Minkowski products of unit quaternion sets

1 Introduction

The Minkowski sum $A \oplus B$ of two point sets $A, B \in \mathbb{R}^n$ is the set of all points generated [16] by the vector sums of points chosen independently from those sets, i.e.,

$$A \oplus B := \{ \mathbf{a} + \mathbf{b} : \mathbf{a} \in A \text{ and } \mathbf{b} \in B \} . \tag{1}$$

The Minkowski sum has applications in computer graphics, geometric design, image processing, and related fields [9, 11, 12, 13, 14, 15, 20]. The validity of the definition (1) in $\mathbb{R}^n$ for all $n \geq 1$ stems from the straightforward extension of the vector sum $\mathbf{a} + \mathbf{b}$ to higher–dimensional Euclidean spaces. However, to define a Minkowski *product* set

$$A \otimes B := \{ \mathbf{a}\,\mathbf{b} : \mathbf{a} \in A \text{ and } \mathbf{b} \in B \} , \tag{2}$$

it is necessary to specify *products* of points in $\mathbb{R}^n$. In the case $n = 1$, this is simply the real–number product — the resulting algebra of point sets in $\mathbb{R}^1$ is called *interval arithmetic* [17, 18] and is used to monitor the propagation of uncertainty through computations in which the initial operands (and possibly also the arithmetic operations) are not precisely determined.

A natural realization of the Minkowski product (2) in $\mathbb{R}^2$ may be achieved [7] by interpreting the points $\mathbf{a}$ and $\mathbf{b}$ as *complex numbers*, with $\mathbf{a}\,\mathbf{b}$ being the usual complex–number product. Algorithms to compute Minkowski products of complex–number sets have been formulated [6], and extended to determine Minkowski roots and powers [3, 8] of complex sets; to evaluate polynomials specified by complex–set coefficients and arguments [4]; and to solve simple equations expressed in terms of complex–set coefficients and unknowns [5]. The Minkowski algebra of complex sets introduces rich geometrical structures and has useful applications to mathematical morphology, geometrical optics, and the stability analysis of linear dynamic systems.

In proceeding to higher dimensions, it is natural to consider next the case of $\mathbb{R}^4$, in which a (non–commutative) "product of points" may be specified by invoking the quaternion algebra. In this context, the study of the Minkowski sum has no obvious and intuitive motivation, but the use of unit quaternions to describe spatial rotations provides a compelling case for the investigation of Minkowski products in $\mathbb{R}^4$. Applications in computer animation, robot path planning, 5–axis CNC machining, and related fields frequently involve compounded sequences of spatial rotations, that are individually subject to certain indeterminacies. The set of all possible outcomes of such compounded sequences of indeterminate spatial rotations possesses a natural description as the (ordered) Minkowski product of unit quaternion sets.

2 Quaternions and spatial rotations

Calligraphic characters $\mathcal{A}, \mathcal{B}, \ldots$ will be used to denote quaternions, which are "four–dimensional numbers" of the form

$$\mathcal{A} \,=\, a + a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} \qquad \text{and} \qquad \mathcal{B} \,=\, b + b_x\mathbf{i} + b_y\mathbf{j} + b_z\mathbf{k}\,.$$

where the basis elements $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ of the quaternion algebra $\mathbb{H}$ are governed by the multiplication rules

$$\mathbf{i}^2 \,=\, \mathbf{j}^2 \,=\, \mathbf{k}^2 \,=\, -1\,,$$

$$\mathbf{i}\,\mathbf{j} \,=\, -\,\mathbf{j}\,\mathbf{i} \,=\, \mathbf{k}\,, \qquad \mathbf{j}\,\mathbf{k} \,=\, -\,\mathbf{k}\,\mathbf{j} \,=\, \mathbf{i}\,, \qquad \mathbf{k}\,\mathbf{i} \,=\, -\,\mathbf{i}\,\mathbf{k} \,=\, \mathbf{j}\,.$$

The quaternion product is non–commutative, i.e., $\mathcal{A}\,\mathcal{B} \neq \mathcal{B}\mathcal{A}$ in general, but it is associative: $(\mathcal{A}\,\mathcal{B})\mathcal{C} = \mathcal{A}(\mathcal{B}\,\mathcal{C})$ for any three quaternions $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$.

A quaternion $\mathcal{A}$ may be regarded as comprising a scalar (or real) part $a = \text{scal}(\mathcal{A})$ and a vector (or imaginary) part $\mathbf{a} = a_x\mathbf{i} + a_y\mathbf{j} + a_z\mathbf{k} = \text{vect}(\mathcal{A})$, and we write $\mathcal{A} = (a, \mathbf{a})$. Real numbers and vectors are subsumed as "pure scalar" and "pure vector" quaternions, of the form $(a, \mathbf{0})$ and $(0, \mathbf{a})$, respectively — for brevity, we often simply write a and $\mathbf{a}$.

The sum and the product of $\mathcal{A} = (a, \mathbf{a})$ and $\mathcal{B} = (b, \mathbf{b})$ can be succinctly expressed using vector sum and dot and cross products as

$$\mathcal{A} + \mathcal{B} \,=\, (a + b\,,\, \mathbf{a} + \mathbf{b})\,, \quad \mathcal{A}\,\mathcal{B} \,=\, (ab - \mathbf{a}\cdot\mathbf{b}\,,\, a\,\mathbf{b} + b\,\mathbf{a} + \mathbf{a}\times\mathbf{b})\,.$$

Each quaternion $\mathcal{A} = (a, \mathbf{a})$ has a *conjugate* $\mathcal{A}^* = (a, -\mathbf{a})$ and a *magnitude* $|\mathcal{A}|$ equal to the non–negative real number defined by

$$|\mathcal{A}|^2 \,=\, \mathcal{A}^*\mathcal{A} \,=\, \mathcal{A}\mathcal{A}^* \,=\, a^2 + |\mathbf{a}|^2\,.$$

One can verify that the quaternion product satisfies

$$(\mathcal{A}\,\mathcal{B})^* \,=\, \mathcal{B}^*\mathcal{A}^* \quad \text{and} \quad |\mathcal{A}\,\mathcal{B}| \,=\, |\mathcal{A}|\,|\mathcal{B}|\,.$$

For each quaternion $\mathcal{A}$ with $|\mathcal{A}| \neq 0$ one can associate an *inverse*

$$\mathcal{A}^{-1} \,:=\, \frac{\mathcal{A}^*}{|\mathcal{A}|^2}\,,$$

such that $\mathcal{A}^{-1}\mathcal{A} = \mathcal{A}\,\mathcal{A}^{-1} = 1$. The *left division* and *right division* of $\mathcal{B}$ by $\mathcal{A}$ are then specified by the expressions $\mathcal{A}^{-1}\mathcal{B}$ and $\mathcal{B}\mathcal{A}^{-1}$ respectively. We also define an *inner product* $\langle \mathcal{A}, \mathcal{B} \rangle$ of $\mathcal{A}$ and $\mathcal{B}$ (regarded as vectors in $\mathbb{R}^4$) by

$$\langle \mathcal{A}, \mathcal{B} \rangle \;:=\; a\,b + \mathbf{a}\cdot\mathbf{b} \;=\; \mathrm{scal}(\mathcal{A}\,\mathcal{B}^*)\,.$$

When $|\mathcal{A}| = 1$, we say that $\mathcal{A}$ is a *unit* quaternion, and identify it with a point on the unit "3–sphere" S^3 in $\mathbb{R}^4$ specified by the equation

$$a^2 \,+\, a_x^2 \,+\, a_y^2 \,+\, a_z^2 \;=\; 1\,.$$

If $\mathcal{A}$ and $\mathcal{B}$ are unit quaternions, their product $\mathcal{A}\,\mathcal{B}$ is also unit and it identifies another point on S^3. Thus, points on the 3–sphere in $\mathbb{R}^4$ have the structure of a (non–commutative) *group* with respect to quaternion multiplication. Any unit quaternion $\mathcal{U}$ may be expressed in the form

$$\mathcal{U} \;=\; (\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta\,\mathbf{n}) \tag{3}$$

for some angle $\theta \in [\,-\pi, \pi\,]$ and unit vector $\mathbf{n}$. This defines a *rotation operator* in $\mathbb{R}^3$ with θ and $\mathbf{n}$ identifying the rotation angle and axis, as follows.

For any pure vector $\mathbf{v}$, the quaternion product $\mathcal{U}\,\mathbf{v}\,\mathcal{U}^*$ also defines a pure vector, corresponding to a rotation of $\mathbf{v}$ through angle θ about an axis defined by $\mathbf{n}$. Carrying out the multiplication yields

$$\mathcal{U}\,\mathbf{v}\,\mathcal{U}^* \;=\; (\mathbf{n}\cdot\mathbf{v})\,\mathbf{n} \,+\, \sin\theta\,\mathbf{n}\times\mathbf{v} \,+\, \cos\theta\,(\mathbf{n}\times\mathbf{v})\times\mathbf{n}\,.$$

Now prior to the rotation, consider the decomposition $\mathbf{v} = \mathbf{v}_{\|} + \mathbf{v}_{\perp}$, where

$$\mathbf{v}_{\|} \;=\; (\mathbf{n}\cdot\mathbf{v})\,\mathbf{n} \quad \text{and} \quad \mathbf{v}_{\perp} \;=\; (\mathbf{n}\times\mathbf{v})\times\mathbf{n}$$

are the components of $\mathbf{v}$ parallel and perpendicular to $\mathbf{n}$. The rotation leaves $\mathbf{v}_{\|}$ unchanged, but $\mathbf{v}_{\perp}$ becomes $\sin\theta\,\mathbf{n}\times\mathbf{v} + \cos\theta\,(\mathbf{n}\times\mathbf{v})\times\mathbf{n}$. Note that $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$ and $-\,\mathcal{U} = (-\cos\frac{1}{2}\theta, -\sin\frac{1}{2}\theta\,\mathbf{n})$ define equivalent rotations — namely, through angle θ about $\mathbf{n}$, and angle $2\pi - \theta$ about $-\,\mathbf{n}$.

Successive spatial rotations, specified by unit quaternions, can be replaced by a single "compounded" rotation — for example, the result of consecutively applying the rotations $\mathcal{U}_1 = (\cos\frac{1}{2}\theta_1, \sin\frac{1}{2}\theta_1\mathbf{n}_1)$ and $\mathcal{U}_2 = (\cos\frac{1}{2}\theta_2, \sin\frac{1}{2}\theta_2\mathbf{n}_2)$ to a vector $\mathbf{v}$ is given by

$$\mathcal{U}_2\,(\,\mathcal{U}_1\mathbf{v}\,\mathcal{U}_1^*\,)\,\mathcal{U}_2^*\,,$$

and since $\mathcal{U}_1^* \mathcal{U}_2^* = (\mathcal{U}_2 \mathcal{U}_1)^*$ this can be expressed as

$$\mathcal{U}\,\mathbf{v}\,\mathcal{U}^*\,,$$

where $\mathcal{U} = \mathcal{U}_2 \mathcal{U}_1$. Hence, the result of applying the rotation $\mathcal{U}_1$ followed by the rotation $\mathcal{U}_2$ is equivalent to a single rotation, specified by $\mathcal{U} = \mathcal{U}_2 \mathcal{U}_1$. The non–commutative quaternion product captures the fact that the outcome of a sequence of rotations depends on the *order* of their application.

If $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$ is the product $\mathcal{U}_1 \mathcal{U}_2$ of $\mathcal{U}_1 = (\cos\frac{1}{2}\theta_1, \sin\frac{1}{2}\theta_1 \mathbf{n}_1)$ and $\mathcal{U}_2 = (\cos\frac{1}{2}\theta_2, \sin\frac{1}{2}\theta_2 \mathbf{n}_2)$, one may verify that the equivalent angle θ and axis $\mathbf{n}$ for the compound rotation $\mathcal{U}$ are given by

$$\cos\tfrac{1}{2}\theta \;=\; \cos\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 - \sin\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2\; \mathbf{n}_1 \cdot \mathbf{n}_2\,, \tag{4}$$

$$\mathbf{n} \;=\; \frac{\sin\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_2\, \mathbf{n}_1 \;+\; \cos\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2\, \mathbf{n}_2 \;+\; \sin\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2\, \mathbf{n}_1 \times \mathbf{n}_2}{\sin\frac{1}{2}\theta}\,. \tag{5}$$

Note that these expressions define two possibilities, since a rotation by angle θ about $\mathbf{n}$ is equivalent to a rotation by $-\theta$ about $-\mathbf{n}$.

3 Minkowski product boundaries

In general, a connected set U of unit quaternions is 3–dimensional, and has a 2–dimensional surface as boundary. As special cases, one may also consider sets U of dimension 2 (surfaces), 1 (curves), or 0 (points) in S^3 — in these cases, U has no interior, and consists entirely of boundary points. Henceforth, unless otherwise stated, we consider connected sets of dimension 3.

Given two connected, 3–dimensional sets of unit quaternions, U_1 and U_2, we wish to determine which products $\mathcal{U}_1 \mathcal{U}_2$ of elements $\mathcal{U}_1 \in U_1$ and $\mathcal{U}_2 \in U_2$ contribute to the boundary $\partial(U_1 \otimes U_2)$ of their Minkowski product. We first consider the behavior of the product $\mathcal{U}_1 \mathcal{U}_2$ with respect to small perturbations imposed on $\mathcal{U}_1$ and $\mathcal{U}_2$.

Proposition 1 *If $\mathcal{U}_1 + \delta\,\mathcal{U}_1$ and $\mathcal{U}_2 + \delta\,\mathcal{U}_2$ are the unit quaternions resulting from perturbations $\delta\theta_1, \delta\mathbf{n}_1$ and $\delta\theta_2, \delta\mathbf{n}_2$ to the rotation angles and axes of two given unit quaternions $\mathcal{U}_1 = (\cos\frac{1}{2}\theta_1, \sin\frac{1}{2}\theta_1\,\mathbf{n}_1)$ and $\mathcal{U}_2 = (\cos\frac{1}{2}\theta_2, \sin\frac{1}{2}\theta_2\,\mathbf{n}_2)$, the product $(\mathcal{U}_1 + \delta\,\mathcal{U}_1)(\mathcal{U}_2 + \delta\,\mathcal{U}_2)$ is always distinct from $\mathcal{U}_1 \mathcal{U}_2$ to first order in $\delta\theta_1, \delta\mathbf{n}_1$ and $\delta\theta_2, \delta\mathbf{n}_2$ — i.e., the unit quaternion product map $S^3 \times S^3 \to S^3$ has no stationary points.*

Proof : Since $\mathcal{U}_1 + \delta\mathcal{U}_1$ and $\mathcal{U}_2 + \delta\mathcal{U}_2$ are defined by

$$\begin{aligned}\mathcal{U}_1 + \delta\mathcal{U}_1 &= (\cos\tfrac{1}{2}(\theta_1 + \delta\theta_1), \sin\tfrac{1}{2}(\theta_1 + \delta\theta_1)(\mathbf{n}_1 + \delta\mathbf{n}_1))\,, \\ \mathcal{U}_2 + \delta\mathcal{U}_2 &= (\cos\tfrac{1}{2}(\theta_2 + \delta\theta_2), \sin\tfrac{1}{2}(\theta_2 + \delta\theta_2)(\mathbf{n}_2 + \delta\mathbf{n}_2))\,,\end{aligned}$$

to first order in $\delta\theta_1, \delta\theta_2$ and $\delta\mathbf{n}_1, \delta\mathbf{n}_2$ we obtain

$$\delta\mathcal{U}_1 \;=\; \tfrac{1}{2}\delta\theta_1\,\tilde{\mathcal{U}}_1 \;+\; \sin\tfrac{1}{2}\theta_1\,(0,\delta\mathbf{n}_1)\,, \quad \delta\mathcal{U}_2 \;=\; \tfrac{1}{2}\delta\theta_2\,\tilde{\mathcal{U}}_2 \;+\; \sin\tfrac{1}{2}\theta_2\,(0,\delta\mathbf{n}_2)\,,$$

where

$$\tilde{\mathcal{U}}_1 \;:=\; (-\sin\tfrac{1}{2}\theta_1, \cos\tfrac{1}{2}\theta_1\,\mathbf{n}_1)\,, \quad \tilde{\mathcal{U}}_2 \;:=\; (-\sin\tfrac{1}{2}\theta_2, \cos\tfrac{1}{2}\theta_2\,\mathbf{n}_2)$$

are the unit quaternions defined by replacing θ_1, θ_2 in $\mathcal{U}_1, \mathcal{U}_2$ with $\theta_1 + \pi, \theta_2 + \pi$. Hence, retaining only first–order terms, the product $(\mathcal{U}_1 + \delta\mathcal{U}_1)(\mathcal{U}_2 + \delta\mathcal{U}_2)$ becomes

$$\mathcal{U}_1\mathcal{U}_2 + \tfrac{1}{2}\delta\theta_1\tilde{\mathcal{U}}_1\mathcal{U}_2 + \tfrac{1}{2}\delta\theta_2\mathcal{U}_1\tilde{\mathcal{U}}_2 + \sin\tfrac{1}{2}\theta_2\,(0,\delta\mathbf{n}_1)\,\mathcal{U}_2 + \sin\tfrac{1}{2}\theta_1\,\mathcal{U}_1\,(0,\delta\mathbf{n}_2)\,.$$

Now the coefficients of the angle perturbations $\delta\theta_1$ and $\delta\theta_2$ never vanish, since $\mathcal{U}_1, \mathcal{U}_2$ and $\tilde{\mathcal{U}}_1, \tilde{\mathcal{U}}_2$ are unit quaternions, and therefore $|\,\tilde{\mathcal{U}}_1\mathcal{U}_2\,| = |\,\mathcal{U}_1\tilde{\mathcal{U}}_2\,| = 1$. Also, for $\theta_1, \theta_2 \in [\,-\pi, +\pi\,]$ the coefficients of the axis perturbations $(0,\delta\mathbf{n}_1)$ and $(0,\delta\mathbf{n}_2)$ can only vanish in the trivial cases $\theta_2 = 0$ and $\theta_1 = 0$. ∎

To first order in $\delta\theta$ and $\delta\mathbf{n}$, the perturbations $\delta\mathcal{U}$ that satisfy

$$|\,\delta\mathcal{U}\,| \;=\; \sqrt{\tfrac{1}{4}(\delta\theta)^2 + \sin^2\tfrac{1}{2}\theta\,|\delta\mathbf{n}|^2} \;<\; \epsilon$$

are equivalent to a spherical set $U(\mathcal{U}_0, \epsilon)$ with any center $\mathcal{U}_0$ and radius $\epsilon \ll 1$ (note that $\mathbf{n}\cdot\delta\mathbf{n} = 0$, since $|\mathbf{n}| = 1$).

Proposition 2 *If U_1 and U_2 are compact 3–dimensional subsets of S^3, only the products of points $\mathcal{U}_1 \in \partial U_1$, $\mathcal{U}_2 \in \partial U_2$ on their boundaries may generate points on the boundary $\partial(U_1 \otimes U_2)$ of their Minkowski product, i.e.,*

$$\partial(U_1 \otimes U_2) \;\subseteq\; \partial U_1 \otimes \partial U_2\,.$$

Proof : For $\mathcal{U}_1 \in U_1$ and $\mathcal{U}_2 \in U_2$, each interior point $\mathcal{U} = \mathcal{U}_1\mathcal{U}_2$ of $U_1 \otimes U_2$ has a neighborhood completely contained within $U_1 \otimes U_2$. On the other hand, every neighborhood of each boundary point $\mathcal{U} = \mathcal{U}_1\mathcal{U}_2$ contains points that belong to $U_1 \otimes U_2$ and points that do not. Consider neighborhoods N_1, N_2 of the points $\mathcal{U}_1, \mathcal{U}_2$ specified by spherical sets $U(\mathcal{U}_1, \epsilon_1), U(\mathcal{U}_2, \epsilon_2)$ with centers $\mathcal{U}_1, \mathcal{U}_2$ and radii ϵ_1, ϵ_2. Then the Minkowski product $N = N_1 \otimes N_2$ of these neighborhoods is simply the spherical set $U(\mathcal{U}_1 \mathcal{U}_2, \epsilon_1 + \epsilon_2)$.

(a) Suppose that $\mathcal{U}_1 \mathcal{U}_2 \in \partial(U_1 \otimes U_2)$ when $\mathcal{U}_1, \mathcal{U}_2$ are interior points of U_1, U_2. Then sufficiently small ϵ_1, ϵ_2 can be chosen, such that N_1, N_2 are interior to U_1, U_2 and therefore $N = N_1 \otimes N_2$ is a subset of $U_1 \otimes U_2$. But this contradicts the fact that every neighborhood of each point on $\partial(U_1 \otimes U_2)$ contains points that do not belong to $U_1 \otimes U_2$. Hence, the supposition that $\mathcal{U}_1 \mathcal{U}_2$ can belong to $\partial(U_1 \otimes U_2)$ when $\mathcal{U}_1, \mathcal{U}_2$ are interior points of U_1, U_2 must be false.

(b) Suppose that $\mathcal{U}_1 \mathcal{U}_2 \in \partial(U_1 \otimes U_2)$ when $\mathcal{U}_1$ is a boundary point of U_1, and $\mathcal{U}_2$ is an interior point of U_2. In this case, we choose $\epsilon_1 = 0$ and ϵ_2 sufficiently small, such that $N_1 = \{ \mathcal{U}_1 \}$ and N_2 is interior to U_2. Then $N = \{ \mathcal{U}_1 \} \otimes N_2$ is a non–degenerate neighborhood of $\mathcal{U}_1 \mathcal{U}_2$, and clearly $N \subset U_1 \otimes U_2$. Again, this contradicts the fact that every neighborhood of each point on $\partial(U_1 \otimes U_2)$ contains points that do not belong to $U_1 \otimes U_2$, so the supposition that $\mathcal{U}_1 \mathcal{U}_2$ can belong to $\partial(U_1 \otimes U_2)$ when $\mathcal{U}_1$ is a boundary point of U_1 and $\mathcal{U}_2$ is an interior point of U_2 must be false. An analogous argument holds when $\mathcal{U}_1$ is an interior point of U_1 and $\mathcal{U}_2$ is boundary point of U_2.

In view of the above arguments, a product $\mathcal{U}_1 \mathcal{U}_2$ can generate a point on the Minkowski sum boundary $\partial(U_1 \otimes U_2)$ only when $\mathcal{U}_1 \in \partial U_1$ and $\mathcal{U}_2 \in \partial U_2$. ■

Remark 1 The requirement that $\mathcal{U}_1 \in \partial U_1$ and $\mathcal{U}_2 \in U_2$ is a *necessary* but not *sufficient* condition for $\mathcal{U}_1 \mathcal{U}_2 \in \partial(U_1 \otimes U_2)$. In general, many products of points from ∂U_1 and ∂U_2 will generate interior points of $U_1 \otimes U_2$, and it is a non–trivial task to identify only those pairs of boundary points such that $\mathcal{U}_1 \mathcal{U}_2 \in \partial(U_1 \otimes U_2)$. In fact, $U_1 \otimes U_2$ may cover all of S^3 (and thus have no boundary) even in cases where U_1 and U_2 are proper subsets of S^3.

4 Stereographic projection to $\mathbb{R}^3$

The set of all unit quaternions occupies the 3–sphere S^3 in $\mathbb{R}^4$. To visualize S^3 a *stereographic projection* can be used to map it into $\mathbb{R}^3$, just as points on the 2–sphere can be imaged onto $\mathbb{R}^2$ to generate a map of the earth's surface. We recall the following definition.

Definition 1 *Consider the conformal map* $\Psi : \mathbb{H} \setminus \{1\} \to \mathbb{H} \setminus \{-1\}$ *defined by*

$$\Psi(\mathcal{Q}) := (1 - \mathcal{Q})^{-1}(1 + \mathcal{Q}) = (1 + \mathcal{Q})(1 - \mathcal{Q})^{-1} \,,$$

and its inverse

$$\Phi(\mathcal{Q}) := (\mathcal{Q} + 1)^{-1}(\mathcal{Q} - 1) = (\mathcal{Q} - 1)(\mathcal{Q} + 1)^{-1} \,.$$

The maps Ψ *and* Φ*, or rather their continuous extensions to the Alexandroff compactification* $\widehat{\mathbb{H}} := \mathbb{H} \cup \{\infty\}$*, are called* quaternionic Cayley transformations*.*

As conformal maps, Ψ and Φ map n–spheres and n–spaces to n–spheres and n–spaces for $0 \leq n \leq 3$. In particular, the next result concerns the unit 3-sphere in $\mathbb{H}$, denoted S^3, and the 3–space of purely imaginary quaternions, denoted $\mathbb{R}^3$.

Proposition 3 *The stereographic projection from the point* -1 *of* S^3 *to* $\mathbb{R}^3$*, defined by*

$$(u, \mathbf{u}) \mapsto \frac{\mathbf{u}}{1+u} \,, \tag{6}$$

is the restriction of Φ *to* S^3*.*

Proof : By direct computation, for all $\mathcal{Q} = (q, \mathbf{q}) \in \mathbb{H} \setminus \{-1\}$ we have

$$\begin{aligned} \Phi(\mathcal{Q}) &:= (|\mathcal{Q}|^2 + 2q + 1)^{-1}(\mathcal{Q}^* + 1)(\mathcal{Q} - 1) \\ &= (|\mathcal{Q}|^2 + 2q + 1)^{-1}(|\mathcal{Q}|^2 + 2\,\mathbf{q} - 1) \,. \end{aligned} \tag{7}$$

Hence, writing $\mathcal{Q} = \mathcal{U} = (u, \mathbf{u})$ when $|\mathcal{Q}| = 1$, we obtain

$$\Phi(\mathcal{U}) = \frac{2\,\mathbf{u}}{2 + 2u} = \frac{\mathbf{u}}{1+u} \,. \quad \blacksquare$$

The set of unit quaternions $\mathcal{U} = (u, u_x, u_y, u_z) \in S^3$ can be parameterized in terms of *hyperspherical coordinates* (α, β, γ) through the expression

$$(u, u_x, u_y, u_z) \;=\; (\cos\alpha, \sin\alpha\cos\beta, \sin\alpha\sin\beta\cos\gamma, \sin\alpha\sin\beta\sin\gamma) \,, \tag{8}$$

where $\alpha, \beta \in [\,0, \pi\,]$ and $\gamma \in [\,0, 2\pi\,]$. In terms of the scalar–vector form (3) with $\mathbf{n} = (n_x, n_y, n_z)$ we have

$$\theta \;=\; 2\alpha \,, \quad n_x \;=\; \cos\beta \,, \quad n_y \;=\; \sin\beta\cos\gamma \,, \quad n_z \;=\; \sin\beta\sin\gamma \,,$$

and conversely

$$\alpha \,=\, \tfrac{1}{2}\theta\,, \quad \beta \,=\, \arccos n_x\,, \quad \gamma \,=\, \arctan(n_y,n_z)\,,$$

where $\arctan^(a,b)$ is the angle with cosine $a/\sqrt{a^2+b^2}$ and sine $b/\sqrt{a^2+b^2}$.

For each unit quaternion $\mathcal{U}\in S^3$, the point $(x,y,z)=\Phi(\mathcal{U})\in\mathbb{R}^3$ defined by the stereographic projection (6) may be identified as the intersection with $\mathbb{R}^3$ of the line in $\mathbb{R}^4$ that passes through $(-1,0,0,0)$ and $\mathcal{U}$. In terms of the hyperspherical coordinates (8) on S^3, this point becomes

$$(x,y,z) \,=\, \tan\tfrac{1}{2}\alpha\,(\cos\beta,\sin\beta\cos\gamma,\sin\beta\sin\gamma)\,. \tag{9}$$

Thus (x,y,z) may be interpreted as the point with the "ordinary" spherical coordinates (β,γ) on the 2–sphere in $\mathbb{R}^3$ with radius $r=\tan\frac{1}{2}\alpha$. The point $\mathcal{U}=(1,0,0,0)$ corresponding to $\alpha=0$ is mapped to the origin of $\mathbb{R}^3$ and the point $\mathcal{U}=(-1,0,0,0)$ corresponding to $\alpha=\pi$ is mapped to infinity.

In terms of the scalar–vector form (3) of $\mathcal{U}$, the stereographic projection to $\mathbb{R}^3$ becomes

$$(x,y,z) \,=\, \tan\tfrac{1}{4}\theta\,(n_x,n_y,n_z)\,, \tag{10}$$

i.e., $\mathcal{U}$ is mapped to the point identified by the unit vector $\mathbf{n}=(n_x,n_y,n_z)$ on the 2–sphere of radius $r=\tan\frac{1}{4}\theta$ in $\mathbb{R}^3$. Note that, although the quaternion $-\mathcal{U}=(-\cos\frac{1}{2}\theta,-\sin\frac{1}{2}\theta\,\mathbf{n})$ specifies a rotation by angle $2\pi-\theta$ about $-\mathbf{n}$, equivalent to that specified by $\mathcal{U}$, it maps to the *distinct* point

$$(x,y,z) \,=\, -\cot\tfrac{1}{4}\theta\,(n_x,n_y,n_z)\,.$$

5 Spherical unit quaternion sets

In the Minkowski algebra of complex sets [7], emphasis was placed on circular disks as set operands, and it seems natural to extend this to the context of unit quaternion sets. We begin by characterizing the subset of S^3 defined by intersecting it with a 4–ball that has a specified radius ρ and center $\mathcal{U}_0$.

Remark 2 *For $\mathcal{U}_0\in S^3$ the intersection of S^3 with the ball of radius ρ and center $\mathcal{U}_0$ in $\mathbb{H}$ is identical to its intersection with the half–space orthogonal to $\mathcal{U}_0$, namely*

$$\{\,\mathcal{U}\in S^3 : |\,\mathcal{U}-\mathcal{U}_0| \le \rho\,\} = \{\,\mathcal{U}\in S^3 : \langle\,\mathcal{U},\mathcal{U}_0\rangle \ge 1-\tfrac{1}{2}\rho^2\,\}\,.$$

For $\rho\ge 2$ this set coincides with S^3; for $0<\rho<2$ it is a proper subset of S^3 whose boundary is a 2–sphere; and for $\rho=0$ it is the singleton set $\{\mathcal{U}_0\}$.

Based on the preceding remark, we set $\rho = 2\sin\frac{1}{2}t$ with $t \in [\,0,\pi\,]$ so that $1 - \frac{1}{2}\rho^2 = \cos t$, and formulate spherical unit quaternion sets as follows.

Definition 2 *For $\mathcal{U}_0 \in S^3$ and $t \in [0,\pi]$, we define*

$$U(\mathcal{U}_0,t) := \{\, \mathcal{U} \in S^3 : \langle \mathcal{U},\mathcal{U}_0 \rangle \geq \cos t \,\}\,.$$

Thus, considering the quaternions $\mathcal{U} \in S^3$ as unit vectors in $\mathbb{R}^4$, we can also interpret $U(\mathcal{U}_0,t)$ as the cone of unit vectors whose inclinations with $\mathcal{U}_0$ do not exceed $t = 2\arcsin\frac{1}{2}\rho$. Note that, if $\mathcal{U}_0 = 1$, then $\langle \mathcal{U},\mathcal{U}_0 \rangle$ is just the scalar part of $\mathcal{U}$, which motivates the following remark.

Remark 3 *Setting* $\exp(s\,\mathbf{n}) = (\cos s, \sin s\,\mathbf{n})$*, we have*

$$U(1,t) = \{\, \exp(s\,\mathbf{n}) : 0 \leq s \leq t, |\mathbf{n}| = 1 \,\}\,.$$

We now consider the stereographic projection of the set $U(\mathcal{U}_0,t)$ onto $\mathbb{R}^3$.

Proposition 4 *Let $\mathcal{U}_0 \in S^3$ and $t \in (0,\pi)$.*

1. *If $-1 \notin U(\mathcal{U}_0,t)$ then $\Phi(U(\mathcal{U}_0,t))$ is a 3–ball in $\mathbb{R}^3$.*

2. *If $-1 \in \partial U(\mathcal{U}_0,t)$ then $\Phi(U(\mathcal{U}_0,t) \setminus \{-1\})$ is a closed half–space in $\mathbb{R}^3$.*

3. *If $-1 \in U(\mathcal{U}_0,t) \setminus \partial U(\mathcal{U}_0,t)$ then $\Phi(U(\mathcal{U}_0,t) \setminus \{-1\})$ is $\mathbb{R}^3$ minus an open 3–ball.*

Finally, for $t = 0$ the image of $U(\mathcal{U}_0,t)$ through Φ is $\{\Phi(\mathcal{U}_0)\}$, and for $t = \pi$ it is $\mathbb{R}^3$.

Proof : Since the statements for the cases $t = 0$ and π are trivial, we focus on the case $t \in (0,\pi)$. The set $U(\mathcal{U}_0,t)$ is the intersection of S^3 with a 4–ball B centered at $\mathcal{U}_0$ in $\mathbb{H}$. Thus,

$$\Phi(U(\mathcal{U}_0,t)) = \Phi(S^3) \cap \Phi(B) = \mathbb{R}^3 \cap \Phi(B)\,.$$

where $\Phi(B)$ is a closed subset of $\widehat{\mathbb{H}}$ that includes $\Phi(\mathcal{U}_0) \in \mathbb{R}^3 \cup \{\infty\}$ as an interior point. The proposition then follows from the following observations:

1. If $-1 \notin U(\mathcal{U}_0,t)$ then $-1 \notin B$ and $\Phi(B)$ is a closed 3–ball in $\mathbb{R}^3$.

2. If $-1 \in \partial U(\mathcal{U}_0,t)$ then $-1 \in \partial B$ and $\Phi(B \setminus \{-1\})$ is a closed half–space in $\mathbb{R}^3$.

3. If $-1 \in U(\mathcal{U}_0,t) \setminus \partial U(\mathcal{U}_0,t)$ then -1 is an interior point of B and $\Phi(B \setminus \{-1\})$ is $\mathbb{H}$ minus an open 4-ball (??? need to check this). ∎

6 Visualization of spherical sets

A spherical unit quaternion set $U(\mathcal{U}_0, t)$ is characterized by a center $\mathcal{U}_0$ and angle $t \in [\,0, \pi\,]$, which specifies its radius as $\rho = 2\sin\frac{1}{2}t \in [\,0, 2\,]$. Such sets are difficult to visualize, since they correspond to subsets of the 3–sphere S^3 in $\mathbb{R}^4$. To clarify the nature of these sets, we describe here some approaches to visualizing their boundaries as 2–surfaces in Euclidean spaces.

Setting $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$ with $\theta \in [\,-\pi, \pi\,]$, elements of the spherical set $U(\mathcal{U}_0, t)$ with center $\mathcal{U}_0 = (\cos\frac{1}{2}\theta_0, \sin\frac{1}{2}\theta_0\,\mathbf{n}_0)$ and radius $\rho = 2\sin\frac{1}{2}\,t$ are characterized by the condition

$$\cos\tfrac{1}{2}\theta_0\cos\tfrac{1}{2}\theta + \sin\tfrac{1}{2}\theta_0\sin\tfrac{1}{2}\theta\,\cos\psi \;\ge\; \cos t\,, \tag{11}$$

where ψ is the angle between $\mathbf{n}$ and $\mathbf{n}_0$. The condition (11) imposes mutual bounds on the rotation angle θ and inclination ψ of $\mathbf{n}$ relative to $\mathbf{n}_0$. Writing $x = \tan\frac{1}{4}\theta \in [\,-1, 1\,]$ and $y = \cos\psi \in [\,-1, 1\,]$ it can be formulated as

$$(\cos t + \cos\tfrac{1}{2}\theta_0)\,x^2 - 2\sin\tfrac{1}{2}\theta_0\,xy + \cos t - \cos\tfrac{1}{2}\theta_0 \;\le\; 0\,. \tag{12}$$

This inequality identifies a subset of the domain $(x, y) \in [\,1, +1\,]^2$ bounded by the two branches of a hyperbola as the pairs of values (θ, ψ) that identify elements of $U(\mathcal{U}_0, t)$. The hyperbola asymptotes have inclinations

$$\arctan\frac{\cos t + \cos\frac{1}{2}\theta_0}{2\sin\frac{1}{2}\theta_0} \qquad \text{and} \qquad \frac{\pi}{2}\,.$$

relative to the x–axis. Figure 1 illustrates this visualization of the spherical set defined by the inequality (12) with $\theta_0 = \frac{1}{2}\,\pi$ and $t < \frac{1}{2}\theta_0$, $t = \frac{1}{2}\theta_0$, $t > \frac{1}{2}\theta_0$.

The boundary of the set $U(\mathcal{U}_0, t)$ is identified by the satisfaction of (11) with equality. In that case, we have

$$\cos\psi \;=\; f(\theta) \;:=\; \frac{\cos t - \cos\frac{1}{2}\theta_0\cos\frac{1}{2}\theta}{\sin\frac{1}{2}\theta_0\sin\frac{1}{2}\theta}\,,$$

but $f(\theta)$ is only defined on a subset of the domain $[\,-\pi, \pi\,]$, since we must have $|f(\theta)| \le 1$. This condition can be formulated as the quadratic inequality

$$g(\theta) \;:=\; \cos^2\tfrac{1}{2}\theta - 2\cos t\cos\tfrac{1}{2}\theta_0\cos\tfrac{1}{2}\theta + \cos^2 t + \cos^2\tfrac{1}{2}\theta_0 - 1 \;\le\; 0 \tag{13}$$

in $\cos\frac{1}{2}\theta$. The function $g(\theta)$ has the factorization

$$g(\theta) \;=\; [\,\cos\tfrac{1}{2}\theta - \cos(\tfrac{1}{2}\theta_0 - t)\,]\,[\,\cos\tfrac{1}{2}\theta - \cos(\tfrac{1}{2}\theta_0 + t)\,]\,,$$

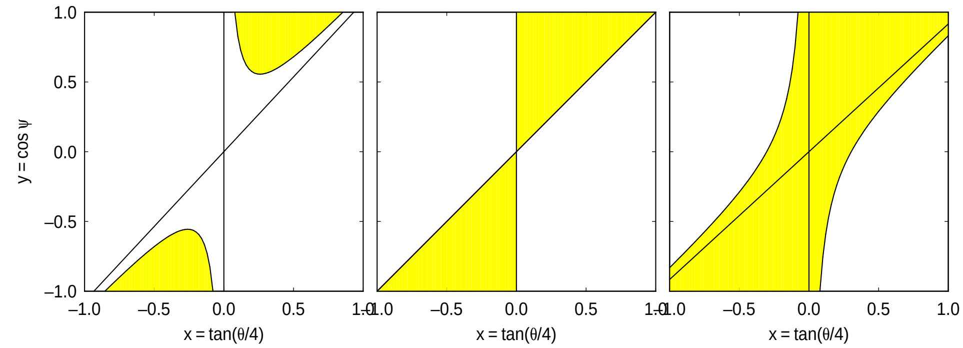

Figure 1: Two–dimensional visualization of a spherical set of unit quaternions $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$ with center $\mathcal{U}_0 = (\cos\frac{1}{2}\theta_0, \sin\frac{1}{2}\theta_0\mathbf{n}_0)$ and radius $\rho = 2\sin\frac{1}{2}t$, in terms of the variables $x = \tan\frac{1}{4}\,\theta$, $y = \mathbf{n}\cdot\mathbf{n}_0 = \cos\psi$ for $\theta_0 = \frac{1}{2}\,\pi$ and $t < \frac{1}{2}\theta_0$ (left), $t = \frac{1}{2}\theta_0$ (center), $t > \frac{1}{2}\theta_0$ (right).

and hence it has the roots $\cos\frac{1}{2}\theta = \cos(\frac{1}{2}\theta_0 \pm t)$, which identify the values

$$\theta \;=\; \pm(\theta_0 - 2t) + 4\pi k\,, \quad \theta \;=\; \pm(\theta_0 + 2t) + 4\pi l\,,$$

where k, l are integers. Since $\theta_0 \in [\,-\pi, \pi\,]$ and $t \in [\,0, \pi\,]$, we have $\theta_0 - 2t \in [\,-3\pi, \pi\,]$ and $\theta_0 + 2t \in [\,-\pi, 3\pi\,]$, and we set

$$\theta_1 \;=\; \begin{cases} \theta_0 - 2t & \text{if } \theta_0 - 2t > -\pi, \\ \theta_0 - 2t + 4\pi & \text{if } \theta_0 - 2t < -\pi, \end{cases}$$

$$\theta_2 \;=\; \begin{cases} \theta_0 + 2t & \text{if } \theta_0 + 2t < \pi, \\ \theta_0 + 2t - 4\pi & \text{if } \theta_0 + 2t > \pi, \end{cases}$$

and define

$$\theta_{\min} = \max(\min(|\theta_1|, |\theta_2|), -\pi)\,, \quad \theta_{\max} = \min(\max(|\theta_1|, |\theta_2|), \pi)\,.$$

Then the domain of $f(\theta)$ is the union of the two intervals

$$[\,-\theta_{\max}, -\theta_{\min}\,] \quad \text{and} \quad [\,\theta_{\max}, \theta_{\min}\,]\,.$$

If $t < \frac{1}{2}\pi$, these intervals are disjoint and have non–zero width, except in the case $\theta_0 = 0$ when they collapse, since $\theta_{\min} = \theta_{\max}$. However, if $t > 0.5\pi$, they are of non–zero width only when $|\theta_0| > |\pi - 2t|$, in which case $\theta_{\max} = \pi$ and by identifying $-\pi$ with π, they may be regarded as a continuous interval.

Now an orthonormal basis $(\mathbf{m}_0, \mathbf{n}_0, \mathbf{m}_0 \times \mathbf{n}_0)$ for $\mathbb{R}^3$ can be constructed from any unit vector $\mathbf{m}_0$ orthogonal to $\mathbf{n}_0$, and using this basis we can express $\mathbf{n}$ in terms of θ and another angular parameter $\phi \in [\,0, 2\pi\,]$ as

$$\mathbf{n}(\theta,\phi) \,=\, f(\theta)\,\mathbf{n}_0 \,+\, \sqrt{1-f^2(\theta)}\,(\mathbf{m}_0\cos\phi + \mathbf{m}_0\times\mathbf{n}_0\sin\phi)\,,$$

where the domain of θ is as described above. The boundary of the spherical set with center $\mathcal{U}_0 = (\cos\frac{1}{2}\theta_0, \sin\frac{1}{2}\theta_0\,\mathbf{n}_0)$ and radius $\rho = 2\sin\frac{1}{2}t$ is therefore the two–parameter family of unit quaternions defined by

$$\mathcal{U}(\theta,\phi) \,=\, (\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta\,\mathbf{n}(\theta,\phi))\,.$$

Using (10), stereographic projection of this set yields the parametric surface in $\mathbb{R}^3$ defined by

$$\mathbf{r}(\theta,\phi) \,=\, \tan\tfrac{1}{4}\theta\;\mathbf{n}(\theta,\phi)\,. \tag{14}$$

This defines a "ringed surface" generated by a one–parameter family of circles — these circles all lie in planes orthogonal to the line through the origin of $\mathbb{R}^3$ in the direction of the vector $\mathbf{n}_0$, with centers $\mathbf{c}(\theta) = \tan\frac{1}{4}\theta\,f(\theta)\,\mathbf{n}_0$ and radii $r(\theta) = \tan\frac{1}{4}\theta\sqrt{1-f^2(\theta)}$, with the domain of θ as described above.

The simple nature of sets of the form $U(1,t)$ introduced in Remark 2 may be elucidated as follows. For such sets, the condition $|\mathcal{U} - 1| \le \rho = 2\sin\frac{1}{2}t$ with $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$, $\theta \in [\,-\pi,\pi\,]$ reduces to

$$\cos\tfrac{1}{2}\theta \,\ge\, 1 - \tfrac{1}{2}\rho^2 \,=\, \cos t\,. \tag{15}$$

This condition depends only on the rotation angle θ, not on the axis $\mathbf{n}$. As noted in Remark 3, $\mathcal{U}(1,t) = \{1\}$ if $t = 0$ ($\rho = 0$), and $\mathcal{U}(1,t) = S^3$ if $t = \pi$ ($\rho = 2$). In the case $t = \frac{1}{2}\pi$ ($\rho = \sqrt{2}$), it is the set of all unit quaternions with any rotation axis $\mathbf{n}$ and rotation angles $\theta \in [\,-\frac{1}{2}\pi, \frac{1}{2}\pi\,]$. The set boundary $\partial U(1,t)$ corresponds to satisfaction of (15) with equality, i.e., $\theta = \pm 2t$.

7 Products of unit quaternion sets

The Minkowski product of unit quaternion sets $U, V \subseteq S^3$ is defined by

$$U \otimes V \,=\, \{\, \mathcal{U}\mathcal{V} : \mathcal{U} \in U\,,\, \mathcal{V} \in V\,\}\,.$$

The elements of the set $U \otimes V$ describe all possible compounded rotations generated by a rotation $\mathcal{V} \in V$ followed by a rotation $\mathcal{U} \in U$.

Remark 4 *As a consequence of the properties of quaternionic multiplication, $\otimes$ is an associative but noncommutative operation on the power set of S^3.*

The following remark will prove useful later.

Remark 5 *Let*

$$\begin{aligned} T : \mathbb{H} \times \mathbb{H} &\to \mathbb{H} \\ (\mathcal{P}, \mathcal{Q}) &\mapsto \mathcal{P}\mathcal{Q} \,. \end{aligned}$$

Then for all $\mathcal{V} \in \mathbb{H}$ the partial derivatives of T in the directions $(\mathcal{V}, 0)$ and $(0, \mathcal{V})$ are

$$\frac{\partial T}{\partial(\mathcal{V}, 0)}(\mathcal{P}, \mathcal{Q}) = \mathcal{V}\mathcal{Q} \quad \textit{and} \quad \frac{\partial T}{\partial(0, \mathcal{V})}(\mathcal{P}, \mathcal{Q}) = \mathcal{P}\mathcal{V} \,.$$

Ideally, for given compact sets $U, V \subset S^3$, we would like to obtain either (i) a "faithful" (one–to-one) parameterization, over a suitable domain in three parameters, of the product set $U \otimes V \subset S^3$; or (ii) a characterization of its boundary $\partial(U \otimes V)$, which amounts to a two–dimensional surface embedded in S^3. Problem (ii) is, in general, more tractable: since $\partial(U \otimes V) \subseteq \partial U \otimes \partial V$, it amounts to identifying corresponding point pairs $\mathcal{U} \in \partial U$ and $\mathcal{V} \in \partial V$ that generate (potential) points on the Minkowski product boundary $\partial(U \otimes V)$.

8 Products of spherical unit quaternion sets

We now compute the product $U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t)$. To this end, the following remark will be useful.

Remark 6 *For all $\mathcal{U}_0, \mathcal{V}_0 \in S^3$ and for all $t \in [\,0, \pi\,]$, we have*

$$U(\mathcal{U}_0, t) \otimes \{\mathcal{V}_0\} = U(\mathcal{U}_0\mathcal{V}_0, t) = \{\mathcal{U}_0\} \otimes U(\mathcal{V}_0, t),$$

since $\langle \mathcal{U}\mathcal{V}_0^*, \mathcal{U}_0 \rangle = \langle \mathcal{U}, \mathcal{U}_0\mathcal{V}_0 \rangle = \langle \mathcal{U}_0^*\mathcal{U}, \mathcal{V}_0 \rangle$.

Before considering the general product $U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t)$, it is instructive to consider a special case.

Lemma 1 *Let $s, t \in [\,0, \pi\,]$. Then*

$$U(1, s) \otimes U(1, t) = \begin{cases} U(1, s+t) & \text{if } s+t \in [\,0, \pi\,], \\ S^3 & \text{if } s+t \in [\,\pi, 2\pi\,]. \end{cases}$$

Proof : By Remark 3, $U(1, s) \otimes U(1, t)$ is the set of all products of the form $\exp(a\,\mathbf{m})\,\exp(b\,\mathbf{n})$ with $|\mathbf{m}| = |\mathbf{n}| = 1$, $0 \le a \le s$, $0 \le b \le t$. Now, the scalar part of $\exp(a\,\mathbf{m})\,\exp(b\,\mathbf{n})$ is equal to

$$\cos a \cos b - \sin a \sin b \; \mathbf{m} \cdot \mathbf{n},$$

which is greater than or equal to $\cos a \cos b - \sin a \sin b = \cos(a+b)$. If $s+t \in [\,0, \pi\,]$ this bound implies that $U(1, s) \otimes U(1, t) \subseteq U(1, s+t)$. If $s+t \in [\,\pi, 2\pi\,]$, the bound only implies the trivial inclusion $U(1, s) \otimes U(1, t) \subseteq S^3$.

On the other hand, let $\mathcal{U} = (u, \mathbf{u}) \in S^3$. If we can identify real numbers a, b with $0 \le a \le s$ and $0 \le b \le t$ such that $u = \cos(a+b)$, then $\mathcal{U} = \exp((a+b)\,\mathbf{p})$ for a suitably chosen $\mathbf{p}$ with $|\mathbf{p}| = 1$, and we conclude that $\mathcal{U} = \exp(a\,\mathbf{p})\exp(b\,\mathbf{p}) \in U(1, s) \otimes U(1, t)$. If $s+t \in [\,0, \pi\,]$, then such a and b exist when $u \ge \cos(s+t)$. If $s+t \in [\,\pi, 2\pi\,]$, then they exist for all $u \ge -1$. This proves that $U(1, s) \otimes U(1, t) \supseteq U(1, s+t)$ in the former case, and that $U(1, s) \otimes U(1, t) \supseteq S^3$ in the latter case. ∎

We are now ready to present the general result for the Minkowski products of spherical unit quaternion sets.

Theorem 1 *Let $\mathcal{U}_0, \mathcal{V}_0 \in S^3$ and $s, t \in [\,0, \pi\,]$. Then*

$$U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t) = \begin{cases} U(\mathcal{U}_0\mathcal{V}_0, s+t) & \text{if } s+t \in [\,0, \pi\,], \\ S^3 & \text{if } s+t \in [\,\pi, 2\pi\,]. \end{cases}$$

Proof : By Remark 6, $U(\mathcal{U}_0, s) = \{\mathcal{U}_0\} \otimes U(1, s)$ and $U(\mathcal{V}_0, t) = U(1, t) \otimes \{\mathcal{V}_0\}$. Taking into account Remark 4, we can write

$$U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t) = \{\mathcal{U}_0\} \otimes U(1, s) \otimes U(1, t) \otimes \{\mathcal{V}_0\}\,.$$

We now apply Lemma 1. If $s+t \in [\,0, \pi\,]$ then

$$U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t) = \{\mathcal{U}_0\} \otimes U(1, s+t) \otimes \{\mathcal{V}_0\}\,,$$

whence

$$U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t) = U(\mathcal{U}_0\mathcal{V}_0, s+t)$$

by two further applications of Remark 6. On the other hand, if $s+t \in [\,\pi, 2\pi\,]$, then

$$U(\mathcal{U}_0, s) \otimes U(\mathcal{V}_0, t) = \{\mathcal{U}_0\} \otimes S^3 \otimes \{\mathcal{V}_0\} = S^3\,. \quad \blacksquare$$

9 Bounded rotation angles and axes

Although spherical unit quaternion sets admit a simple and elegant theory for their Minkowski products, they correspond to rather complicated and non–intuitive relations among the feasible rotation axes and angles. We consider now different types of sets, of greater relevance to the physical actuators used in robot manipulators, 5–axis milling machines, and related contexts.

Specifically, we shall analyze the Minkowski products of sets defined by (1) fixed rotation axes, and rotation angles that vary over prescribed subsets of $[\,-\pi, \pi\,]$; and (2) fixed rotation angles, and rotation axes that deviate from prescribed directions by no more than a given angle. In case (1) the operand sets are curves, and their product is a 2–surface, in S^3; while in case (2) the sets are 2–surfaces, and their product is of full dimension, in S^3.

9.1 Fixed rotation axis, bounded angle

Let us consider the following sets.

Definition 3 *For each quaternionic imaginary unit* $\mathbf{c}$*, for all* $\phi \in \mathbb{R}$*, and for all* $\delta \in [\,0, \pi\,]$*,*

$$C(\mathbf{c}, \phi, \delta) := \{\, \exp(s\,\mathbf{c}) : |s - \phi| \le \delta \,\}\,.$$

$C(\mathbf{c}, \phi, \pi)$ is the great circle in S^3 that passes through 1 and $\exp(\phi\,\mathbf{c})$, and for all $\delta \in [\,0, \pi)$ the set $C(\mathbf{c}, \phi, \delta)$ is an arc of this great circle.

Remark 7 *For each quaternionic imaginary unit* $\mathbf{c}$*, for all* $\phi \in \mathbb{R}$*, and for all* $\delta \in [\,0, \pi\,]$*,*

$$C(\mathbf{c}, 0, \delta) \otimes \{\exp(\phi\,\mathbf{c})\} = C(\mathbf{c}, \phi, \delta) = \{\exp(\phi\,\mathbf{c})\} \otimes C(\mathbf{c}, 0, \delta)\,.$$

Theorem 2 *Fix a quaternionic imaginary unit* $\mathbf{c}_1$*, angles* $\phi_1, \phi_2 \in \mathbb{R}$*, and errors* $\delta_1, \delta_2 \in [\,0, \pi\,]$*. Then the product* $C(\mathbf{c}_1, \phi_1, \delta_1) \otimes C(\mathbf{c}_1, \phi_2, \delta_2)$ *is a circle or a circular arc:*

$$C(\mathbf{c}_1, \phi_1, \delta_1) \otimes C(\mathbf{c}_1, \phi_2, \delta_2) = \begin{cases} C(\mathbf{c}_1, \phi_1 + \phi_2, \delta_1 + \delta_2) & \text{if } \delta_1 + \delta_2 \in [\,0, \pi\,], \\ C(\mathbf{c}_1, \phi_1 + \phi_2, \pi) & \text{if } \delta_1 + \delta_2 \in [\,\pi, 2\pi\,]. \end{cases}$$

For each quaternionic imaginary unit $\mathbf{c}_2 \neq \pm\mathbf{c}_1$*, the product* $C(\mathbf{c}_1, \phi_1, \delta_1) \otimes C(\mathbf{c}_2, \phi_2, \delta_2) \subset S^3$ *is an immersed* 2*–surface in* $\mathbb{H}$*, possibly with boundary. The least value of* η *for which it is included in* $U(\exp(\phi_1 \mathbf{c}_1) \exp(\phi_2 \mathbf{c}_2), \eta)$ *is* $\eta_0 := \arccos(r_0)$*, where*

$$r_0 := \min_{|s| \leq \delta_1, |t| \leq \delta_2} (\cos s \cos t - \sin s \sin t \, \langle \mathbf{c}_1, \mathbf{c}_2 \rangle) \,.$$

In particular, if $\delta_1, \delta_2 \in [\,0, \frac{1}{2}\pi\,]$ *then*

$$\eta_0 = \arccos\left(\cos \delta_1 \cos \delta_2 - \sin \delta_1 \sin \delta_2 |\langle \mathbf{c}_1, \mathbf{c}_2 \rangle|\right) < \delta_1 + \delta_2 \,.$$

Finally, if neither δ_1 *nor* δ_2 *equals* π *and at least one of them is less than* $\frac{1}{2}\pi$ *then* $C(\mathbf{c}_1, \phi_1, \delta_1) \otimes C(\mathbf{c}_2, \phi_2, \delta_2) \subset S^3$ *is an embedded* 2*–surface in* $\mathbb{H}$*, with boundary. Its boundary consists of four circular arcs which have pairwise intersections at the four points* $\exp((\phi_1 \pm \delta_1)\mathbf{c}_1) \exp((\phi_2 \pm \delta_2)\mathbf{c}_2)$*, two of which belong to* $\partial U(\exp(\phi_1 \mathbf{c}_1) \exp(\phi_2 \mathbf{c}_2), \eta_0)$*.*

Proof : By Remarks 6 and 7, it suffices to consider the case $\phi_1 = \phi_2 = 0$. The first statement is an immediate consequence of the fact that

$$\exp(s\,\mathbf{c}_1) \exp(t\,\mathbf{c}_1) = \exp((s+t)\mathbf{c}_1) \quad \text{for all } s, t \in \mathbb{R} \,.$$

We now focus on the second part of the theorem, under the hypothesis that $\mathbf{c}_2 \neq \pm\mathbf{c}_1$. Consider the surjective map

$$\begin{aligned} P : [-\delta_1, \delta_1] \times [-\delta_2, \delta_2] &\to C(\mathbf{c}_1, 0, \delta_1) \otimes C(\mathbf{c}_2, 0, \delta_2) \\ (s, t) &\mapsto \exp(s\,\mathbf{c}_1) \exp(t, \mathbf{c}_2) \,. \end{aligned}$$

Note that P is non–singular, since the s–derivative $\exp(s\,\mathbf{c}_1)\,\mathbf{c}_1 \exp(t\,\mathbf{c}_2)$ and t–derivative $\exp(s\,\mathbf{c}_1) \exp(t\,\mathbf{c}_2)\,\mathbf{c}_2 = \exp(s\,\mathbf{c}_1)\,\mathbf{c}_2 \exp(t\,\mathbf{c}_2)$ cannot be linearly dependent over $\mathbb{R}$: if they were, then $\mathbf{c}_1, \mathbf{c}_2$ would also be linearly dependent, contradicting the hypothesis $\mathbf{c}_2 \neq \pm\mathbf{c}_1$.

Now, let us determine for which η the inclusion

$$C(\mathbf{c}_1, 0, \delta_1) \otimes C(\mathbf{c}_2, 0, \delta_2) \subseteq U(1, \eta)$$

holds. The scalar part of the product $e^{s\mathbf{c}_1} e^{t\mathbf{c}_2}$ equals

$$\cos s \cos t - \sin s \sin t \, \langle \mathbf{c}_1, \mathbf{c}_2 \rangle \, .$$

This quantity spans the whole interval $[r_0, 1]$ with

$$r_0 := \min_{|s| \le \delta_1, |t| \le \delta_2} (\cos s \cos t - \sin s \sin t \, \langle \mathbf{c}_1, \mathbf{c}_2 \rangle) \, .$$

When $\delta_1, \delta_2 \in [0, \pi/2]$, the minimum is attained either at the two angles $(\pm\delta_1, \pm\delta_2)$ or at $(\pm\delta_1, \mp\delta_2)$ and it equals

$$r_0 = \cos \delta_1 \cos \delta_2 - \sin \delta_1 \sin \delta_2 |\langle \mathbf{c}_1, \mathbf{c}_2 \rangle| \, .$$

Moreover, P is an embedding if, and only if, P is injective. The equality $\exp(p\,\mathbf{c}_1) \exp(r\,\mathbf{c}_2) = \exp(s\,\mathbf{c}_1) \exp(t\,\mathbf{c}_2)$ holds if, and only if, $\exp((p-s)\mathbf{c}_1) = \exp((t-r)\mathbf{c}_2)$, i.e., $p-s, t-r \in \{-2\pi, 0, 2\pi\}$ or $p-s, t-r \in \{\pm\pi\}$. Thus, P is injective if, and only if, neither δ_1 nor δ_2 equals π and at least one of them is less than $\frac{1}{2}\pi$.

Finally, when P is an embedding, the boundary of $C(\mathbf{c}_1, 0, \delta_1) \otimes C(\mathbf{c}_2, 0, \delta_2)$ consists of the four circular arcs

$$[-\delta_1, \delta_1] \to C(\mathbf{c}_1, 0, \delta_1) \otimes C(\mathbf{c}_2, 0, \delta_2) \quad s \mapsto \exp(s\,\mathbf{c}_1) \exp(\pm\delta_2 \mathbf{c}_2) \, ,$$

$$[-\delta_2, \delta_2] \to C(\mathbf{c}_1, 0, \delta_1) \otimes C(\mathbf{c}_2, 0, \delta_2) \quad t \mapsto \exp(\pm\delta_1 \mathbf{c}_1) \exp(t\,\mathbf{c}_2) \, ,$$

as desired.

9.2 Bounded rotation axis, fixed angle

Let us consider the following sets.

Definition 4 *For each quaternionic imaginary unit* $\mathbf{c}$*, for all* $\phi \in (0, \pi)$ *and for all* $\xi \in [\,0, \pi\,]$*, we define*

$$S(\mathbf{c}, \phi, \xi) := \{\, \exp(\phi\,\mathbf{m}) = \cos \phi + \mathbf{m} \sin \phi : \langle \mathbf{m}, \mathbf{c} \rangle \ge \cos \xi \,\} \, .$$

$S(\mathbf{c},\phi,\pi)$ is the 2–sphere obtained by intersecting S^3 with the 3–space of quaternions whose scalar part is equal to $\cos\phi$. For all $\xi \in (0,\pi)$, the set $S(\mathbf{c},\phi,\xi)$ is a spherical cap of that 2–sphere, with boundary

$$\partial S(\mathbf{c},\phi,\xi) = \{\, \exp(\phi\,\mathbf{m}) = \cos\phi + \mathbf{m}\,\sin\phi : \langle \mathbf{m},\mathbf{c}\rangle = \cos\xi \,\}$$

in the 2-sphere. Finally, $S(\mathbf{c},\phi,0) = \{\, \exp(\phi\,\mathbf{c})\,\}$.

Proposition 5 *Choose quaternionic imaginary units* $\mathbf{c}_1, \mathbf{c}_2$ *and let* $\phi_1, \phi_2 \in (0,\pi)$ *and* $\xi_1, \xi_2 \in (0,\pi\,]$. *The rank of the real differential of the map*

$$\begin{aligned} &S(\mathbf{c}_1,\phi_1,\xi_1) \times S(\mathbf{c}_2,\phi_2,\xi_2) \to S^3 \\ &\qquad (\exp(\phi_1\mathbf{m}), \exp(\phi_2\mathbf{n})) \mapsto \exp(\phi_1\mathbf{m})\exp(\phi_2\mathbf{n}) \end{aligned}$$

at a point $(\exp(\phi_1\mathbf{m}), \exp(\phi_2\mathbf{n}))$ *is less than* 3 *if, and only if,* $\mathbf{m} = \pm\mathbf{n}$.

Proof : We denote the map by σ and fix a point $(\exp(\phi_1\mathbf{m}), \exp(\phi_2\mathbf{n})) \in S(\mathbf{c}_1,\phi_1,\xi_1) \times S(\mathbf{c}_2,\phi_2,\xi_2)$. By Remark 5, we have

$$\frac{\partial\sigma}{\partial(\mathbf{v},0)}(\exp(\phi_1\mathbf{m}), \exp(\phi_2\mathbf{n})) = \mathbf{v}\exp(\phi_2\mathbf{n})\,,$$

$$\frac{\partial\sigma}{\partial(0,\mathbf{w})}(\exp(\phi_1\mathbf{m}), \exp(\phi_2\mathbf{n})) = \exp(\phi_1\mathbf{m})\,\mathbf{w}\,,$$

for all $(\mathbf{v},0)$ and $(0,\mathbf{w})$ in the tangent space to $S(\mathbf{c}_1,\phi_1,\xi_1) \times S(\mathbf{c}_2,\phi_2,\xi_2)$ at the point $(\exp(\phi_1\mathbf{m}), \exp(\phi_2\mathbf{n}))$, i.e., for all purely imaginary quaternions with $\mathbf{v} \perp \mathbf{m}, \mathbf{w} \perp \mathbf{n}$. The span of all such $\mathbf{v}\exp(\phi_2\mathbf{n})$ is a 2–plane Π_1 through the origin, and the span of all such $\exp(\phi_1\mathbf{m})\,\mathbf{w}$ is a 2–plane Π_2 through the origin. The sum $\Pi_1 + \Pi_2$ has dimension less than 3 if, and only if, $\Pi_1 = \Pi_2$ — which is equivalent to $\{\mathbf{v} : \mathbf{v} \perp \mathbf{m}\} = \{\mathbf{w} : \mathbf{w} \perp \mathbf{n}\}$, i.e., $\mathbf{m} = \pm\mathbf{n}$.

Corollary 1 *Choose quaternionic imaginary units* $\mathbf{c}_1, \mathbf{c}_2$ *with* $\mathbf{c}_1 \neq \pm\mathbf{c}_2$ *and let* $\phi_1, \phi_2 \in (0,\pi)$. *If* $\xi_1, \xi_2 \in (0,\pi\,]$ *are small enough, then:*

1. *the map*

$$\begin{aligned} &\sigma : S(\mathbf{c}_1,\phi_1,\xi_1) \times S(\mathbf{c}_2,\phi_2,\xi_2) \to S^3 \\ &\qquad (\exp(\phi_1\mathbf{m}), \exp(\phi_2\mathbf{n})) \mapsto \exp(\phi_1\mathbf{m})\exp(\phi_2\mathbf{n}) \end{aligned}$$

   is a submersion;

2. *its image* $S(\mathbf{c}_1,\phi_1,\xi_1)\otimes S(\mathbf{c}_2,\phi_2,\xi_2)$ *is (the closure in* S^3 *of) an open subset of* S^3*;*

3. *the boundary of* $S(\mathbf{c}_1,\phi_1,\xi_1)\otimes S(\mathbf{c}_2,\phi_2,\xi_2)$ *is included in*

$$(\partial S(\mathbf{c}_1,\phi_1,\xi_1)\otimes S(\mathbf{c}_2,\phi_2,\xi_2))\cup(S(\mathbf{c}_1,\phi_1,\xi_1)\otimes\partial S(\mathbf{c}_2,\phi_2,\xi_2))\ ,$$

where the two members of the union intersect in the set $\partial S(\mathbf{c}_1,\phi_1,\xi_1)\otimes\partial S(\mathbf{c}_2,\phi_2,\xi_2)$*.*

We can obtain a rough estimate of which sets $U(\exp(\phi_1\mathbf{c}_1)\exp(\phi_2\mathbf{c}_2),t)$ contain $S(\mathbf{c}_1,\phi_1,\xi_1)\otimes S(\mathbf{c}_2,\phi_2,\xi_2)$ by first including each factor $S(\mathbf{c}_\ell,\phi_\ell,\xi_\ell)$ in a certain $U(\exp)(\phi_\ell\mathbf{c}_\ell),t_\ell)$ and then applying Theorem 1.

Remark 8 *The least value of* t *such that* $S(\mathbf{c},\phi,\xi)\subseteq U(\exp(\phi\,\mathbf{c}),t)$ *is*

$$t_0:=\arccos(\cos^2\phi+\sin^2\phi\cos\xi)$$

because

$$\langle\exp(\phi\,\mathbf{m}),\exp(\phi\,\mathbf{c})\rangle=\cos^2\phi+\sin^2\phi\,\langle\mathbf{m},\mathbf{c}\rangle$$

attains its minimum when $\langle\mathbf{m},\mathbf{c}\rangle=\cos\xi$*. The inequality* $t_0\leq\xi$ *holds and it is strict if, and only if,* $\phi\neq\frac{1}{2}\pi$*.*

Remark 9 *Let* $t_\ell:=\arccos(\cos^2\phi_\ell+\sin^2\phi_\ell\cos\xi_\ell)$*,* $\ell=1,2$*. By the previous remark and Theorem 1, if* $t_1+t_2\in[\,0,\pi\,]$ *then*

$$S(\mathbf{c}_1,\phi_1,\xi_1)\otimes S(\mathbf{c}_2,\phi_2,\xi_2)\subseteq U(\exp(\phi_1\mathbf{c}_1)\exp(\phi_2\mathbf{c}_2),t_1+t_2)\,.$$

In some cases, the estimate presented in Remark 9 is sharp.

Example 1 *For all quaternionic imaginary units* $\mathbf{c}$ *and for all* $\xi\in[\,0,\frac{1}{2}\pi\,]$*,*

$$S\left(\mathbf{c},\tfrac{1}{2}\pi,\xi\right)\otimes S\left(\mathbf{c},\tfrac{1}{2}\pi,\xi\right)\subseteq U(-1,2\xi)$$

by Remark 9. The product is not included in $U(-1,t)$ *for any* $t<2\xi$ *because we can consider two elements*

$$\cos(\xi)\,\mathbf{c}\pm\sin(\xi)\,\mathbf{v}\in S\left(\mathbf{c},\tfrac{1}{2}\pi,\xi\right)$$

with $\mathbf{v}$ *unitary and orthogonal to* $\mathbf{c}$*, and observe that their product belongs to* $\partial U(-1,2\xi)$*, since its scalar product with* -1 *is equal to* $\cos^2\xi-\sin^2\xi=\cos 2\xi$*.*

10 Lie algebra representation

As an alternative to stereographic projection, the Lie algebra $\mathfrak{so}(3)$ associated with the Lie group $SO(3)$ provides a more intuitive visualization in $\mathbb{R}^3$ of the Minkowski products of unit quaternion sets. In this algebra, spatial rotations are represented by *Euler vectors* of the form $\theta\,\mathbf{n}$, where θ is the rotation angle and the unit vector $\mathbf{n}$ defines the rotation axis. With $\theta \in [\,-\pi, \pi\,]$, any set of spatial rotations lies inside the sphere with center at the origin and radius π in $\mathbb{R}^3$, and we avoid the problem of finite points of S^3 being mapped to infinity associated with stereographic projection.

The product of two elements $\theta_1\mathbf{n}_1$ and $\theta_2\mathbf{n}_2$ of $\mathfrak{so}(3)$ is defined by the Baker–Campbell–Hausdorff (BCH) formula [10, 19]. We give a brief synopsis of this approach here, and illustrate its use in visualizing Minkowski products of unit quaternion sets through some computed examples.

A rotation in $\mathbb{R}^3$ through angle $\theta \in [\,-\pi, \pi\,]$ about an axis defined by a unit vector $\mathbf{n} = (n_x, n_y, n_z)$ can be specified by a 3×3 matrix $\mathbf{M}$ corresponding to an element of the Lie group $SO(3)$. The components of this matrix are

$$
\begin{aligned}
M_{11} &= n_x^2 + (1 - n_x^2)\cos\theta\,, \\
M_{12} &= n_x n_y(1 - \cos\theta) - n_z \sin\theta\,, \\
M_{13} &= n_z n_x(1 - \cos\theta) + n_y \sin\theta\,, \\
M_{21} &= n_x n_y(1 - \cos\theta) + n_z \sin\theta\,, \\
M_{22} &= n_y^2 + (1 - n_y^2)\cos\theta\,, \\
M_{23} &= n_y n_z(1 - \cos\theta) - n_x \sin\theta\,, \\
M_{31} &= n_z n_x(1 - \cos\theta) - n_y \sin\theta\,, \\
M_{32} &= n_y n_z(1 - \cos\theta) + n_x \sin\theta\,, \\
M_{33} &= n_z^2 + (1 - n_z^2)\cos\theta\,.
\end{aligned}
$$

With each $\mathbf{M}$ we may uniquely identify a skew–symmetric matrix

$$
\mathbf{A} := \tfrac{1}{2}(\mathbf{M} - \mathbf{M}^T) = \sin\theta\,\mathbf{N}\,, \quad \mathbf{N} := \begin{bmatrix} 0 & -n_z & n_y \\ n_z & 0 & -n_x \\ -n_y & n_x & 0 \end{bmatrix},
$$

and a corresponding vector

$$
\mathbf{a} := \sin\theta\,\mathbf{n}\,.
$$

The product of the matrix $\mathbf{N}$ with a column vector $\mathbf{v}$ yields the cross product of $\mathbf{n}$ with $\mathbf{v}$, i.e., $\mathbf{N}\,\mathbf{v} = \mathbf{n} \times \mathbf{v}$.

The element $\mathbf{X}$ of the Lie algebra $\mathfrak{so}(3)$ that corresponds to an element $\mathbf{M}$ of the Lie group SO(3) may be represented by the skew–symmetric matrix specified by the logarithmic map

$$\mathbf{X} \;=\; \log(\mathbf{M}) \;:=\; \frac{\arcsin(\|\mathbf{a}\|)}{\|\mathbf{a}\|}\,\mathbf{A}\,, \tag{16}$$

where the range of arcsin() is $[\,-\frac{1}{2}\pi, \frac{1}{2}\pi\,]$, and for $\theta \neq 0$ or $\pm\pi$ this gives

$$\mathbf{X} \;=\; \frac{\arcsin(|\sin\theta|)}{|\sin\theta|}\,\sin\theta\,\mathbf{N} \;=\; \theta\,\mathbf{N}\,.$$

An Euler (or rotation) vector $\mathbf{x} \in \mathbb{R}^3$ may be identified with each element $\mathbf{X}$ of the Lie algebra, through the relation

$$\mathbf{x} \;:=\; \theta\,\mathbf{n}\,.$$

Note that the Lie bracket or commutator

$$[\,\mathbf{X}_1, \mathbf{X}_2\,] \;:=\; \mathbf{X}_1\mathbf{X}_2 - \mathbf{X}_2\mathbf{X}_1$$

of two elements $\mathbf{X}_1 = \theta_1\,\mathbf{N}_1$ and $\mathbf{X}_2 = \theta_2\,\mathbf{N}_2$ of $\mathfrak{so}(3)$ then corresponds to the cross product

$$\mathbf{x}_1 \times \mathbf{x}_2 \;=\; \theta_1\theta_2\;\mathbf{n}_1 \times \mathbf{n}_2$$

of the corresponding Euler vectors $\mathbf{x}_1 = \theta_1\mathbf{n}_1$ and $\mathbf{x}_2 = \theta_2\mathbf{n}_2$ in $\mathbb{R}^3$.

The Lie group element $\mathbf{M} \in \mathrm{SO}(3)$ corresponding to a Lie algebra element $\mathbf{X} = \theta\,\mathbf{N} \in \mathfrak{so}(3)$ is obtained via the exponential map

$$\mathbf{M} \;=\; \exp(\theta\,\mathbf{N}) \;:=\; \mathbf{I} \,+\, \theta\,\mathbf{N} \,+\, \frac{\theta^2}{2!}\,\mathbf{N}^2 \,+\, \frac{\theta^3}{3!}\,\mathbf{N}^3 \,+\, \cdots\,,$$

and since for any $\mathbf{n}$ the matrix $\mathbf{N}$ satisfies $\mathbf{N}^3 = -\,\mathbf{N}$, this reduces to

$$\begin{aligned}\mathbf{M} \;&=\; \mathbf{I} + \left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \cdots\right)\mathbf{N} \,+\, \left(\frac{\theta^2}{2!} - \frac{\theta^4}{4!} + \frac{\theta^6}{6!} - \cdots\right)\mathbf{N}^2 \\ &=\; \mathbf{I} \,+\, \sin\theta\,\mathbf{N} \,+\, (1 - \cos\theta)\,\mathbf{N}^2\,. \end{aligned} \tag{17}$$

Let $\mathbf{M}_1, \mathbf{M}_2$ be matrices representing rotations through angles θ_1, θ_2 about axes $\mathbf{n}_1, \mathbf{n}_2$ as elements of the Lie group SO(3), and let $\mathbf{X}_1 = \log(\mathbf{M}_1) = \theta_1 \mathbf{N}_1$ and $\mathbf{X}_2 = \log(\mathbf{M}_2) = \theta_2 \mathbf{N}_2$ be the corresponding elements of the Lie algebra $\mathfrak{so}(3)$. Then the element $\mathbf{X} = \theta \mathbf{N} \in \mathfrak{so}(3)$ that corresponds to the product $\mathbf{M}_1\mathbf{M}_2 \in \mathrm{SO}(3)$ is identified by the Baker–Campbell–Hausdorff formula

$$\mathbf{X} \;=\; \mathrm{BCH}(\mathbf{X}_1, \mathbf{X}_2) \;=\; \alpha\, \mathbf{X}_1 + \beta\, \mathbf{X}_2 + \gamma\, [\, \mathbf{X}_1, \mathbf{X}_2\,]$$

for appropriate real coefficients α, β, γ dependent on $\mathbf{X}_1, \mathbf{X}_2$. In terms of the corresponding vectors $\mathbf{x} = \theta\, \mathbf{n}$, $\mathbf{x}_1 = \theta_1\, \mathbf{n}_1$, $\mathbf{x}_2 = \theta_2\, \mathbf{n}_2$ in $\mathbb{R}^3$ this becomes

$$\mathbf{x} \;=\; \mathrm{BCH}(\mathbf{x}_1, \mathbf{x}_2) \;=\; \alpha\, \mathbf{x}_1 + \beta\, \mathbf{x}_2 + \gamma\, \mathbf{x}_1 \times \mathbf{x}_2 \,. \tag{18}$$

The bilinear map $\mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}^3$ defined by $\mathbf{x} = \mathrm{BCH}(\mathbf{x}_1, \mathbf{x}_2)$ can be regarded as a kind of "weighted non–commutative vector sum" in $\mathbb{R}^3$.

Now the product $\mathcal{U} = \mathcal{U}_1 \mathcal{U}_2 = (\cos \frac{1}{2}\theta, \sin \frac{1}{2}\theta\, \mathbf{n})$ of $\mathcal{U}_1 = (\cos \frac{1}{2}\theta_1, \sin \frac{1}{2}\theta_1 \mathbf{n}_1)$ and $\mathcal{U}_2 = (\cos \frac{1}{2}\theta_2, \sin \frac{1}{2}\theta_2 \mathbf{n}_2)$ has rotation angle θ and axis $\mathbf{n}$ defined by (4) and (5). Using $\cos\theta = 2\, \cos^2 \frac{1}{2}\theta - 1$, we have

$$\theta \;=\; \arccos[\, (\cos \tfrac{1}{2}\theta_1 \cos \tfrac{1}{2}\theta_2 - \sin \tfrac{1}{2}\theta_1 \sin \tfrac{1}{2}\theta_2\ \mathbf{n}_1 \cdot \mathbf{n}_2)^2 - 1\,] \,.$$

For $\theta \in [\, -\pi, +\pi\,]$ this yields two equal magnitude θ values of opposite signs, with corresponding opposite sign values for $\sin \frac{1}{2}\theta$ in (5). Writing (18) as

$$\theta\, \mathbf{n} \;=\; \mathrm{BCH}(\theta_1 \mathbf{n}_1, \theta_2 \mathbf{n}_2) \;=\; \alpha\, \theta_1\, \mathbf{n}_1 + \beta\, \theta_2\, \mathbf{n}_2 + \gamma\, \theta_1 \theta_2\, \mathbf{n}_1 \times \mathbf{n}_2 \,,$$

and comparing with equations (4) and (5), the coefficients of $\mathbf{n}_1$, $\mathbf{n}_2$, $\mathbf{n}_1 \times \mathbf{n}_2$ may be identified as

$$\begin{aligned} \alpha\, \theta_1 \;&=\; \theta\, \frac{\sin \frac{1}{2}\theta_1 \cos \frac{1}{2}\theta_2}{\sin \frac{1}{2}\theta} \,, \\ \beta\, \theta_2 \;&=\; \theta\, \frac{\cos \frac{1}{2}\theta_1 \sin \frac{1}{2}\theta_2}{\sin \frac{1}{2}\theta} \,, \\ \gamma\, \theta_1 \theta_2 \;&=\; \theta\, \frac{\sin \frac{1}{2}\theta_1 \sin \frac{1}{2}\theta_2}{\sin \frac{1}{2}\theta} \,. \end{aligned}$$

In general α, β, γ depend on 6 scalar variables: the three components of each of $\mathbf{x}_1, \mathbf{x}_2$ — or, equivalently, the two angles θ_1, θ_2 and the two scalar freedoms characterizing each of $\mathbf{n}_1, \mathbf{n}_2$. For the case in which θ_1, θ_2 are fixed and only $\mathbf{n}_1, \mathbf{n}_2$ vary (i.e., $\mathbf{x}_1, \mathbf{x}_2$ have fixed magnitude but variable direction) these coefficients maintain fixed proportions, since only the common factor $f = \theta / \sin \frac{1}{2}\theta$ depends on $\mathbf{n}_1, \mathbf{n}_2$.

$$\begin{aligned} \theta\,\mathbf{n} &= \mathrm{BCH}(\theta_1 \mathbf{n}_1, \theta_2 \mathbf{n}_2) \\ &= f(\mathbf{n}_1, \mathbf{n}_2)\, \sin \tfrac{1}{2}\theta_1 \sin \tfrac{1}{2}\theta_2 \,[\, \cot \tfrac{1}{2}\theta_1\, \mathbf{n}_1 + \cot \tfrac{1}{2}\theta_2\, \mathbf{n}_2 + \mathbf{n}_1 \times \mathbf{n}_2 \,] \end{aligned}$$

We illustrate the Lie algebra approach by several computed examples, and the accompanying Figures 2–7 — note that, for reference, these figures also show the spheres of radius 1 and π).

Example 2 We consider sets of rotations V_1 and V_2 about the y and z axes, parameterized by the Euler vectors

$$V_1 : \mathbf{c}_1(s) = (0, s, 0),\ s \in [\,-\pi, \pi\,] \quad \text{and} \quad V_2 : \mathbf{c}_2(t) = (0, 0, t),\ t \in [\,-\pi, \pi\,]\,.$$

These correspond to the unit quaternion sets $C(\mathbf{j}, s, \pi)$ and $C(\mathbf{k}, t, \pi)$ defined in Section 9, and the Minkowski product $V_1 \otimes V_2 = \mathrm{BCH}(V_1, V_2)$ is a surface within the sphere of center 0 and radius π, as illustrated in Figure 2.

Example 3 In this case we consider a subset of the surface constructed in the preceding example, defined by restricting the range of the parameters s and t — namely,

$$V_1 : \mathbf{c}(s) = (0, s, 0),\ s \in [\,0, \pi\,] \quad \text{and} \quad V_2 : \mathbf{c}(t) = (0, 0, t),\ t \in [\,0, \pi\,]\,.$$

The Minkowski product $V_1 \otimes V_2 = \mathrm{BCH}(V_1, V_2)$ is illustrated in Figure 3.

Example 4 In this case we keep the fixed–axis set V_1 from the preceding example, but replace V_2 by a set V_3 corresponding to a fixed rotation angle but variable rotation axes:

$$\begin{aligned} V_1 &: \mathbf{c}_1(s) = (0, s, 0),\ s \in [\,0, \pi\,]\,, \\ V_3 &: \mathbf{c}_3(t) = \frac{\pi}{4}\,\frac{(\cos t, \sin t, 1)}{\sqrt{2}},\ t \in [\,-\pi, \pi\,]\,. \end{aligned}$$

The elements of the set V_3 correspond to the constant rotation angle $\theta = \pi/4$, but the rotation axis $\mathbf{n}(s)$ traces a small–circle on the unit sphere, parallel to the (x, y)–plane and at distance $z = 1/\sqrt{2}$ from it. The Minkowski product $V_1 \otimes V_3 = \mathrm{BCH}(V_1, V_3)$ is illustrated in Figure 4.

Example 5 To illustrate the non–commutative nature of the product, we employ the same sets as in the preceding example, but in converse order. The Minkowski product $V_3 \otimes V_1 = \mathrm{BCH}(V_3, V_1)$, shown in Figure 5, is seen to be qualitatively different from the product $V_1 \otimes V_3 = \mathrm{BCH}(V_1, V_3)$ in Figure 4.

Example 6 This example illustrates product of two sets corresponding to fixed rotation angles and variable rotation axes, namely

$$
\begin{aligned}
V_3 \ : \ \mathbf{c}_3(s) &= \frac{\pi}{4}\,\frac{(\cos s, \sin s, 1)}{\sqrt{2}}, \ s \in [\,-\pi, \pi\,]\,, \\
V_4 \ : \ \mathbf{c}_4(t) &= \frac{\pi}{4}\,\frac{(\cos t, 1, \sin t)}{\sqrt{2}}, \ t \in [\,-\pi, \pi\,]\,.
\end{aligned}
$$

Both sets correspond to the constant rotation angle $\theta = \pi/4$. The rotation axis $\mathbf{n}(s)$ for V_3 traces a small–circle on the unit sphere, parallel to the (x, y)–plane and at distance $z = 1/\sqrt{2}$ from it, and the rotation axis $\mathbf{n}(s)$ for V_4 traces a small–circle parallel to the (z, x)–plane and at distance $y = 1/\sqrt{2}$ from it. The Minkowski product $V_3 \otimes V_4 = \mathrm{BCH}(V_3, V_4)$ is shown in Figure 6.

Example 7 In this example we reverse the order of the sets in the previous example. Comparing Figure 6 with $V_4 \otimes V_3 = \mathrm{BCH}(V_4, V_3)$, shown in Figure 7, again illustrates the non–commutative nature of the product.

11 Minkowski product boundary evaluation

As observed in Proposition 2, the boundary $\partial(U_1 \otimes \mathcal{U}_2)$ of two 3–dimensional subsets U_1, U_2 of S^3 is, in general, a *subset* of the product $\partial U_1 \otimes \partial U_2$ of their individual boundaries. Certain elements $\mathcal{U} \in \partial U_1 \otimes \partial U_2$ may correspond to *interior* points of $\partial(U_1 \otimes \mathcal{U}_2)$, and it is desirable to identify a criterion that will distinguish these points from true boundary points.

For the case of Minkowski sums $S_1 \otimes S_2$ of point sets S_1, S_2 in $\mathbb{R}^2$ or $\mathbb{R}^3$, such a criterion is well known — when $\mathbf{p}_1 \in \partial S_1$ and $\mathbf{p}_2 \in \partial S_2$, a necessary condition for $\mathbf{p}_1 + \mathbf{p}_2$ to belong to $\partial(S_1 \oplus S_2)$ is that the normals $\mathbf{n}_1, \mathbf{n}_2$ to $\partial S_1, \partial S_2$ at $\mathbf{p}_1, \mathbf{p}_2$ must be linearly dependent. This principle was extended to Minkowksi products of point sets in $\mathbb{R}^2$ by interpreting points as complex numbers $\mathbf{p}_1 = x_1 + \mathrm{i}\, y_1, \ \mathbf{p}_2 = x_2 + \mathrm{i}\, y_2$ and invoking the logarithm map $\mathbf{z} \to \log \mathbf{z}$ to transform products into sums [6, 7].

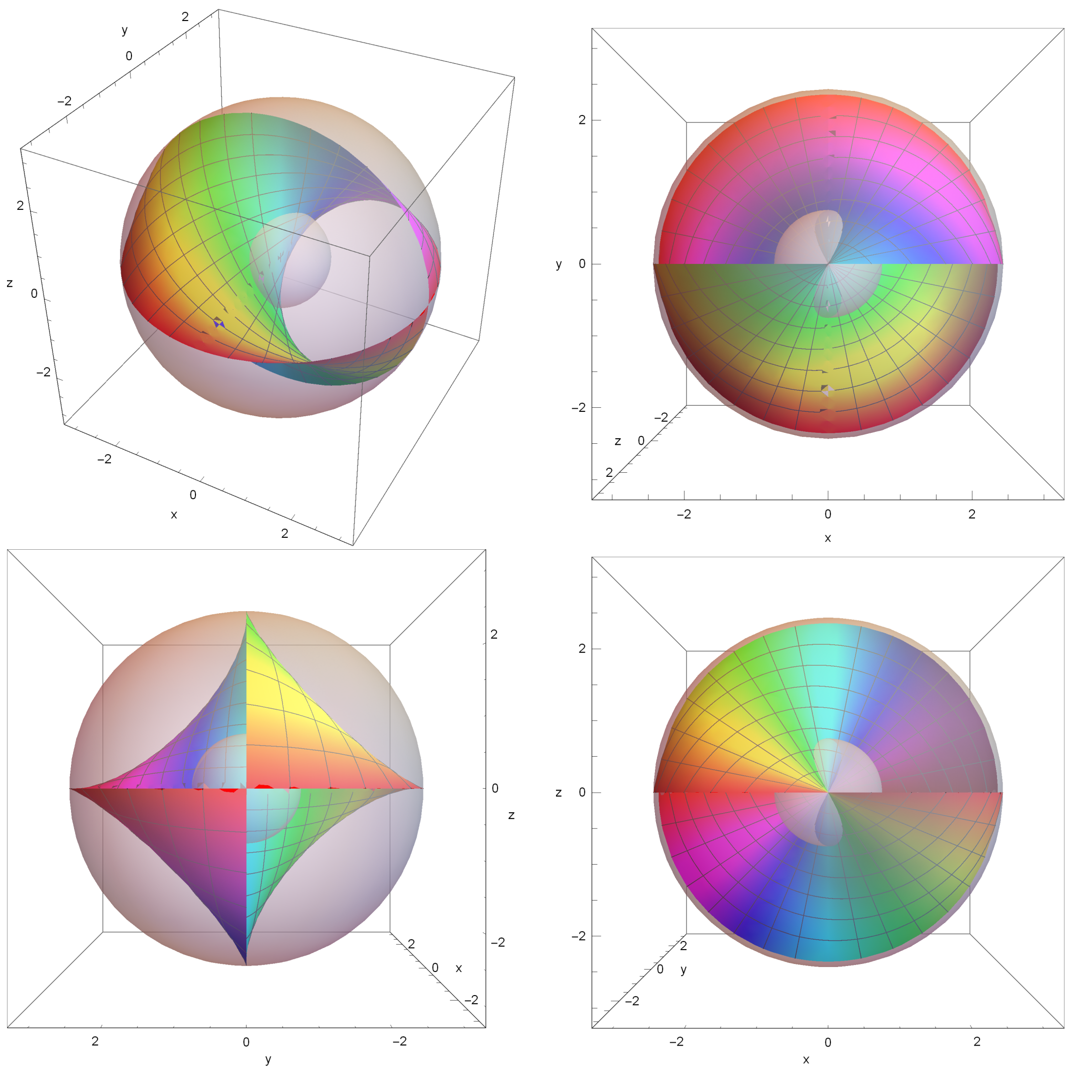

Figure 2: General view (upper left), top view (upper right), left view (lower left), and front view (lower right) of the Minkowski product in Example 2.

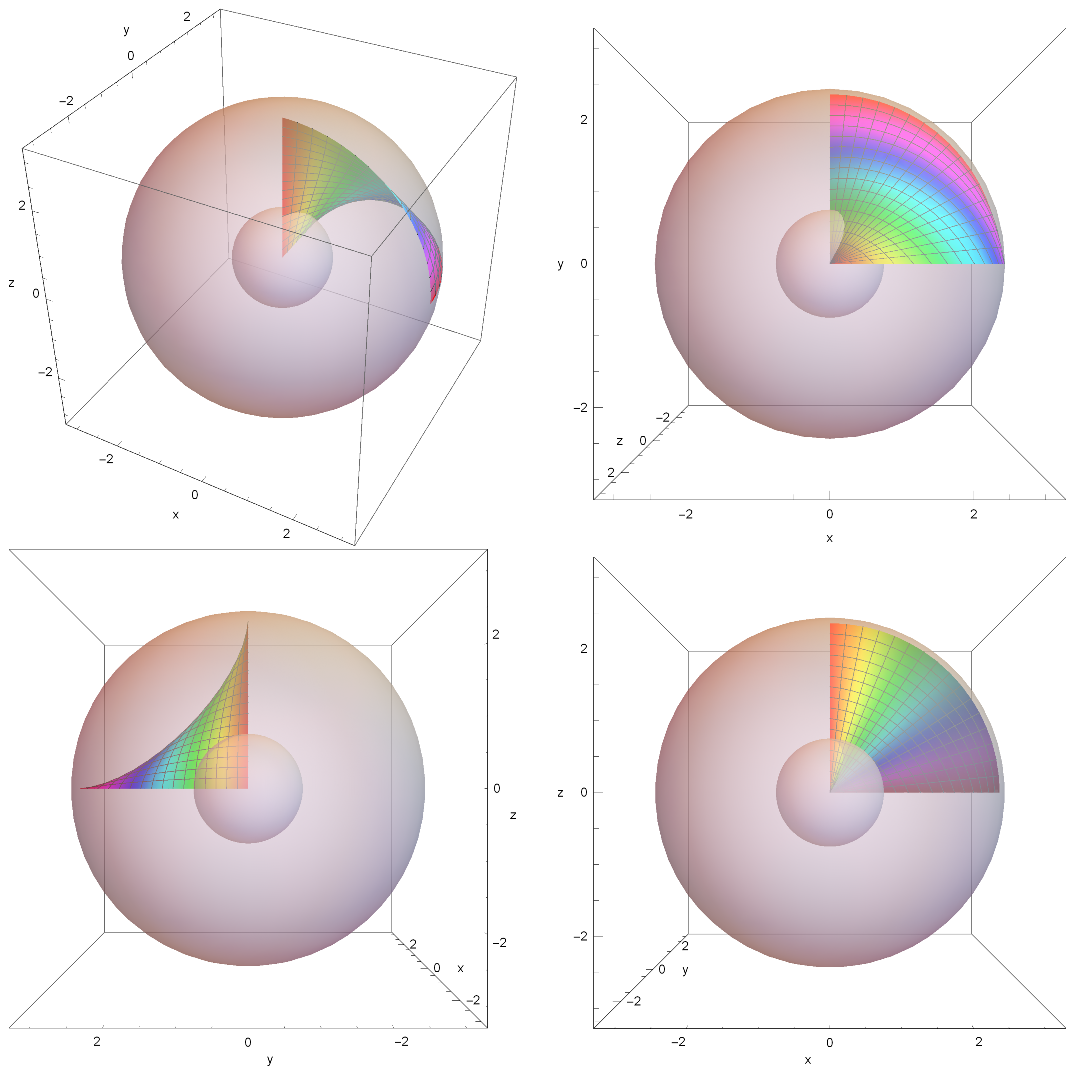

Figure 3: General view (upper left), top view (upper right), left view (lower left), and front view (lower right) of the Minkowski product in Example 3.

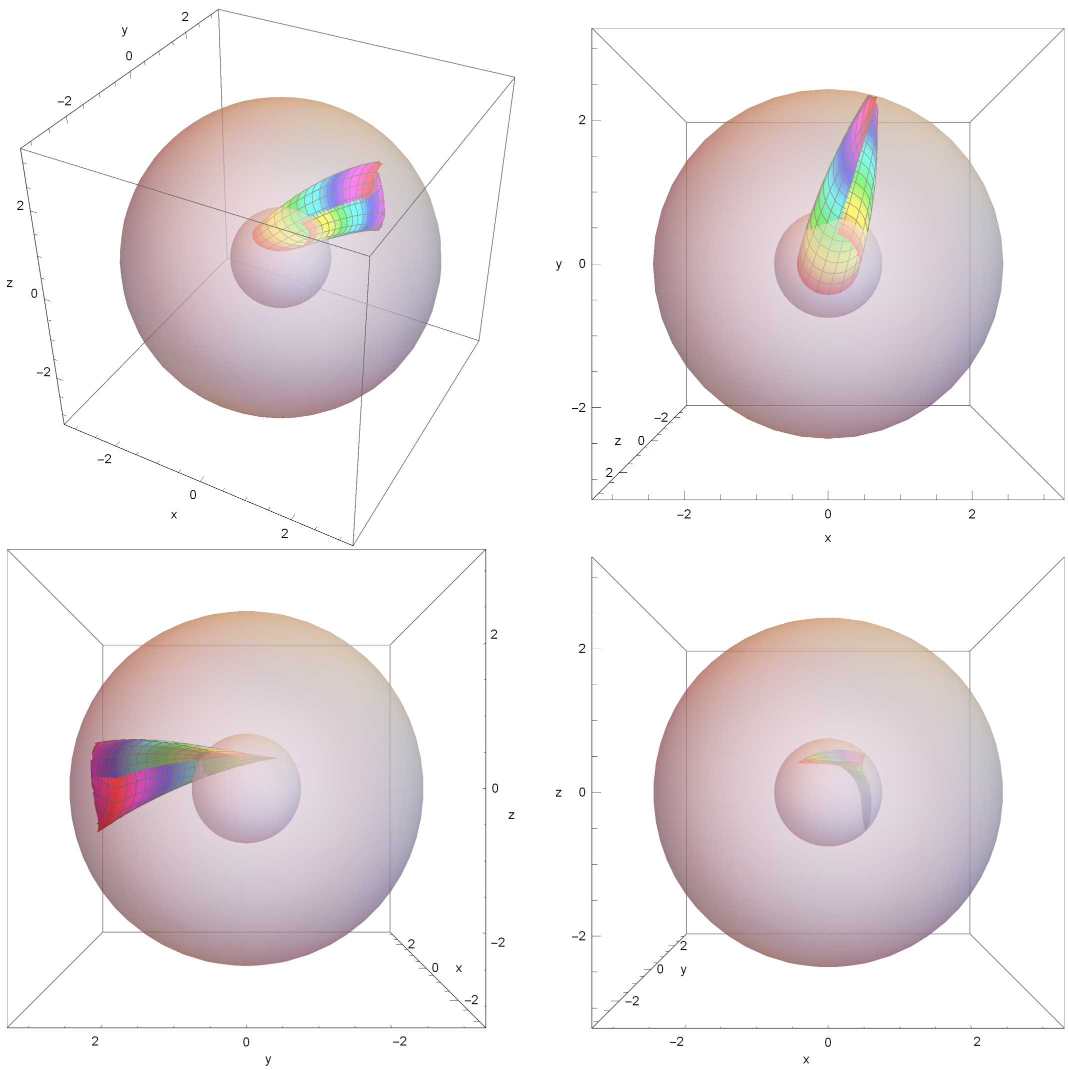

Figure 4: General view (upper left), top view (upper right), left view (lower left), and front view (lower right) of the Minkowski product in Example 4.

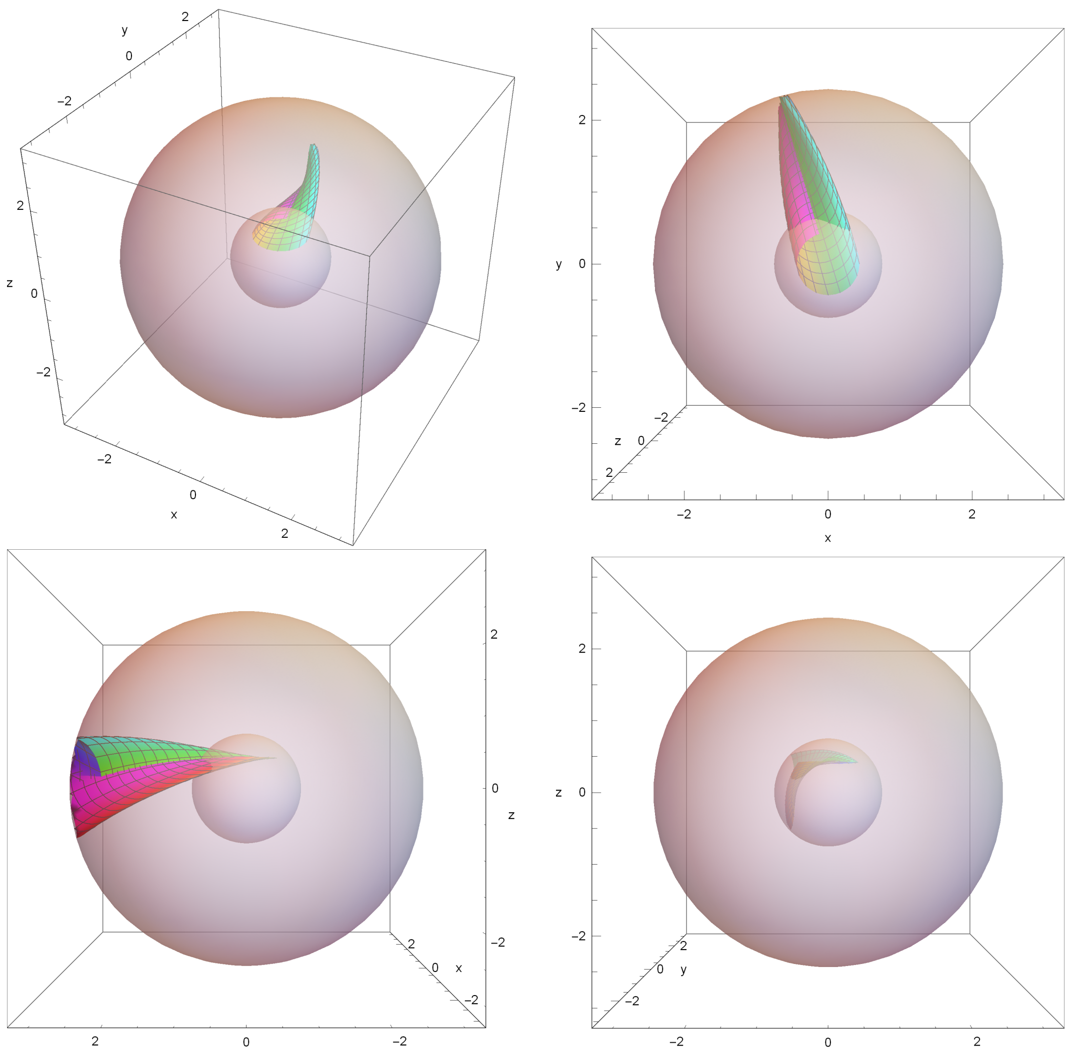

Figure 5: General view (upper left), top view (upper right), left view (lower left), and front view (lower right) of the Minkowski product in Example 5.

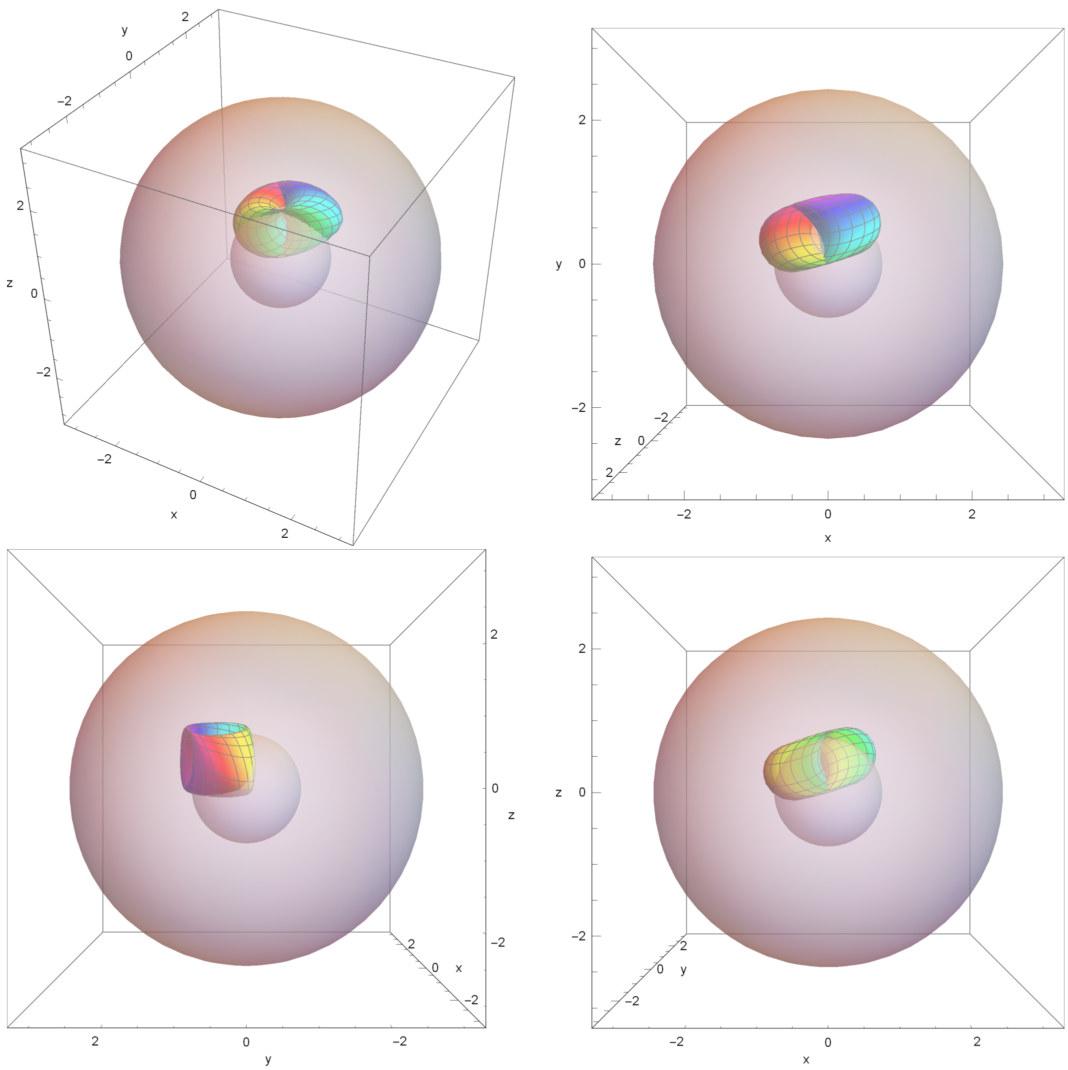

Figure 6: General view (upper left), top view (upper right), left view (lower left), and front view (lower right) of the Minkowski product in Example 6.

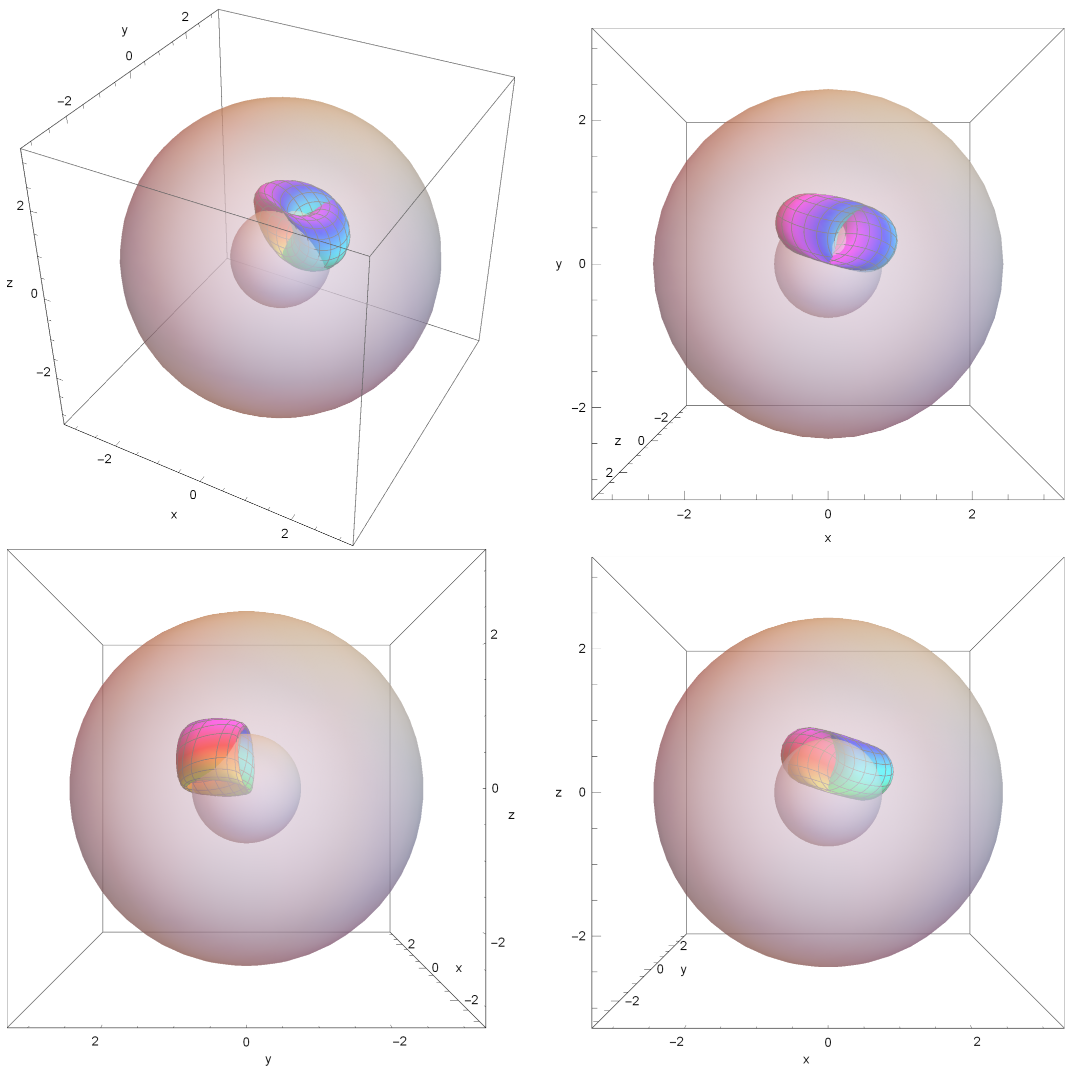

Figure 7: General view (upper left), top view (upper right), left view (lower left), and front view (lower right) of the Minkowski product in Example 7.

The situation with Minkowski products $U_1 \otimes U_2$ of unit quaternion sets is more subtle, because of the complexity of the logarithm map defined by the BCH formula. However, it is expected that a necessary condition for the product $\mathcal{U}\,\mathcal{V}$ of elements $\mathcal{U} \in \partial U$, $\mathcal{V} \in \partial V$ to belong to $\partial(U \otimes V)$ is that they identify *critical points* of the map $S^3 \times S^3 \to S^3$ defined by $(\mathcal{U}, \mathcal{V}) \to \mathcal{U}\,\mathcal{V}$, i.e., they occur when $\mathcal{U}$, $\mathcal{V}$ cause the Jacobian of this map to be rank deficient.

12 Closure

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13 Bounded rotation axes and angles — extra material (will probably delete)

Consider the set of all compound rotations $\mathcal{U} = \mathcal{U}_2\mathcal{U}_1$ generated by two unit quaternions $\mathcal{U}_1 = (\cos\frac{1}{2}\theta_1, \sin\frac{1}{2}\theta_1\,\mathbf{n}_1)$ and $\mathcal{U}_2 = (\cos\frac{1}{2}\theta_2, \sin\frac{1}{2}\theta_2\,\mathbf{n}_2)$ when the rotation axes $\mathbf{n}_1, \mathbf{n}_2$ and angles θ_1, θ_2 are subject to prescribed uncertainties — namely, $\mathbf{n}_1, \mathbf{n}_2$ have angular deviations from nominal unit vectors $\mathbf{c}_1$, $\mathbf{c}_2$ no more than a specified amount ξ, i.e.,

$$\mathbf{c}_1 \cdot \mathbf{n}_1 \,\ge\, \cos\xi \quad \text{and} \quad \mathbf{c}_2 \cdot \mathbf{n}_2 \,\ge\, \cos\xi\,, \tag{19}$$

while θ_1, θ_2 deviate from nominal values ϑ_1, ϑ_2 by no more than a specified amount $\delta\theta$, i.e.,

$$\theta_1 \in [\,\vartheta_1 - \delta\theta, \vartheta_1 + \delta\theta\,] \quad \text{and} \quad \theta_2 \in [\,\vartheta_2 - \delta\theta, \vartheta_2 + \delta\theta\,]\,.$$

For simplicity, we focus on the case $\xi > 0$ and $\delta\theta = 0$ — i.e., the rotation angles θ_1, θ_2 are precisely determined, but the axes $\mathbf{n}_1, \mathbf{n}_2$ span cones of half–angle ξ about their nominal positions $\mathbf{c}_1, \mathbf{c}_2$ and thus cover spherical caps C_1, C_2 of angular radius ξ on the unit sphere in $\mathbb{R}^3$. Without loss of generality, we may choose a coordinate system with $\mathbf{c}_1, \mathbf{c}_2$ symmetrically disposed about the z–axis, i.e., they are of the form

$$\mathbf{c}_1 \,=\, (\sin\tau, 0, \cos\tau) \quad \text{and} \quad \mathbf{c}_2 \,=\, (-\sin\tau, 0, \cos\tau)\,. \tag{20}$$

Note that C_1, C_2 are disjoint and do not encompass the "north pole" $(0, 0, 1)$ of S^2 if $\tau > \xi$. However, we do not require them to be disjoint. To determine the Minkowski product boundary $\partial(U_2 \otimes U_1)$, it suffices to consider quaternions on the boundaries of the sets

$$U_i \,=\, \{\,(\cos\tfrac{1}{2}\theta_i, \sin\tfrac{1}{2}\theta_i\,\mathbf{n}_i) \,:\, \mathbf{c}_i \cdot \mathbf{n}_i \ge \cos\xi\,\}\,, \quad i = 1, 2\,. \tag{21}$$

These boundaries are characterized by fixed angles θ_1, θ_2 and vectors $\mathbf{n}_1, \mathbf{n}_2$ that trace circles on the unit sphere in $\mathbb{R}^3$ — the boundaries ∂C_1, ∂C_2 of the caps C_1, C_2. In the adopted coordinate system, explicit parameterizations of ∂C_1, ∂C_2 may be derived in terms of angular variables ϕ_1, ϕ_2 as

$$\mathbf{n}_1(\phi_1) \,=\, \cos\xi\;\mathbf{c}_1 + \sin\xi\;\mathbf{v}_1(\phi_1)\,, \quad \mathbf{n}_2(\phi_2) \,=\, \cos\xi\;\mathbf{c}_2 + \sin\xi\;\mathbf{v}_2(\phi_2)\,, \tag{22}$$

where

$$\begin{aligned}\mathbf{v}_1(\phi_1) &= -\cos\xi\cos\phi_1\,\mathbf{i} + \sin\phi_1\,\mathbf{j} + \sin\xi\cos\phi_1\,\mathbf{k}\,,\\ \mathbf{v}_2(\phi_2) &= -\cos\xi\cos\phi_2\,\mathbf{i} + \sin\phi_2\,\mathbf{j} - \sin\xi\cos\phi_2\,\mathbf{k}\,.\end{aligned} \tag{23}$$

Note that $|\mathbf{v}_1(\phi_1)| = |\mathbf{v}_2(\phi_2)| = 1$ and $\mathbf{c}_1\cdot\mathbf{v}_1(\phi_1) = \mathbf{c}_2\cdot\mathbf{v}_2(\phi_2) = 0$, and hence

$$|\mathbf{n}_1(\phi_1)| = |\mathbf{n}_2(\phi_2)| = 1 \quad \text{and} \quad \mathbf{c}_1\cdot\mathbf{n}_1(\phi_1) = \mathbf{c}_2\cdot\mathbf{n}_2(\phi_2) = \cos\xi\,.$$

The dot product of $\mathbf{n}_1(\phi_1)$ and $\mathbf{n}_2(\phi_2)$ may be expressed as

$$\begin{aligned}\mathbf{n}_1(\phi_1)\cdot\mathbf{n}_2(\phi_2) &= \cos^2\!\xi\cos 2\xi + \tfrac{1}{2}\sin 2\xi\sin 2\xi(\cos\phi_1 - \cos\phi_2)\\ &\quad + \sin^2\!\xi\,(\sin\phi_1\sin\phi_2 + \cos 2\xi\cos\phi_1\cos\phi_2)\,,\end{aligned} \tag{24}$$

and the cross product as

$$\begin{aligned}\mathbf{n}_1(\phi_1)\times\mathbf{n}_2(\phi_2) &= \cos^2\!\xi\,\mathbf{c}_1\times\mathbf{c}_2 + \sin^2\!\xi\,\mathbf{v}_1(\phi_1)\times\mathbf{v}_2(\phi_2)\\ &\quad + \sin\xi\cos\xi\,[\,\mathbf{c}_1\times\mathbf{v}_2(\phi_2) - \mathbf{c}_2\times\mathbf{v}_1(\phi_1)\,]\,.\end{aligned} \tag{25}$$

In terms of components, $\mathbf{n}_1(\phi_1)\times\mathbf{n}_2(\phi_2)$ is given by

$$\begin{aligned}&[\,\tfrac{1}{2}\sin 2\xi\cos\xi\,(\sin\phi_1 - \sin\phi_2) - \sin^2\!\xi\sin\xi\sin(\phi_1+\phi_2)\,]\,\mathbf{i}\\ +\;&[\,\tfrac{1}{2}\sin 2\xi\cos 2\xi\,(\cos\phi_1 + \cos\phi_2) + \sin 2\xi\,(\sin^2\!\xi\cos\phi_1\cos\phi_2 - \cos^2\!\xi)\,]\,\mathbf{j}\\ +\;&[\,\tfrac{1}{2}\sin 2\xi\sin\xi\,(\sin\phi_1 + \sin\phi_2) + \sin^2\!\xi\cos\xi\sin(\phi_1-\phi_2)\,]\,\mathbf{k}\,.\end{aligned}$$

In general, the Minkowski product boundary $\partial(U_2\otimes U_1)$ is a subset of all the points on the unit sphere S^3 generated by products of the unit quaternions

$$\mathcal{U}_2(\phi_2) = (\cos\tfrac{1}{2}\theta_2, \sin\tfrac{1}{2}\theta_2\,\mathbf{n}_2(\phi_2))\,, \quad \mathcal{U}_1(\phi_1) = (\cos\tfrac{1}{2}\theta_1, \sin\tfrac{1}{2}\theta_1\,\mathbf{n}_1(\phi_1))\,.$$

For $0 \le \phi_1, \phi_2 \le 2\pi$ this product yields a unit quaternion defining a rotation with angle and axis obtained by substituting (24) and (25) into (4) and (5). The product $\mathcal{U}(\phi_1,\phi_2) = \mathcal{U}_2(\phi_2)\,\mathcal{U}_1(\phi_1)$ can be written in the form

$$\mathcal{U}(\phi_1,\phi_2) = (\cos\tfrac{1}{2}\theta(\phi_1,\phi_2), \sin\tfrac{1}{2}\theta(\phi_1,\phi_2)\,\mathbf{n}(\phi_1,\phi_2))\,, \tag{26}$$

with a rotation angle $\theta(\phi_1,\phi_2)$ and axis $\mathbf{n}(\phi_1,\phi_2)$ determined by substituting from (24) and (25) into (4) and (5). The expression (26) defines a map from the planar domain $(\phi_1,\phi_2)\in[\,0,2\pi\,]^2$ to some subset of the unit sphere S^3 in $\mathbb{R}^4$, whose boundary coincides with $\partial(U_2\otimes U_1)$. In general, this map is not one–to–one, and $\partial(U_2\otimes U_2)$ consists of segments where its image "folds" on S^3 — i.e., where the Jacobian of the map is rank–deficient.

$\partial(U_2\otimes U_1)$ is evidently difficult to visualize, since it is a two–dimensional surface embedded in the 3–sphere S^3 in $\mathbb{R}^4$. To address this, we shall consider separately the behavior of the rotation angle $\theta(\phi_1,\phi_2)$, which can be viewed as the graph of a bivariate function, and the rotation axis $\mathbf{n}(\phi_1,\phi_2)$, which covers an area C on the unit sphere S^2 in $\mathbb{R}^3$. As a further aid to visualization, we invoke stereographic projection to map S^2 to the Euclidean plane.

13.1 Stereographic projection $S^2\to\mathbb{R}^2$

Points $\mathbf{n}=(n_x,n_y,n_z)$ on the unit sphere S^2 in $\mathbb{R}^3$ defined by $n_x^2+n_y^2+n_z^2=1$ may be mapped to the (x,y) plane by the stereographic projection

$$(n_x,n_y,n_z)\;\to\;(x,y)\;=\;\frac{(n_x,n_y)}{1-n_z}\,. \tag{27}$$

The image point (x,y) corresponds to the intersection with the (x,y) plane of a line drawn through the "north pole" $(0,0,1)$ and a given point (n_x,n_y,n_z) of S^2. The "equator" $n_z=0$ on S^2 is mapped into itself, while points in the "northern/southern hemispheres" of S^2 are mapped to its interior/exterior.

Under stereographic projection, circles on S^2 that does not pass through the polar point $(0,0,1)$ are mapped to circles in the (x,y) plane, while circles that pass through it are mapped to straight lines. The point $(0,0,1)$ is itself considered to be mapped to the "point at infinity." The inverse stereographic projection maps points of the plane to the unit sphere according to

$$(x,y)\;\to\;(n_x,n_y,n_z)\;=\;\frac{(2x,2y,x^2+y^2-1)}{x^2+y^2+1}\,.$$

One approach to visualizing the two–parameter set $\mathbf{n}(\phi_1,\phi_2)$ of rotation axes generated by (26) is to interpret it as the area C on S^2 covered by a one–parameter family of curves obtained with (say) ϕ_1 as the curve

Figure 8: Stereographic projection of the set of rotation axes (red) generated by a Minkowski product of sets U_1, U_2 (blue) defined by (21), where $\theta_1 = \theta_2 = \pi/8$ and $\mathbf{n}_1, \mathbf{n}_2$ satisfy (19) and (20) with $\xi = \pi/8$, $\tau = 2\pi/3$. The two cases $U_2 \otimes U_1$ and $U_1 \otimes U_2$ are shown on the left and right, respectively.

parameter, and ϕ_2 as the family parameter. Figure 8 illustrates this approach by means of stereographic projection, and shows that $U_1 \otimes U_2 \neq U_2 \otimes U_1$ (the black circle indicates the equator on S^2). Figure 9 shows the sets of rotation axes when $\theta_1 = \theta_2 = \pi/10$, $\xi = \pi/12$, and $\tau = 0.85\pi$, 0.75π, 0.65π, 0.55π.

Figure 9: Stereographic projection of the set of rotation axes (red) generated by a Minkowski product of sets (blue) defined by (21), where $\theta_1 = \theta_2 = \pi/10$ and $\mathbf{n}_1, \mathbf{n}_2$ satisfy (19)–(20) with $\tau = \pi/12$, $\xi = 0.85\pi$, 0.75π, 0.65π, 0.55π.

Substituting for $\mathbf{n}_1(\phi_1)$, $\mathbf{n}_2(\phi_2)$, $\mathbf{n}_1(\phi_1) \cdot \mathbf{n}_2(\phi_2)$, $\mathbf{n}_1(\phi_1) \times \mathbf{n}_2(\phi_2)$ into (4) and (5), it is possible to explicitly determine the (n_x, n_y, n_z) components of $\mathbf{n}(\phi_1, \phi_2)$ and then map it to the plane through the stereographic projection (27). This image can be regarded as a one–parameter family of plane curves $\mathbf{r}(\phi_1, \phi_2) = (x(\phi_1, \phi_2), y(\phi_1, \phi_2))$ with ϕ_1 as the curve parameter and ϕ_2 as the family parameter (or vice–versa). The envelope is (a subset of) the *silhouette* of the parametric surface $\mathbf{s}(\phi_1, \phi_2) = (x(\phi_1, \phi_2), y(\phi_1, \phi_2), \phi_2)$ constructed by "stacking" each curve at height $z = \phi_2$, viewed along the z–direction [1] — i.e., it is the projection of the set of points where the surface normal

$$\frac{\mathbf{s}_{\phi_1} \times \mathbf{s}_{\phi_2}}{|\, \mathbf{s}_{\phi_1} \times \mathbf{s}_{\phi_2} \,|}$$

is orthogonal to the z–direction. Hence, a necessary condition for a point to lie on the stereographic image of the boundary ∂C of the set of rotation axes can be formulated as

$$\frac{\partial x}{\partial \phi_1} \frac{\partial y}{\partial \phi_2} - \frac{\partial y}{\partial \phi_1} \frac{\partial x}{\partial \phi_2} = 0\,. \tag{28}$$

This condition establishes a correspondence between the parameters ϕ_1, ϕ_2 that identifies a superset of the boundary — it is a superset since, in general, certain pairs ϕ_1, ϕ_2 satisfying (28) identify *interior* points of the set.

13.2 Stereographic projection $S^3 \to \mathbb{R}^3$

Writing the product of $\mathcal{U}_2(\phi_2)$ and $\mathcal{U}_1(\phi_1)$ in the form

$$\begin{aligned}\mathcal{U}(\phi_1,\phi_2) &= \mathcal{U}_2(\phi_2)\mathcal{U}_1(\phi_1)\\ &= u(\phi_1,\phi_2) + u_x(\phi_1,\phi_2)\,\mathbf{i} \ + u_y(\phi_1,\phi_2)\,\mathbf{j} + u_z(\phi_1,\phi_2)\,\mathbf{k}\,,\end{aligned}$$

its components may be expressed as

$$\begin{aligned}
u(\phi_1,\phi_2) &= \cos\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2 - \sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\,[\cos^2\!\tau\cos 2\xi\\
&+ \tfrac{1}{2}\sin 2\tau\sin 2\xi\,(\cos\phi_1 - \cos\phi_2)\\
&+ \sin^2\!\tau\,(\sin\phi_1\sin\phi_2 + \cos 2\xi\cos\phi_1\cos\phi_2)\,]\,,\\
u_x(\phi_1,\phi_2) &= \cos\tau\sin\xi\sin\tfrac{1}{2}(\theta_1-\theta_2)\\
&- \sin\tau\cos\xi\,(\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\phi_1 + \cos\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\phi_2)\\
&- \tfrac{1}{2}\sin 2\tau\cos\xi\,(\sin\phi_1+\sin\phi_2) - \sin^2\!\tau\sin\xi\sin(\phi_1+\phi_2)\,,\\
u_y(\phi_1,\phi_2) &= \cos^2\!\tau\sin 2\xi\sin\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\\
&+ \sin\tau\,(\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\sin\phi_1 + \cos\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\sin\phi_2)\\
&- \tfrac{1}{2}\sin 2\tau\cos 2\xi\,(\cos\phi_1+\cos\phi_2) - \sin^2\!\tau\sin 2\xi\cos\phi_1\cos\phi_2\,,\\
u_z(\phi_1,\phi_2) &= \cos\tau\sin\xi\sin\tfrac{1}{2}(\theta_1+\theta_2)\\
&+ \sin\tau\sin\xi\,(\sin\tfrac{1}{2}\theta_1\cos\tfrac{1}{2}\theta_2\cos\phi_1 - \cos\tfrac{1}{2}\theta_1\sin\tfrac{1}{2}\theta_2\cos\phi_2)\\
&- \tfrac{1}{2}\sin 2\tau\sin\xi\,(\sin\phi_1-\sin\phi_2) - \sin^2\!\tau\cos\xi\sin(\phi_1+\phi_2)\,.
\end{aligned}$$

For fixed $\tau, \xi, \theta_1, \theta_2$ these expressions determine a 2–dimensional parametric surface $\mathcal{U}(\phi_1,\phi_2)$ in $\mathbb{R}^4$ that is embedded in the unit sphere S^3. Moreover, the isoparametric curves on $\mathcal{U}(\phi_1,\phi_2)$ defined by fixing either of ϕ_1, ϕ_2 and varying the other are circles in $\mathbb{R}^4$. This can be seen as follows. When ϕ_2 is fixed, the expressions assume the form

$$\begin{aligned}
u &= a + b\,\cos\phi_1 + c\,\sin\phi_1\,,\\
u_x &= a_x + b_x\cos\phi_1 + c_x\sin\phi_1\,,\\
u_y &= a_y + b_y\cos\phi_1 + c_y\sin\phi_1\,,\\
u_z &= a_z + b_z\cos\phi_1 + c_z\sin\phi_1\,,
\end{aligned}$$

where $a.b, c, \ldots$ etc. are known constants. The last two equations can be used to solve for $\sin\phi_1, \cos\phi_1$ in terms af u_y, u_z as

$$(\sin\phi_1,\cos\phi_1) = \frac{(c_z u_y - c_y u_z + c_y a_z - c_z a_y,\, b_y u_z - b_z u_y + a_y b_z - a_z b_y)}{b_y c_z - b_z c_y}\,.$$

Substituting into the first two equations, the coordinates $(u, u_x.u_y, u_z)$ of a locus $\phi_2 =$ constant on $\mathcal{U}(\phi_1, \phi_2)$ must satisfy the constraints

$$
\begin{aligned}
&(b_y c_z - b_z c_y)u + (b_z c - b c_z)u_y + (b c_y - b_y c)u_z \\
&\quad = \; a(b_y c_z - b_z c_y) + b(c_y a_z - c_z a_y) + c(a_y b_z - a_z b_y)\,, \\
&(b_y c_z - b_z c_y)u_x + (b_z c - b c_z)u_y + (b c_y - b_y c)u_z \\
&\quad = \; a_x(b_y c_z - b_z c_y) + b_x(c_y a_z - c_z a_y) + c_x(a_y b_z - a_z b_y)\,.
\end{aligned}
$$

These equations define 3–dimensional hyperplanes in $\mathbb{R}^4$ whose intersection is, in general, a 2–dimensional plane. Since a 2–dimensional plane intersects S^3 in a circle, the locus $\phi_2 =$ constant is a circle on S^3. Analogous arguments hold when ϕ_1 is fixed and ϕ_2 varies. Since it is difficult to visualize the surface $\mathcal{U}(\phi_1, \phi_2)$ in $\mathbb{R}^4$, one can invoke the stereographic projection (6) to $\mathbb{R}^3$.

Under stereographic projection, the images of circles on S^3 are circles in $\mathbb{R}^3$. Substituting for u, u_x, u_y, u_z into (6) yields an explicit parameterization

$$
\mathbf{r}(\phi_1, \phi_2) \; = \; (x(\phi_1, \phi_2), y(\phi_1, \phi_2), z(\phi_1, \phi_2))
$$

for the image in $\mathbb{R}^3$ of the surface $\mathcal{U}(\phi_1, \phi_2)$ in $\mathbb{R}^4$. By the circle–preserving property, the sets of isoparametric curves $\phi_1 =$ constant and $\phi_2 =$ constant on $\mathbf{r}(\phi_1, \phi_2)$ are also circles.

This surface is a characterization of the set of unit quaternions generated as products of $\mathcal{U}_2 = (\cos\frac{1}{2}\theta_2, \sin\frac{1}{2}\theta_2\,\mathbf{n}_2(\phi_2))$ and $\mathcal{U}_1 = (\cos\frac{1}{2}\theta_1, \sin\frac{1}{2}\theta_1\,\mathbf{n}_1(\phi_1))$ for $\phi_1, \phi_2 \in [\,0, 2\pi\,]$ with $\mathbf{n}_1(\phi_1)$, $\mathbf{n}_2(\phi_2)$ defined by (20), (22), and (23) with fixed $\tau, \xi, \theta_1, \theta_2$. Note that $\mathcal{U}(\phi_1, \phi_2)$ and its stereographic image $\mathbf{r}(\phi_1, \phi_2)$ are *rational* surfaces, as can be seen by invoking the parameter transformations

$$
(\sin\phi_1, \cos\phi_1) \; \rightarrow \; \frac{(2\,s, 1 - s^2)}{1 + s^2}\,, \quad (\sin\phi_2, \cos\phi_2) \; \rightarrow \; \frac{(2\,t, 1 - t^2)}{1 + t^2}\,.
$$