\mathbf{w}(\xi) = a(\xi) + \mathrm{i}\,b(\xi)$ such that

$$\frac{\mathrm{Im}(\overline{\boldsymbol{\alpha}}\boldsymbol{\alpha}' + \overline{\boldsymbol{\beta}}\boldsymbol{\beta}')}{|\boldsymbol{\alpha}|^2 + |\boldsymbol{\beta}|^2} = \frac{\mathrm{Im}(\overline{\mathbf{w}}\mathbf{w}')}{|\mathbf{w}|^2}$$

Han (2008): no RRMF cubics exist, except degenerate (planar) curves

characterization of RRMF quintics

Farouki, Giannelli, Manni, Sestini (2009): use Hopf map form with

$$\begin{aligned}\boldsymbol{\alpha}(t) &= \boldsymbol{\alpha}_0\,(1-t)^2 + \boldsymbol{\alpha}_1\,2(1-t)t + \boldsymbol{\alpha}_2\,t^2\,,\\ \boldsymbol{\beta}(t) &= \boldsymbol{\beta}_0\,(1-t)^2 + \boldsymbol{\beta}_1\,2(1-t)t + \boldsymbol{\beta}_2\,t^2\,.\end{aligned}$$

defines RRMF quintic $\iff$ $\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2 \in \mathbb{C}$, $\gamma \in \mathbb{R}$ exist such that

$$\begin{aligned}|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2 &= \gamma\,|\mathbf{w}_0|^2\,,\\ \overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_1 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_1 &= \gamma\,\overline{\mathbf{w}}_0\mathbf{w}_1\,,\\ \overline{\boldsymbol{\alpha}}_0\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_0\boldsymbol{\beta}_2 \;+\; 2\,(|\boldsymbol{\alpha}_1|^2 + |\boldsymbol{\beta}_1|^2) &= \gamma\,(\overline{\mathbf{w}}_0\mathbf{w}_2 + 2\,|\mathbf{w}_1|^2)\,,\\ \overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2 &= \gamma\,\overline{\mathbf{w}}_1\mathbf{w}_2\,,\\ |\boldsymbol{\alpha}_2|^2 + |\boldsymbol{\beta}_2|^2 &= \gamma\,|\mathbf{w}_2|^2\,.\end{aligned}$$

NOTE: can take $\mathbf{w}_0 = 1$ without loss of generality

sufficient–and–necessary conditions

Proposition 1. *A PH quintic has a rational rotation–minimizing frame if and only if the coefficients* $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ *and* $\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2$ *of the two quadratic complex polynomials* $\boldsymbol{\alpha}(t)$ *and* $\boldsymbol{\beta}(t)$ *satisfy the constraints*

$$(|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2)\,|\overline{\boldsymbol{\alpha}}_1\boldsymbol{\alpha}_2 + \overline{\boldsymbol{\beta}}_1\boldsymbol{\beta}_2|^2 \;=\; (|\boldsymbol{\alpha}_2|^2 + |\boldsymbol{\beta}_2|^2)\,|\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_1 + \boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_1|^2\,,$$

$$(|\boldsymbol{\alpha}_0|^2 + |\boldsymbol{\beta}_0|^2)\,(\boldsymbol{\alpha}_0\boldsymbol{\beta}_2 - \boldsymbol{\alpha}_2\boldsymbol{\beta}_0) \;=\; 2\,(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_1 + \boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_1)(\boldsymbol{\alpha}_0\boldsymbol{\beta}_1 - \boldsymbol{\alpha}_1\boldsymbol{\beta}_0)\,.$$

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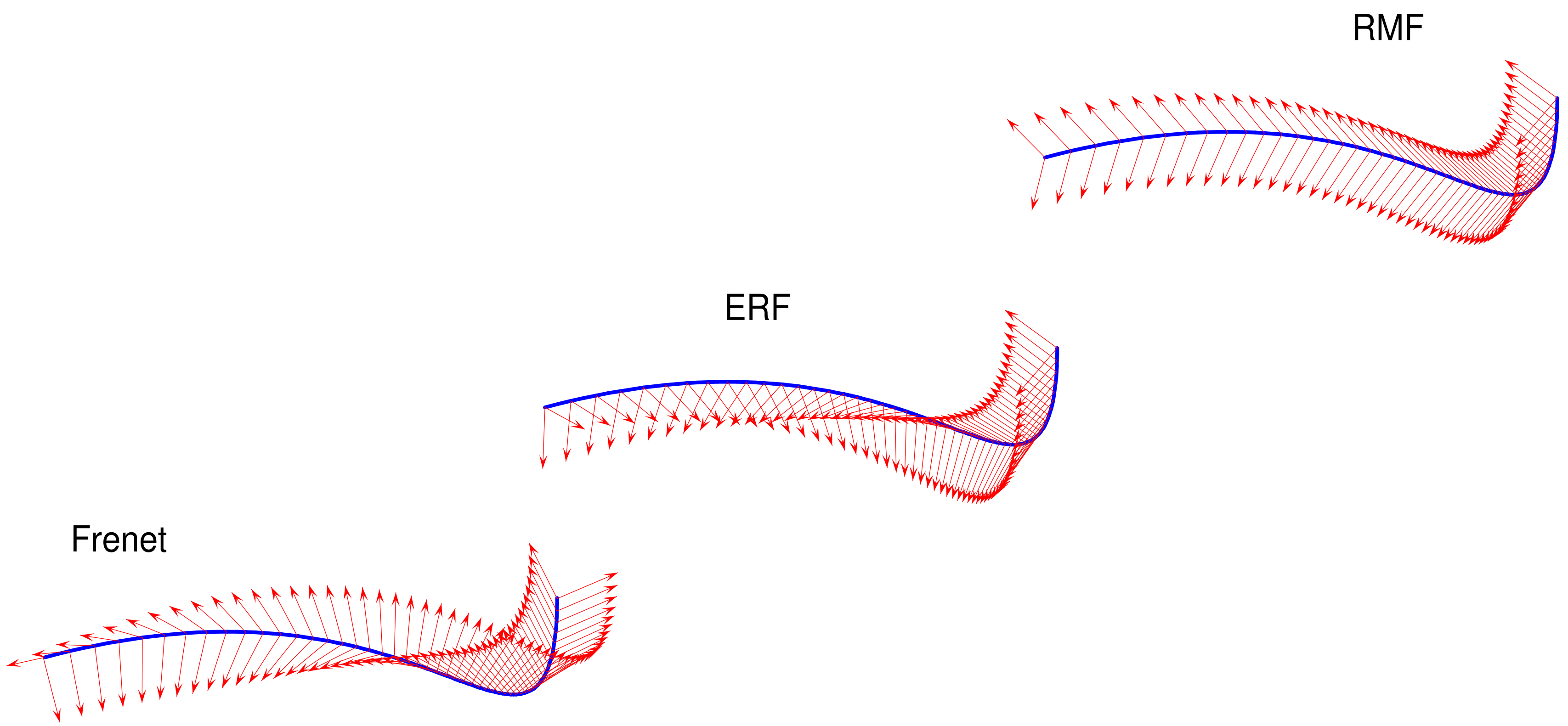
one real $+$ one complex constraint on $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_1, \boldsymbol{\alpha}_2$ and $\boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\beta}_2$
$\Rightarrow$ RRMF quintics have three less freedoms than general PH quintics

Algorithm to construct RRMF quintics: freely choose $\boldsymbol{\alpha}_0, \boldsymbol{\alpha}_2$ and $\boldsymbol{\beta}_0, \boldsymbol{\beta}_2$
& obtain $\boldsymbol{\alpha}_1, \boldsymbol{\beta}_1$ in terms of one free parameter, from RRMF constraints

example RRMF quintic construction

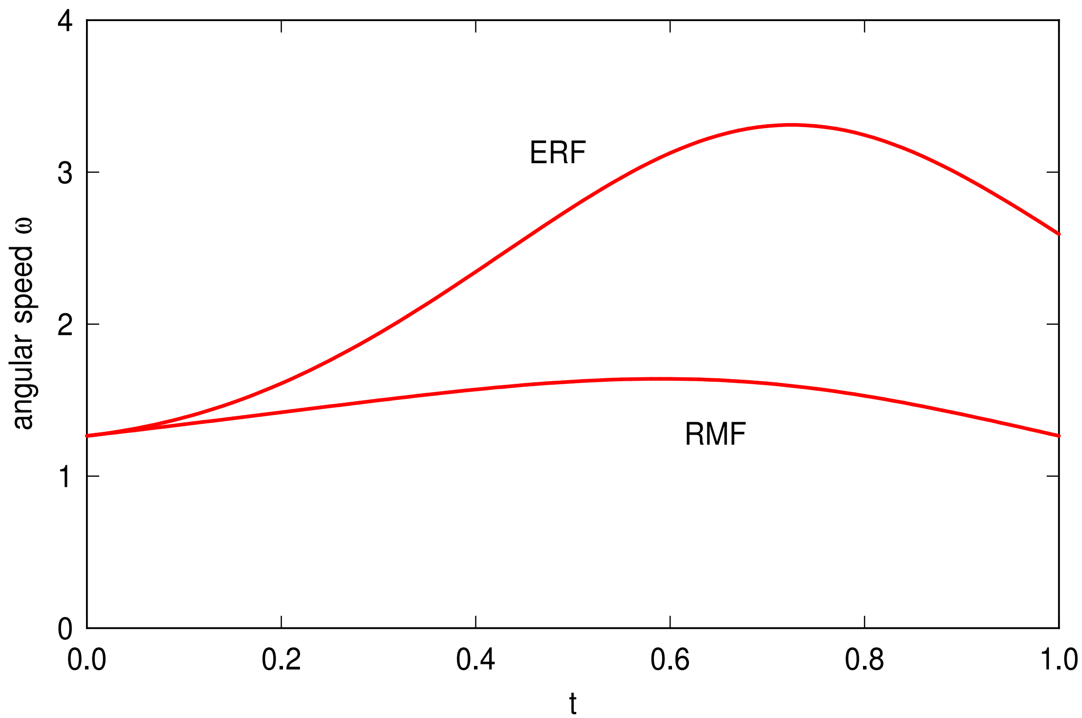
choose $\boldsymbol{\alpha}_0 = 1 + 2\,\mathrm{i}\,, \quad \boldsymbol{\beta}_0 = -2 + \mathrm{i}\,, \quad \boldsymbol{\alpha}_2 = 2 - \mathrm{i}\,, \quad \boldsymbol{\beta}_2 = -1 + 2\,\mathrm{i}$

$$\Longrightarrow \quad \boldsymbol{\alpha}_1 = \frac{1+\mathrm{i}}{\sqrt{2}}, \quad \boldsymbol{\beta}_1 = \frac{-3+\mathrm{i}}{\sqrt{2}} \quad \text{and} \quad (\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2) = \left(1, \frac{1}{\sqrt{2}}, \frac{3-4\,\mathrm{i}}{5}\right)$$



polynomials defining RMF vectors $(\mathbf{u},\mathbf{v})$ in terms of ERF vectors $(\mathbf{p},\mathbf{q})$

$$a(t) \,=\, (1-t)^2+\frac{1}{\sqrt{2}}\,2(1-t)t+\frac{3}{5}\,t^2\,, \qquad b(t) \,=\, -\frac{4}{5}\,t^2\,.$$



comparison of angular speeds for ERF and RMF

“lingering doubts” about RRMF quintic conditions

- constraints are of rather high degree — 4 and 6
- not invariant when “0” and “2” subscripts swapped
  (corresponds to the re–parameterization $t \to 1 - t$)
- do not easily translate to quaternion representation

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problem revisited in Farouki (2010), *Adv. Comp. Math.* **33**, 331–348

- to avoid asymmetry, do not assume $\mathbf{w}_0 = 1$
- consider PH quintics in canonical form with $\mathbf{r}'(0) = (1, 0, 0)$
- strategic switching between quaternion & Hopf map forms

improved sufficient–and–necessary conditions

Proposition 2. *A spatial PH quintic defined by the quaternion polynomial* $\mathcal{A}_0(1-\xi)^2 + \mathcal{A}_1 2(1-\xi)\xi + \mathcal{A}_2\,\xi^2$ *has a rational RMF if and only if*

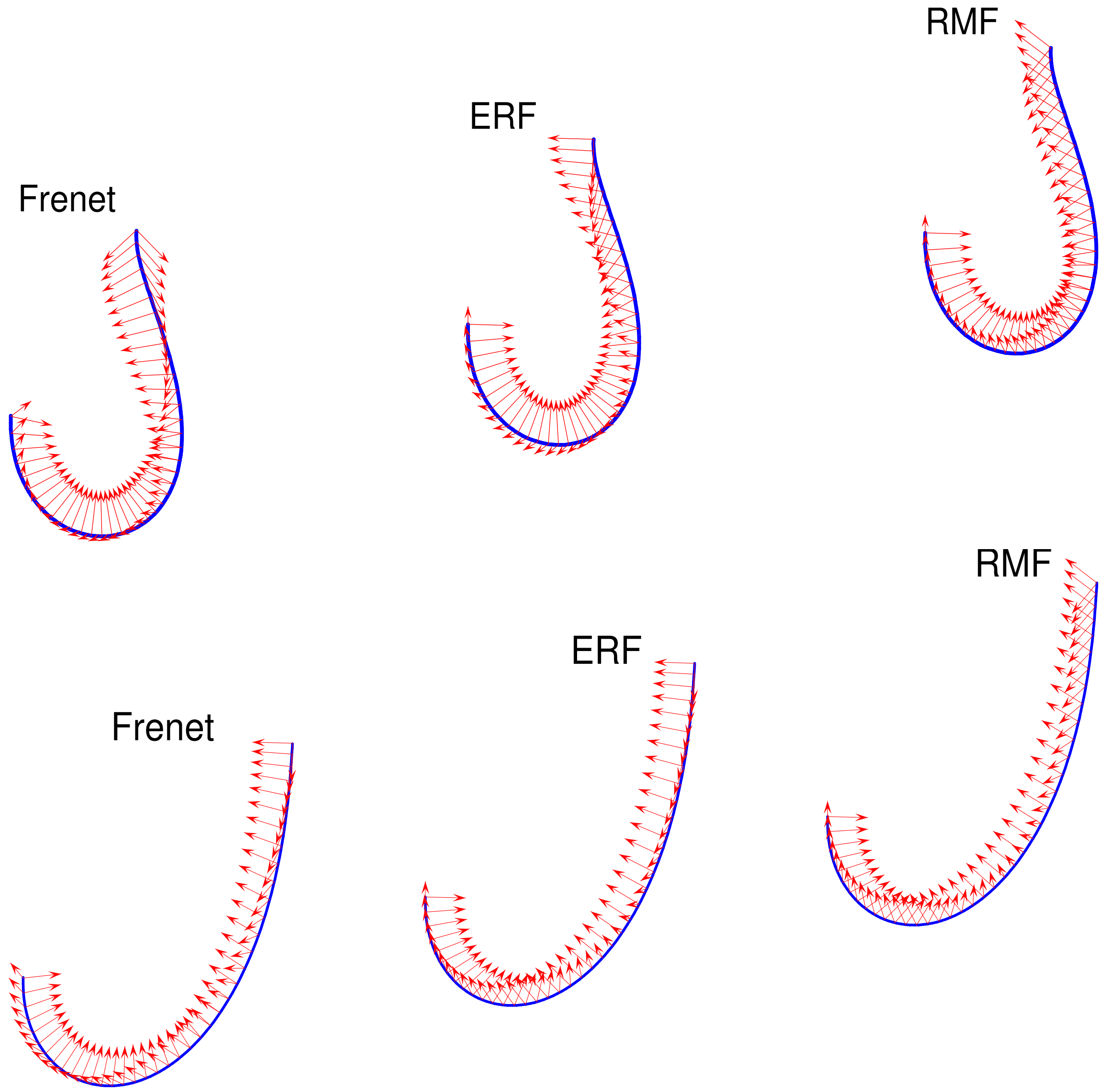
$$\mathcal{A}_0\,\mathbf{i}\,\mathcal{A}_2^* \,+\, \mathcal{A}_2\,\mathbf{i}\,\mathcal{A}_0^* \,=\, 2\,\mathcal{A}_1\,\mathbf{i}\,\mathcal{A}_1^*\,.$$

Proposition 3. *A spatial PH quintic defined by the complex polynomials* $\boldsymbol{\alpha}_0(1-\xi)^2 + \boldsymbol{\alpha}_1 2(1-\xi)\xi + \boldsymbol{\alpha}_2\,\xi^2$ *and* $\boldsymbol{\beta}_0(1-\xi)^2 + \boldsymbol{\beta}_1 2(1-\xi)\xi + \boldsymbol{\beta}_2\,\xi^2$ *has a rational RMF if and only if*

$$\mathrm{Re}(\boldsymbol{\alpha}_0\overline{\boldsymbol{\alpha}}_2 - \boldsymbol{\beta}_0\overline{\boldsymbol{\beta}}_2) \,=\, |\boldsymbol{\alpha}_1|^2 - |\boldsymbol{\beta}_1|^2\,, \quad \boldsymbol{\alpha}_0\overline{\boldsymbol{\beta}}_2 + \boldsymbol{\alpha}_2\overline{\boldsymbol{\beta}}_0 \,=\, 2\,\boldsymbol{\alpha}_1\overline{\boldsymbol{\beta}}_1\,.$$

- new conditions are only quadratic in coefficients
- easy transformation quaternion $\leftrightarrows$ Hopf map forms
- obvious invariance on swapping “0” and “2” subscripts

RRMF quintics constructed from new conditions



rational RMFs on space curves of any degree

R. T. Farouki and T. Sakkalis (2010), Rational rotation–minimizing frames on polynomial space curves of arbitrary degree, *Journal of Symbolic Computation* **45**, 844–856

Proposition 4. *For* $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$*, the condition*

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{ab' - a'b}{a^2 + b^2}$$

can be satisfied if and only if a polynomial $h(t)$ *exists, such that*

$$(uv' - u'v - pq' + p'q)^2 + (uq' - u'q - vp' + v'p)^2 \,=\, h(u^2 + v^2 + p^2 + q^2)\,.$$

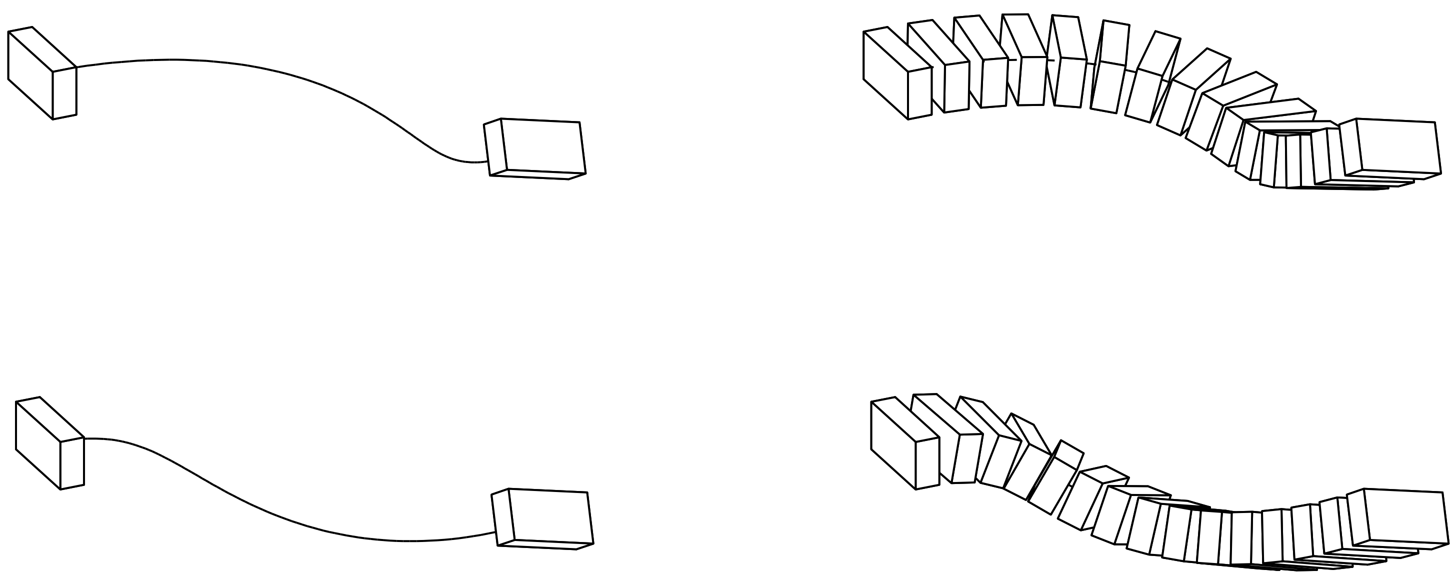
Polynomial $\rho = (uv' - u'v - pq' + p'q)^2 + (uq' - u'q - vp' + v'p)^2$ plays a key role in the theory of double PH curves, with $|\mathbf{r}'(t)|$ and $|\mathbf{r}'(t) \times \mathbf{r}''(t)|$ both *polynomials* in t — rational Frenet frames and rational curvatures.

Hermite interpolation by quintic RRMF curves

R. T. Farouki, C. Giannelli, C. Manni, A. Sestini (2011), Design of rational rotation–minimizing rigid body motions by Hermite interpolation, *Mathematics of Computation*, to appear

given initial, final positions & frames $\mathbf{p}_i$ & $(\mathbf{t}_i, \mathbf{u}_i, \mathbf{v}_i)$ and $\mathbf{p}_f$ & $(\mathbf{t}_f, \mathbf{u}_f, \mathbf{v}_f)$

compute RRMF quintic $\mathbf{r}(\xi)$ & frame $(\mathbf{t}(\xi), \mathbf{u}(\xi), \mathbf{v}(\xi))$ interpolating data



two distinct rational rotation–minimizing motions interpolating given data

RRMF Hermite interpolation problem decomposable into four phases

(1) interpolation of the end tangents $\mathbf{t}_i$ and $\mathbf{t}_f$

(2) satisfaction of RRMF constraints on the coefficients

(3) interpolation of normal–plane vectors $(\mathbf{u}_i, \mathbf{v}_i)$ and $(\mathbf{u}_f, \mathbf{v}_f)$

(4) interpolation of end–point displacement $\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$

phases (1)–(3) possess closed–form algebraic solutions

unique solutions for interpolation of $(\mathbf{t}_i, \mathbf{u}_i, \mathbf{v}_i)$ and $(\mathbf{t}_f, \mathbf{u}_f, \mathbf{v}_f)$

interpolation of $\Delta\mathbf{p} = \mathbf{p}_f - \mathbf{p}_i$ requires a certain degree 6 polynomial to have a positive real root (not always true), but solutions always exist for data sampled asymptotically from a smooth analytic curve

since (1)–(3) are independent of (4), new freedoms (e.g., multiplying $\mathbf{r}'(\xi)$ by scalar polynomial) can be introduced to facilitate existence of solutions

distinct classes of RRMF curves

R. T. Farouki and T. Sakkalis (2011), A complete classification of quintic space curves with rational rotation–minimizing frames *J. Symb. Comp.*, to appear

so far $\deg(a,b) = \deg(u,v,p,q)$ assumed in satisfying RRMF condition

$$\frac{uv' - u'v - pq' + p'q}{u^2 + v^2 + p^2 + q^2} = \frac{ab' - a'b}{a^2 + b^2}$$

- $\deg(a,b) = \deg(u,v,p,q)$ defines Class 1 RRMF curves
- RRMF condition can also be satisfied with $\deg(a,b) < \deg(u,v,p,q)$
- $\deg(a,b) = \deg(u,v,p,q) - k + 1$ defines Class k RRMF curves
- Class 2 RRMF quintics exist as true space curves (same d.o.f. as Class 1 quintics, but more complicated algebraic characterization)
- Class 3 RRMF curves of degree 7 exist as true space curves — for these curves, the ERF is rotation–minimizing ($ab' - a'b \equiv 0$)

adapted & directed frames on space curve $\mathbf{r}(\xi)$

- adapted frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \;\Rightarrow\; \mathbf{e}_1$ is the unit curve tangent, $\mathbf{t} = \mathbf{r}'/|\mathbf{r}'|$
- infinitely many choices of normal plane vectors $\mathbf{e}_2, \mathbf{e}_3$ orthogonal to $\mathbf{t}$
- angular velocity $\boldsymbol{\omega}$ of rotation-minimizing adapted frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is characterized by $\boldsymbol{\omega} \cdot \mathbf{t} \equiv 0$

- directed frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) \;\Rightarrow\; \mathbf{e}_1$ is the unit polar vector, $\mathbf{o} = \mathbf{r}/|\mathbf{r}|$
- infinitely many choices of image plane vectors $\mathbf{e}_2, \mathbf{e}_3$ orthogonal to $\mathbf{o}$
- angular velocity $\boldsymbol{\omega}$ of rotation-minimizing directed frame $(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3)$ is characterized by $\boldsymbol{\omega} \cdot \mathbf{o} \equiv 0$

rotation-minimizing directed frames — applications

R. T. Farouki and C. Giannelli (2009), Spatial camera orientation control by rotation–minimizing directed frames, *Computer Animation and Virtual Worlds* **20**, 457–472

- camera orientation planning for cinematography, video inspection, computer games, virtual reality, etc.
- minimize surgeon disorientation in endoscopic surgery
- related problem: field de-rotator for altazimuth telescope
- maintenance for aircraft engines, gas turbines, pipes, etc.
- for many applications, RMDF image orientation can be achieved through software transformations

camera orientation frame along space curve $\mathbf{r}(\xi)$

- assume target object fixed at origin (for moving target, consider only relative motion between camera & target)
- unit polar vector $\mathbf{o}(\xi) = \dfrac{\mathbf{r}(\xi)}{|\mathbf{r}(\xi)|}$ defines camera optical axis
- let camera image plane, orthogonal to $\mathbf{o}(\xi)$, be spanned by unit vectors $\mathbf{u}(\xi)$ and $\mathbf{v}(\xi)$
- if $\mathbf{r}(\xi), \mathbf{r}'(\xi)$ linearly independent, set $\mathbf{v}(\xi) = \dfrac{\mathbf{r}(\xi) \times \mathbf{r}'(\xi)}{|\mathbf{r}(\xi) \times \mathbf{r}'(\xi)|}$
- set $\mathbf{u}(\xi) = \mathbf{v}(\xi) \times \mathbf{o}(\xi)$ — $(\mathbf{o}(\xi), \mathbf{u}(\xi), \mathbf{v}(\xi))$ defines a right-handed orthonormal directed frame along $\mathbf{r}(\xi)$

compare directed frame defined above

$$\mathbf{o} = \frac{\mathbf{r}}{|\,\mathbf{r}\,|}, \quad \mathbf{u} = \frac{\mathbf{r} \times \mathbf{r}'}{|\,\mathbf{r} \times \mathbf{r}'\,|} \times \mathbf{o}\,, \quad \mathbf{v} = \frac{\mathbf{r} \times \mathbf{r}'}{|\,\mathbf{r} \times \mathbf{r}'\,|} \tag{1}$$

with Frenet frame from differential geometry

$$\mathbf{t} = \frac{\mathbf{r}'}{|\,\mathbf{r}'\,|}, \quad \mathbf{n} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\,\mathbf{r}' \times \mathbf{r}''\,|} \times \mathbf{t}\,, \quad \mathbf{b} = \frac{\mathbf{r}' \times \mathbf{r}''}{|\,\mathbf{r}' \times \mathbf{r}''\,|} \tag{2}$$

note that $(\mathbf{t},\mathbf{n},\mathbf{b}) \rightarrow (\mathbf{o},\mathbf{u},\mathbf{v})$ under map $(\mathbf{r}',\mathbf{r}'') \rightarrow (\mathbf{r},\mathbf{r}')$

call (1) the *Frenet directed frame*, (2) the *Frenet adapted frame*

define anti-hodograph (indefinite integral) $\mathbf{s}(\xi) = \int \mathbf{r}(\xi)\,\mathrm{d}\xi$

$\Rightarrow$ Frenet *directed* frame of a curve $\mathbf{r}(\xi)$

$=$ Frenet *adapted* frame of its anti-hodograph, $\mathbf{s}(\xi)$

properties of "anti-hodograph" — $\mathbf{s}(\xi) = \int \mathbf{r}(\xi)\,\mathrm{d}\xi$

- curve hodographs (derivatives) $\mathbf{r}'(\xi)$ are widely used in CAGD
- anti-derivative of function $f(\xi)$ is indefinite integral, $s(\xi) = \int f(\xi)\,\mathrm{d}\xi$
- infinitely many anti-hodographs — just translates of each other
- $\mathbf{s}(\xi_*)$ is a cusp of anti-hodograph $\Rightarrow$ $\mathbf{r}(\xi)$ traverses origin at $\xi = \xi_*$
- $\mathbf{s}(\xi_*)$ is an inflection of anti-hodograph $\Rightarrow$ tangent line to $\mathbf{r}(\xi)$ goes through origin for $\xi = \xi_*$
- polynomial curve $\Longleftrightarrow$ polynomial anti-hodograph, but this correspondence does not extend to rational anti-hodographs (integral of rational function may incur transcendental terms)

polar differential geometry of space curve $\mathbf{r}(\xi)$

$$\rho = |\,\mathbf{r}\,|, \qquad \lambda = \frac{|\,\mathbf{r}\times\mathbf{r}'\,|}{|\,\mathbf{r}\,|^3}, \qquad \upsilon = \frac{(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}''}{|\,\mathbf{r}\times\mathbf{r}'\,|^2}$$

polar distance, polar curvature, polar torsion of $\mathbf{r}(\xi)$

$=$ parametric speed, curvature, torsion of anti-hodograph, $\mathbf{s}(\xi) = \int \mathbf{r}(\xi)$

- polar curvature $\lambda(\xi) \equiv 0 \iff \mathbf{r}(\xi) =$ line through origin
- polar torsion $\upsilon(\xi) \equiv 0 \iff \mathbf{r}(\xi) =$ in plane through origin
- hence, $\lambda(\xi) \equiv 0 \;\Rightarrow\; \kappa(\xi) \equiv 0$ and $\upsilon(\xi) \equiv 0 \;\Rightarrow\; \tau(\xi) \equiv 0$
- $\lambda = 0$ identifies polar inflection — $\mathbf{r}$ and $\mathbf{r}'$ linearly dependent
- polar helix $\dfrac{\lambda(\xi)}{\upsilon(\xi)} =$ constant $\iff \mathbf{r}(\xi) =$ on cone with apex at origin

Frenet-Serret equations for directed frame $(\mathbf{o}, \mathbf{u}, \mathbf{v})$

$$\begin{bmatrix} \mathbf{o}' \\ \mathbf{u}' \\ \mathbf{v}' \end{bmatrix} = \rho \begin{bmatrix} 0 & \lambda & 0 \\ -\lambda & 0 & \upsilon \\ 0 & -\upsilon & 0 \end{bmatrix} \begin{bmatrix} \mathbf{o} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix}$$

polar distance, polar curvature, polar torsion of $\mathbf{r}(\xi)$

$$\rho = |\,\mathbf{r}\,|, \qquad \lambda = \frac{|\,\mathbf{r} \times \mathbf{r}'\,|}{|\,\mathbf{r}\,|^3}, \qquad \upsilon = \frac{(\mathbf{r} \times \mathbf{r}') \cdot \mathbf{r}''}{|\,\mathbf{r} \times \mathbf{r}'\,|^2}$$

arc-length derivatives of $(\mathbf{o}, \mathbf{u}, \mathbf{v})$

$$\frac{\mathrm{d}\mathbf{o}}{\mathrm{d}s} = \mathbf{e} \times \mathbf{o}\,, \qquad \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}s} = \mathbf{e} \times \mathbf{u}\,, \qquad \frac{\mathrm{d}\mathbf{v}}{\mathrm{d}s} = \mathbf{e} \times \mathbf{v}\,.$$

polar Darboux vector $\mathbf{e} = \dfrac{\rho}{\sigma}(\lambda\,\mathbf{v} + \upsilon\,\mathbf{o})$

angular velocity of directed frame $\omega = |\,\mathbf{e}\,| = \dfrac{\rho}{\sigma}\sqrt{\lambda^2 + \upsilon^2}$

corresponding properties of the Frenet *adapted* and *directed* frames on space curves

Frenet adapted frame	Frenet directed frame
tangent vector $\mathbf{t}$	polar vector $\mathbf{o}$
principal normal $\mathbf{n}$	principal axis $\mathbf{u}$
binormal vector $\mathbf{b}$	bi-axis vector $\mathbf{v}$
normal plane = span$(\mathbf{n}, \mathbf{b})$	image plane = span$(\mathbf{u}, \mathbf{v})$
osculating plane = span$(\mathbf{t}, \mathbf{n})$	motion plane = span$(\mathbf{o}, \mathbf{u})$
rectifying plane = span$(\mathbf{b}, \mathbf{t})$	orthogonal plane = span$(\mathbf{v}, \mathbf{o})$
parametric speed σ	polar distance ρ
curvature κ	polar curvature λ
torsion τ	polar torsion υ

each property of the Frenet *directed* frame of $\mathbf{r}(\xi)$ coincides with the corresponding property of the Frenet *adapted* frame of its anti-hodograph, $\mathbf{s}(\xi) = \int \mathbf{r}(\xi)\, \mathrm{d}\xi$

connection between Frenet adapted & directed frames

$(\mathbf{t}, \mathbf{n}, \mathbf{b})$ and $(\mathbf{o}, \mathbf{u}, \mathbf{v})$ are both orthonormal frames for $\mathbb{R}^3$

$$\begin{bmatrix} \mathbf{o} \\ \mathbf{u} \\ \mathbf{v} \end{bmatrix} = \begin{bmatrix} \mathbf{o}\cdot\mathbf{t} & \mathbf{o}\cdot\mathbf{n} & \mathbf{o}\cdot\mathbf{b} \\ \mathbf{u}\cdot\mathbf{t} & \mathbf{u}\cdot\mathbf{n} & \mathbf{u}\cdot\mathbf{b} \\ \mathbf{v}\cdot\mathbf{t} & \mathbf{v}\cdot\mathbf{n} & \mathbf{v}\cdot\mathbf{b} \end{bmatrix} \begin{bmatrix} \mathbf{t} \\ \mathbf{n} \\ \mathbf{b} \end{bmatrix}$$

elements of matrix $\mathrm{M} \in \mathrm{SO}(3)$ in terms of $\mathbf{r}$, $\mathbf{r}'$, $\mathbf{r}''$, $\rho = |\mathbf{r}|$, $\sigma = |\mathbf{r}'|$:

$$\mathbf{o}\cdot\mathbf{t} \;=\; \frac{\mathbf{r}\cdot\mathbf{r}'}{\rho\,\sigma}\,, \quad \mathbf{o}\cdot\mathbf{n} = -\,\frac{(\mathbf{r}\times\mathbf{r}')\cdot(\mathbf{r}'\times\mathbf{r}'')}{\rho\,\sigma\,|\,\mathbf{r}'\times\mathbf{r}''\,|}\,, \quad \mathbf{o}\cdot\mathbf{b} \;=\; \frac{(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}''}{\rho\,|\,\mathbf{r}'\times\mathbf{r}''\,|}\,,$$

$$\mathbf{u}\cdot\mathbf{t} \;=\; \frac{|\,\mathbf{r}\times\mathbf{r}'\,|}{\rho\,\sigma}\,, \quad \mathbf{u}\cdot\mathbf{n} \;=\; \frac{\mathbf{r}\cdot\mathbf{r}'}{\rho\,\sigma}\,\frac{(\mathbf{r}\times\mathbf{r}')\cdot(\mathbf{r}'\times\mathbf{r}'')}{|\,\mathbf{r}\times\mathbf{r}'\,|\,|\,\mathbf{r}'\times\mathbf{r}''\,|}\,, \quad \mathbf{u}\cdot\mathbf{b} \;=\; -\,\frac{(\mathbf{r}\cdot\mathbf{r}')\,(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}''}{\rho\,|\,\mathbf{r}\times\mathbf{r}'\,|\,|\,\mathbf{r}'\times\mathbf{r}''\,|}\,,$$

$$\mathbf{v}\cdot\mathbf{t} \;=\; 0\,, \quad \mathbf{v}\cdot\mathbf{n} \;=\; \frac{\sigma\,(\mathbf{r}\times\mathbf{r}')\cdot\mathbf{r}''}{|\,\mathbf{r}\times\mathbf{r}'\,|\,|\,\mathbf{r}'\times\mathbf{r}''\,|}\,, \quad \mathbf{v}\cdot\mathbf{b} \;=\; \frac{(\mathbf{r}\times\mathbf{r}')\cdot(\mathbf{r}'\times\mathbf{r}'')}{|\,\mathbf{r}\times\mathbf{r}'\,|\,|\,\mathbf{r}'\times\mathbf{r}''\,|}\,.$$

computation of rotation-minimizing directed frames

let $(\mathbf{o}, \mathbf{p}, \mathbf{q})$ be rotation-minimizing directed frame on $\mathbf{r}(\xi)$

obtain $(\mathbf{p}, \mathbf{q})$ from $(\mathbf{u}, \mathbf{v})$ by rotation in image plane

$$\left[\begin{array}{c} \mathbf{p} \\ \mathbf{q} \end{array} \right] = \left[\begin{array}{cc} \cos\psi & \sin\psi \\ -\sin\psi & \cos\psi \end{array} \right] \left[\begin{array}{c} \mathbf{u} \\ \mathbf{v} \end{array} \right]$$

using anti-hodograph transformation, $\psi = -\int \upsilon\, \rho\, \mathrm{d}\xi$

(i.e., integral of polar torsion w.r.t. anti-hodograph arc length)

- RMDF angular velocity $\boldsymbol{\omega}$ omits $\upsilon\, \mathbf{o}$ term from polar Darboux vector
- infinitely many directed RMFs, corresponding to different integration constants (maintain fixed angles relative to each other)
- angle function $\psi(\xi)$ can be determined exactly for spatial P curves by rational function integration

example: circular camera path $\mathbf{r}(\theta) = (r\cos\theta, r\sin\theta, h)$

$$\mathbf{o} = \frac{(r\cos\theta, r\sin\theta, h)}{\sqrt{r^2+h^2}}\,, \quad \mathbf{u} = (-\sin\theta, \cos\theta, 0)\,, \quad \mathbf{v} = \frac{(-h\cos\theta, -h\sin\theta, r)}{\sqrt{r^2+h^2}}\,.$$

note — principal axis vector $\mathbf{u}$ coincides with curve tangent $\mathbf{t}$

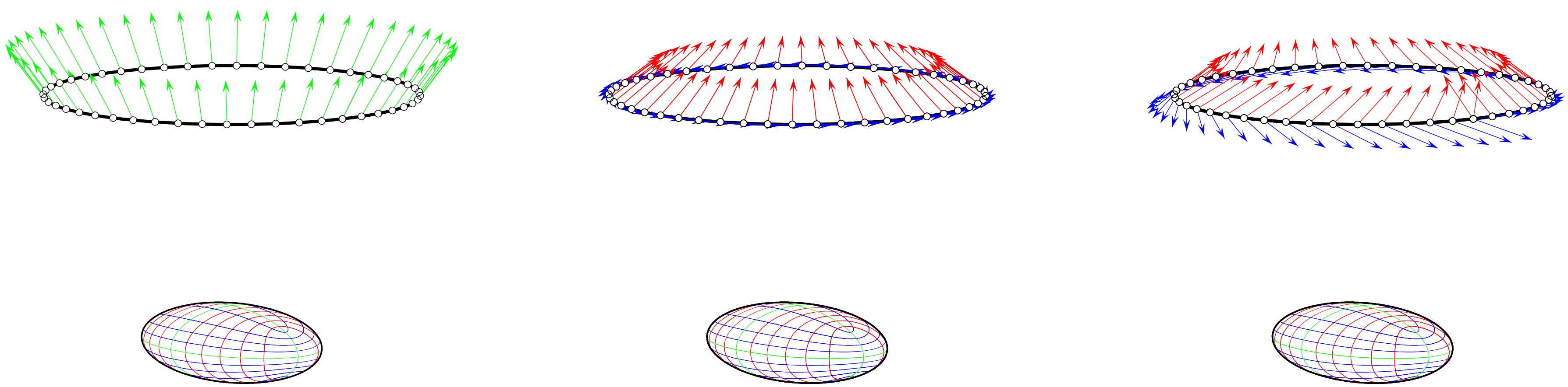
$$\rho \,=\, r\sqrt{r^2+h^2}\,, \qquad \lambda \,=\, \frac{r}{r^2+h^2}\,, \qquad \upsilon \,=\, \frac{h}{r^2+h^2}\,.$$

polar distance, polar curvature, polar torsion — all constant

$$\psi \,=\, -\,\frac{\theta}{\sqrt{1+(r/h)^2}}\,.$$

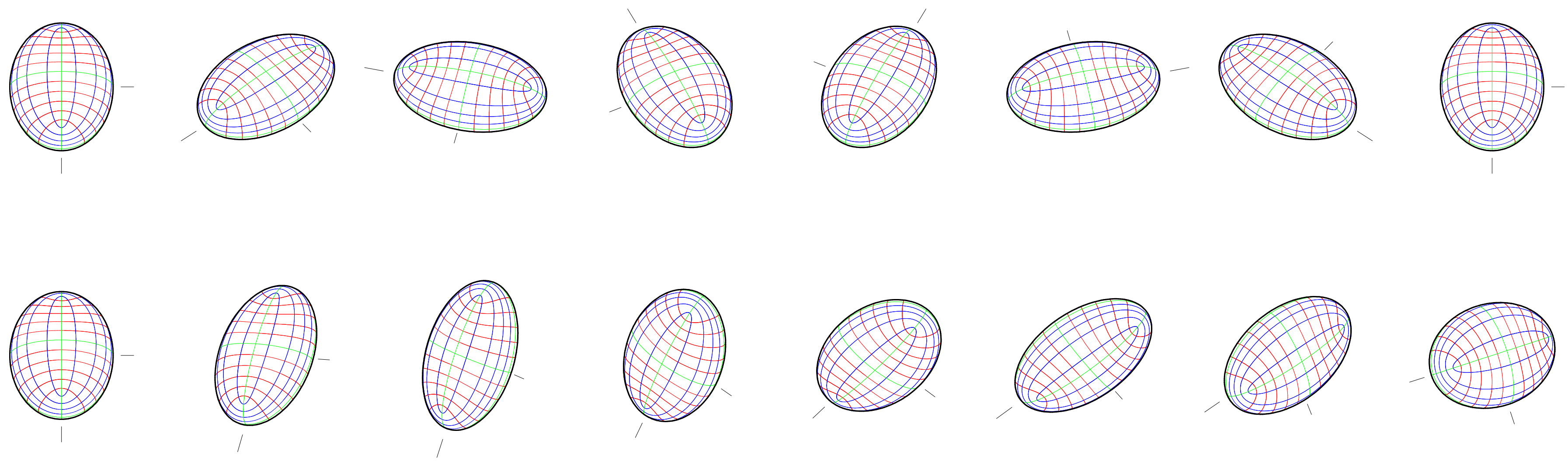
RMDF orientation relative to Frenet directed frame — linear in θ

directed frames on circular path, $\mathbf{r}(\theta) = (r\cos\theta, r\sin\theta, h)$



Left: polar vectors. Center: image-plane vectors for directed Frenet frame.
Right: image-plane vectors for the rotation-minimizing directed frame.

views of ellipsoid with camera image plane oriented using Frenet directed frame (upper) and rotation-minimizing directed frame (lower)



closure

- theory, algorithms, applications for rotation-minimizing frames
- RRMF curve $=$ PH curve with rational rotation–minimizing frame
- quaternion and Hopf map characterizations of RRMF quintics
- divisibility characterization for RRMF curves of any degree
- distinct classes of RRMF curves of any given degree
- rotation-minimizing directed frames in camera orientation control
- anti-hodograph and polar differential geometry of space curves