Minkowski geometric algebra of complex sets — theory, algorithms, applications

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synopsis

- introduction, motivation, historical background
- Minkowski sums, products, roots, implicitly-defined sets
- *connections 1.* complex interval arithmetic, planar shape operators
- bipolar coordinates and geometry of Cartesian ovals
- *connections 2.* anticaustics in geometrical optics
- Minkowski products — logarithmic Gauss map, curvature, convexity
- implicitly-defined sets (inclusion relations) & solution of equations
- *connections 3.* stability of linear dynamic systems —
  Hurwitz & Kharitonov theorems, Γ-stability

geometric algebras in $\mathbb{R}^N$

algebras of points

- $N = 1$: *real numbers* $N = 2$: *complex numbers*
- $N \geq 4$: *quaternions*, *octonions*, *Grassmann & Clifford algebras*
- elements are finitely-describable, closed under arithmetic operations

algebras of point sets

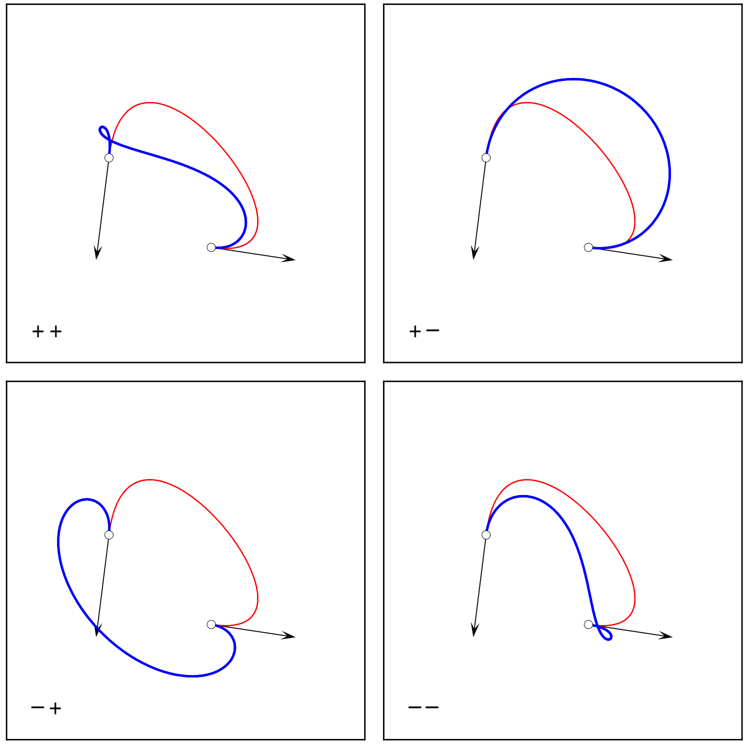
- *real interval arithmetic* (finite descriptions, exhibit closure)
- *Minkowski algebra of complex sets* (closure impossible for any family of finitely-describable sets)
- must relinquish distributive law for algebra of sets

bibliography

Schwerdtfeger: *Geometry of Complex Numbers*, Needham: *Visual Complex Analysis*

- Farouki, Moon, Ravani: Algorithms for Minkowski products and implicitly-defined complex sets, *Advances in Computational Mathematics* **13**, 199-229 (2000)
- Farouki, Gu, Moon: Minkowski roots of complex sets, *Geometric Modeling & Processing 2000*, IEEE Computer Society Press, 287-300 (2000)
- Farouki, Moon, Ravani: Minkowski geometric algebra of complex sets, *Geometriae Dedicata* **85**, 283-315 (2001)
- Farouki & Pottmann: Exact Minkowski products of N complex disks, *Reliable Computing* **8**, 43-66 (2002)
- Farouki & Moon: Minkowski geometric algebra and the stability of characteristic polynomials, *Visualization and Mathematics III*, Springer, 163-188 (2003)
- Farouki & Han: Computation of Minkowski values of polynomials over complex sets, *Numerical Algorithms* **36**, 13-29 (2004)
- Farouki & Han: Solution of elementary equations in the Minkowski geometric algebra of complex sets, *Advances in Computational Mathematics* **22**, 301-323 (2005)
- Farouki, Han, Hass: Boundary evaluation algorithms for Minkowski combinations of complex sets using topological analysis of implicit curves, *Journal of Computational and Applied Mathematics* **209**, 246-266 (2007)
- Farouki & Han: Root neighborhoods, generalized lemniscates, and robust stability of dynamic systems, *Applicable Algebra in Engineering, Communication, and Computing* **18**, 169–189 (2007)

selection of PH quintic Hermite interpolants



$$\mathcal{D} = \{\, \mathbf{z} \mid \mathrm{Re}(\mathbf{z}) > |\mathrm{Im}(\mathbf{z})| \ \& \ |\mathbf{z}| < \sqrt{3}\,\}$$

show that $\mathcal{D} \, ⓕ \, \mathcal{D} = \{\, \mathbf{f}(\mathbf{z}_0, \mathbf{z}_2) \mid \mathbf{z}_0, \mathbf{z}_2 \in \mathcal{D}\,\} \subset \mathcal{D}$

where $\mathbf{f}(\mathbf{z}_0, \mathbf{z}_2) = \frac{1}{4}\left[\, \mathbf{z}_0 - 3\,\mathbf{z}_2 + \sqrt{120 - 15(\mathbf{z}_0^2 + \mathbf{z}_2^2) + 10\,\mathbf{z}_0\mathbf{z}_2}\,\right]$

Caspar Wessel 1745-1818, Norwegian surveyor

Om

Directionens analytiske Betegning,

et Forsøg,

anvendt fornemmelig

til

plane og sphæriske Polygoners Opløsning.

Af

Caspar Wessel,

Landmaaler.

Kiøbenhavn 1798.

Trykt hos Johan Rudolph Thiele.

Wessel’s algebra of line segments

sums of directed line segments

Two right lines are added if we unite them in such a way that the second line begins where the first one ends, and then pass a right line from the first to the last point of the united lines.

products of directed line segments

As regards length, the product shall be to one factor as the other factor is to the unit. As regards direction, it shall diverge from the one factor as many degrees, and on the same side, as the other factor diverges from the unit, so that the direction angle of the product is the sum of the direction angles of the factors.

identification with complex numbers

Let $+1$ *be the positive unit, and* $+\epsilon$ *a unit perpendicular to it. Then the direction angle of* $+1$ *is* 0°*, that of* -1 *is* 180°*, that of* $+\epsilon$ *is* 90°*, and that of* $-\epsilon$ *is* 270°*. By the rule that the angle of a product is the sum of the angles of the factors, we have*

$$(+1)(+1) = +1\,,\ (+1)(-1) = -1\,,\ \cdots\ (+\epsilon)(+\epsilon) = -1\,,\ \cdots$$

From this, it is seen that ϵ *is equal to* $\sqrt{-1}$*.*

extension to other geometric algebras

There are other quantities besides right lines among which such relations exist . . . But I have accepted the advice of men of judgement, that in this paper both the nature of the contents and the plainness of exposition demand that the reader be not burdened with concepts so abstract.

sad fate of Caspar Wessel, Norwegian surveyor

moral #1: don’t expect mathematicians to pay any attention to your work if you’re just a humble surveyor

moral #2: don’t expect anyone to read your scientific papers if you publish in Norwegian (Danish, actually)

basic operations

$a, b \,=\,$ reals $\quad \mathbf{a}, \mathbf{b} \,=\,$ complex numbers $\quad \mathcal{A}, \mathcal{B} \,=\,$ subsets of $\mathbb{C}$

$$\text{Minkowski sum}: \quad \mathcal{A} \oplus \mathcal{B} \;=\; \{\, \mathbf{a} + \mathbf{b} \mid \mathbf{a} \in \mathcal{A} \text{ and } \mathbf{b} \in \mathcal{B} \,\}$$

$$\text{Minkowski product}: \quad \mathcal{A} \otimes \mathcal{B} \;=\; \{\, \mathbf{a} \times \mathbf{b} \mid \mathbf{a} \in \mathcal{A} \text{ and } \mathbf{b} \in \mathcal{B} \,\}$$

$$\text{subdistributive law}: \quad (\mathcal{A} \oplus \mathcal{B}) \otimes \mathcal{C} \,\subset\, (\mathcal{A} \otimes \mathcal{C}) \oplus (\mathcal{B} \otimes \mathcal{C})$$

negation and reciprocal of a set:

$$-\mathcal{B} \,=\, \{\, -\mathbf{b} \mid \mathbf{b} \in \mathcal{B} \,\}\,, \quad \mathcal{B}^{-1} \,=\, \{\, \mathbf{b}^{-1} \mid \mathbf{b} \in \mathcal{B} \,\}$$

Minkowski difference and division:

$$\mathcal{A} \ominus \mathcal{B} \,=\, \mathcal{A} \oplus (-\mathcal{B})\,, \quad \mathcal{A} \oslash \mathcal{B} \,=\, \mathcal{A} \otimes \mathcal{B}^{-1}$$

$\oplus, \ominus$ and $\otimes, \oslash$ not inverses — $(\mathcal{A} \oplus \mathcal{B}) \ominus \mathcal{B} \neq \mathcal{A}\,,\; (\mathcal{A} \otimes \mathcal{B}) \oslash \mathcal{B} \neq \mathcal{A}$

“implicitly-defined” complex sets

$$\mathcal{A} \textcircled{f} \mathcal{B} \;=\; \{\, \mathbf{f}(\mathbf{a},\mathbf{b}) \mid \mathbf{a} \in \mathcal{A},\; \mathbf{b} \in \mathcal{B} \,\}$$

$$\mathcal{A} \oplus \mathcal{B} \quad = \quad \bigcup_{\mathbf{a}\in\mathcal{A}} \text{translations of } \mathcal{B} \text{ by } \mathbf{a}$$

$$\mathcal{A} \otimes \mathcal{B} \quad = \quad \bigcup_{\mathbf{a}\in\mathcal{A}} \text{scalings/rotations of } \mathcal{B} \text{ by } \mathbf{a}$$

$$\mathcal{A} \textcircled{f} \mathcal{B} \quad = \quad \bigcup_{\mathbf{a}\in\mathcal{A}} \text{conformal mappings of } \mathcal{B} \text{ by } \mathbf{f}(\mathbf{a},\cdot)$$

$\mathcal{A} \textcircled{f} \mathcal{B}$ can be difficult to evaluate — sometimes use bounding Minkowski combination, e.g., for $\mathbf{f}(\mathbf{a},\mathbf{b}) = \mathbf{a}^2 + \mathbf{a}\mathbf{b}$

$$\mathcal{A} \textcircled{f} \mathcal{B} \;\subset\; \mathcal{A} \otimes (\mathcal{A} \oplus \mathcal{B}) \;\subset\; (\mathcal{A} \otimes \mathcal{A}) \oplus (\mathcal{A} \otimes \mathcal{B})$$

Minkowski powers and roots

$\otimes$ commutative, associative $\Rightarrow$ define Minkowski power by

$$\begin{aligned} \otimes^n\mathcal{A} \;&=\; \overbrace{\mathcal{A}\otimes\mathcal{A}\otimes\cdots\otimes\mathcal{A}}^{n\ \text{times}} \\ &=\; \{\, \mathbf{z}_1\mathbf{z}_2\cdots\mathbf{z}_n \mid \mathbf{z}_i\in\mathcal{A}\ \ \text{for}\ \ i=1,\ldots,n\,\} \end{aligned}$$

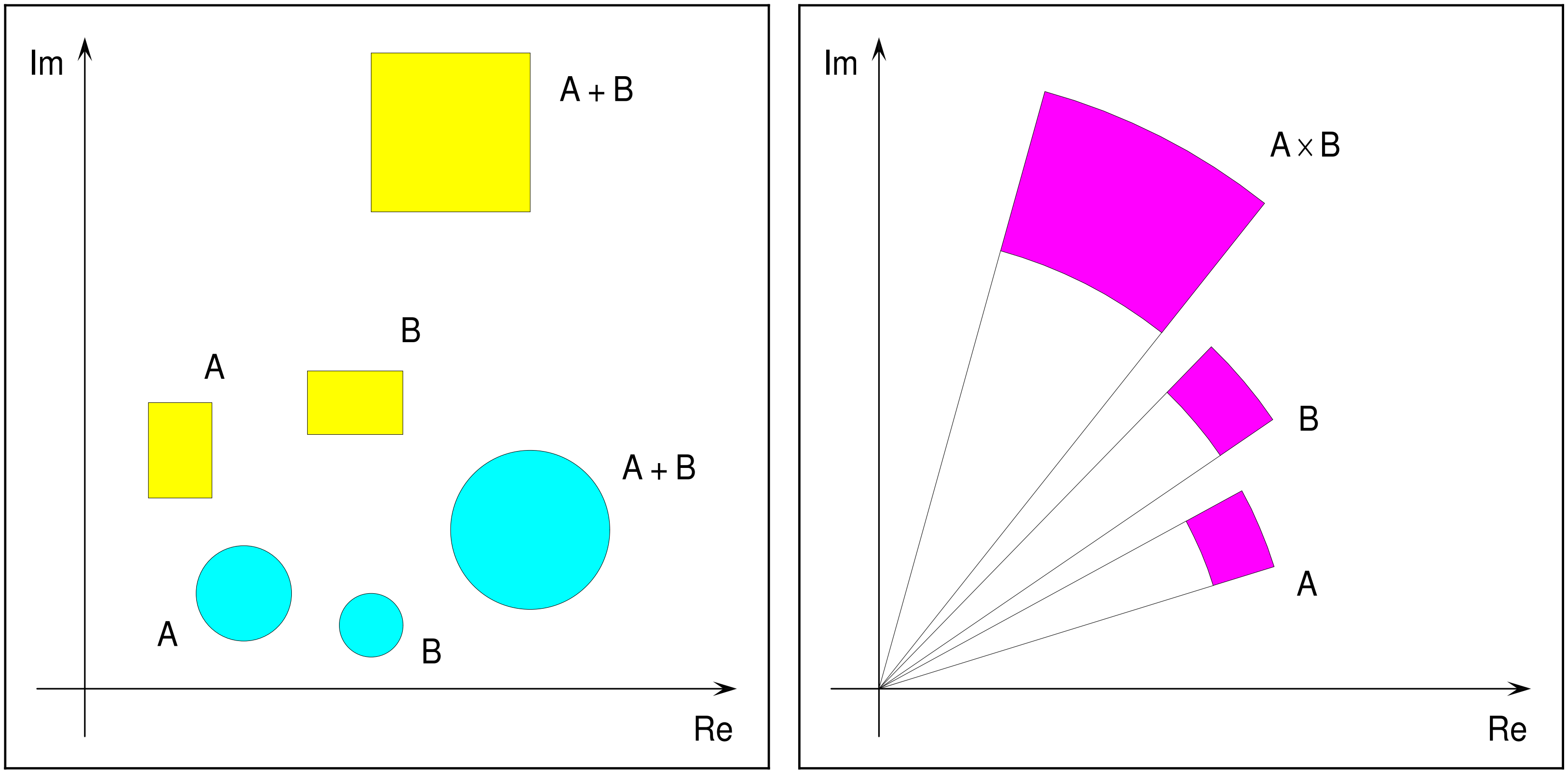
correspondingly, define Minkowski root by $\quad \otimes^n(\otimes^{1/n}\mathcal{A}) \;=\; \mathcal{A}$

$$\{\, \mathbf{z}_1\mathbf{z}_2\cdots\mathbf{z}_n \mid \mathbf{z}_i\in\otimes^{1/n}\mathcal{A}\ \ \text{for}\ \ i=1,\ldots,n\,\} \;=\; \mathcal{A}$$

do not confuse with “ordinary” powers & roots

$$\mathcal{A}^n \;=\; \{\,\mathbf{z}^n \mid \mathbf{z}\in\mathcal{A}\,\}\,, \quad \mathcal{A}^{1/n} \;=\; \{\,\mathbf{z} \mid \mathbf{z}^n\in\mathcal{A}\,\}$$

inclusion relations: $\quad \mathcal{A}^n \subseteq \otimes^n\mathcal{A}\,, \quad \otimes^{1/n}\mathcal{A} \,\subseteq\, \mathcal{A}^{1/n}$



Nickel (1980): no closure under both + and × for sets specified by finite number of parameters

complex interval arithmetic

$$
\begin{aligned}
[\,a,b\,]+[\,c,d\,] &= [\,a+c,b+d\,] \\
[\,a,b\,]-[\,c,d\,] &= [\,a-d,b-c\,] \\
[\,a,b\,]\times[\,c,d\,] &= [\,\min(ac,ad,bc,bd),\max(ac,ad,bc,bd)\,] \\
[\,a,b\,]\div[\,c,d\,] &= [\,a,b\,]\times[\,1/d,1/c\,]
\end{aligned}
$$

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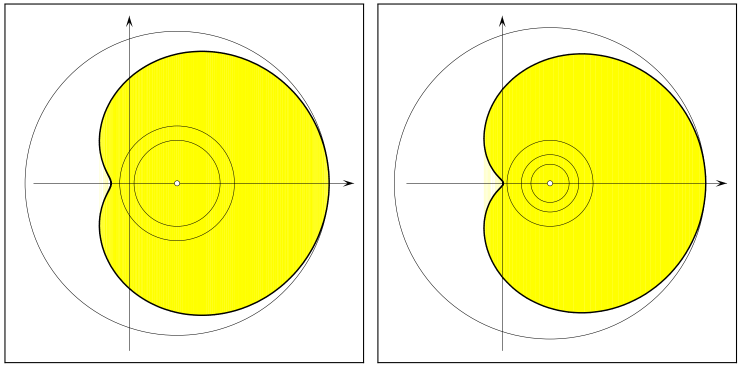
extend to “complex intervals” (rectangles, disks, . . .)

disk $\otimes$ disk $\neq$ disk $\;\rightarrow\;$ $(\mathbf{c}_1,R_1)\otimes(\mathbf{c}_2,R_2)$ “=” $(\,\mathbf{c}_1\mathbf{c}_2\,,\;|\mathbf{c}_1|R_2+|\mathbf{c}_2|R_1+R_1R_2\,)$



exact complex interval arithmetic $\equiv$ *Minkowski geometric algbera*

geometrical applications: 2D shape operators

$$\mathcal{S}_d \;=\; \text{complex disk of radius } d$$

offset at distance $d > 0$ of planar domain $\mathcal{A}$: $\quad \mathcal{A}_d \;=\; \mathcal{A} \oplus \mathcal{S}_d$

for negative offset, use set complementation: $\quad \mathcal{A}_{-d} \;=\; (\mathcal{A}^c \oplus \mathcal{S}_d)^c$

dilation & erosion operators in mathematical morphology (image processing)

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scaled Minkowski sum (f = real function on $\mathcal{A}$):

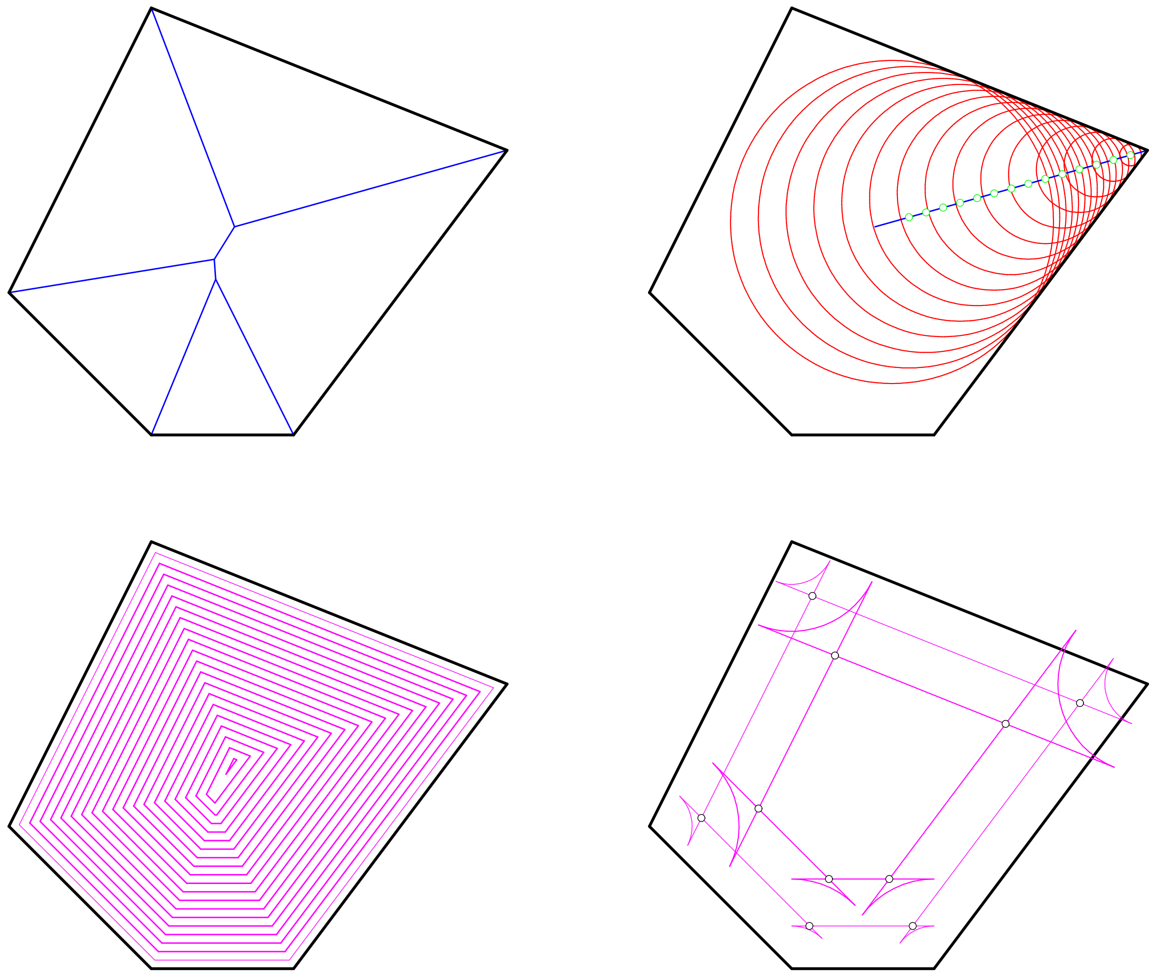
$$\mathcal{A} \oplus_f \mathcal{B} \;=\; \{\, \mathbf{a} + f(\mathbf{a})\mathbf{b} \mid \mathbf{a} \in \mathcal{A},\; \mathbf{b} \in \mathcal{B} \,\}$$

recover domain $\mathcal{D}$ from medial-axis transform:

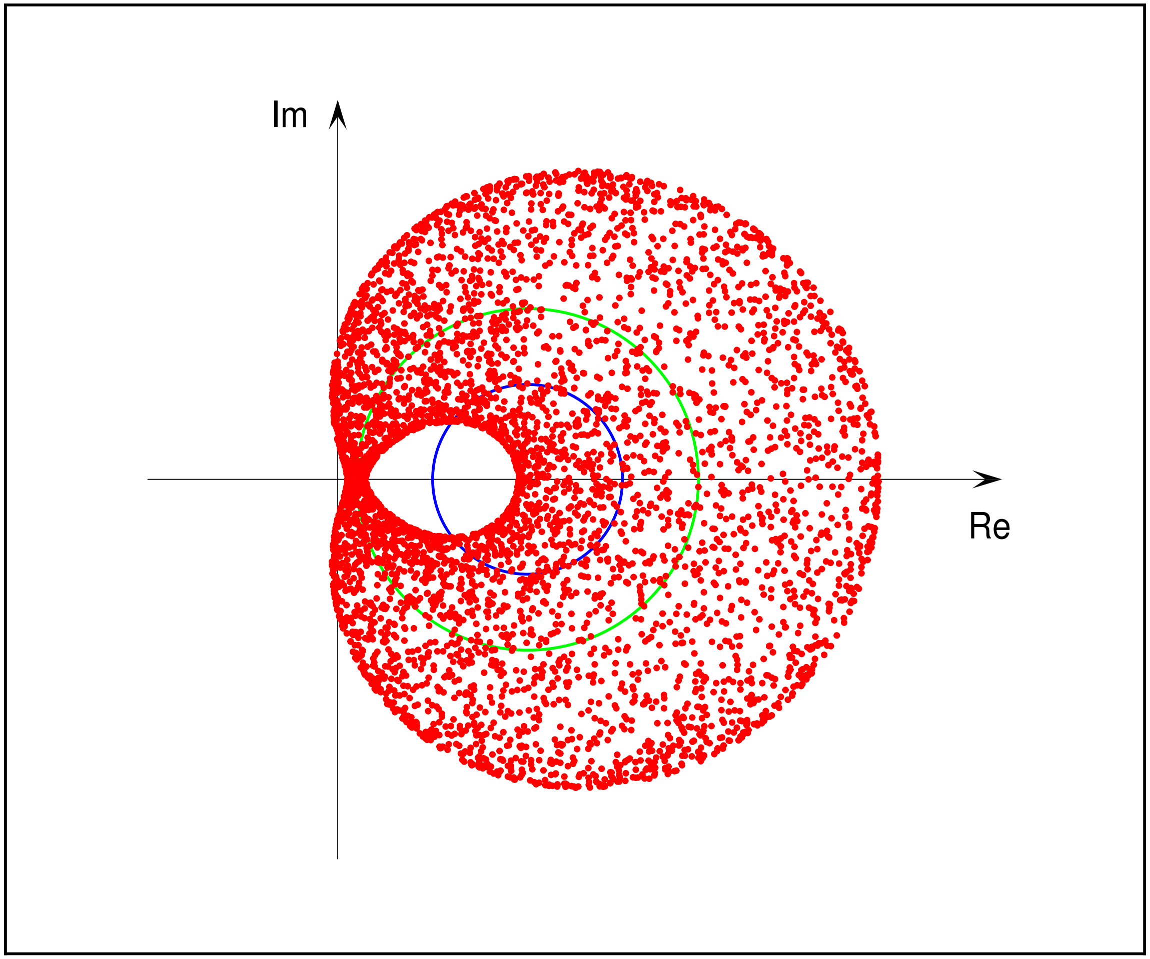
$$\mathcal{D} \;=\; \mathcal{M} \oplus_r \mathcal{S}_1 \;=\; \{\, \mathbf{m} + r(\mathbf{m})\mathbf{s} \mid \mathbf{m} \in \mathcal{M},\; \mathbf{s} \in \mathcal{S}_1 \,\}$$

$\mathcal{M}$ = medial axis, r = radius function on $\mathcal{M}$

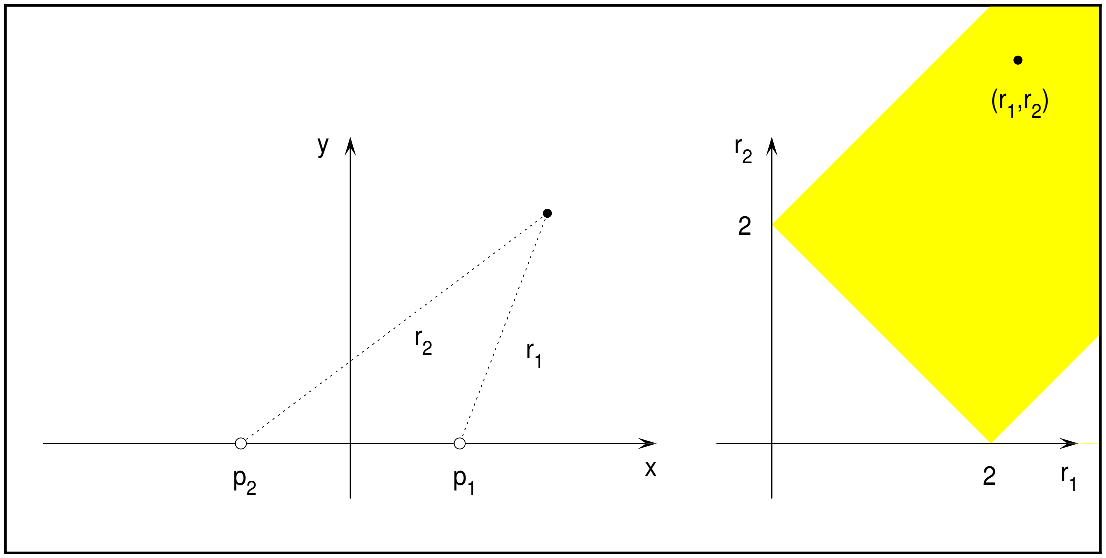
offset curves & medial axis transform



Monte Carlo experiment – product of two circles



bipolar coordinates



ellipse & hyperbola : $\quad r_1 \pm r_2 = k$

the ovals of Cassini : $\quad r_1 r_2 = k$

the Cartesian oval(s) : $\quad m r_1 \pm n r_2 = \pm 1$

generalize to (redundant) multipolar coordinates

Cartesian oval $\mathcal{C}_1 \otimes \mathcal{C}_2$

$\mathcal{C}_1, \mathcal{C}_2$ have center (1,0) and radii R_1, R_2

poles $(0,0)$, $(a_1, 0)$, $(a_2, 0)$ where $a_1 = 1 - R_1^2$, $a_2 = 1 - R_2^2$

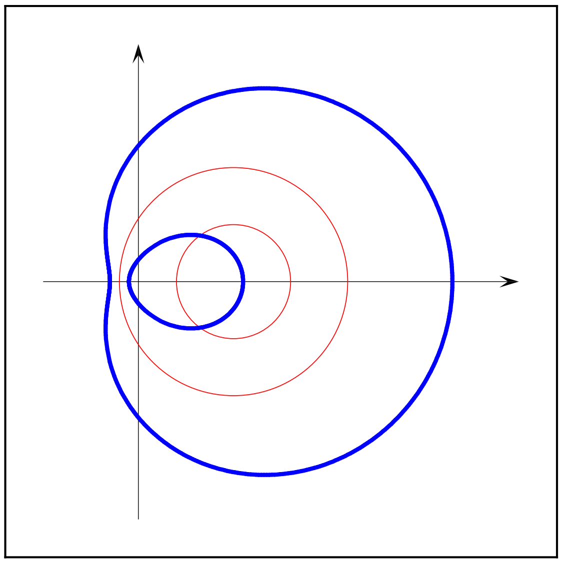
(a_1, a_2 = images of 0 under inversion in $\mathcal{C}_1, \mathcal{C}_2$)

three different representations in bipolar coordinates:

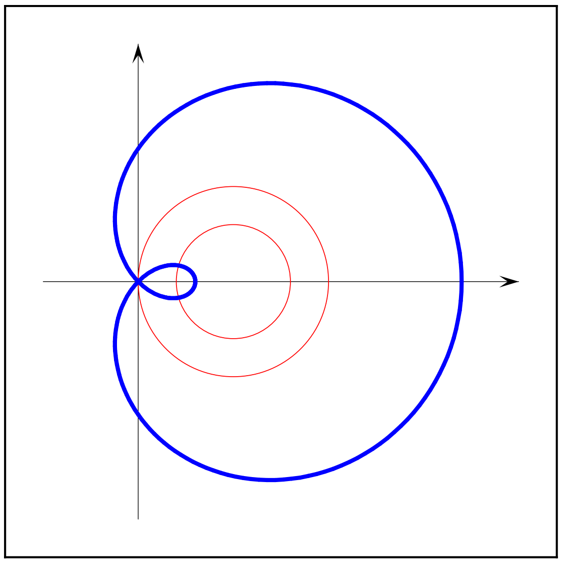
$$
\begin{aligned}
R_1\rho_0 \pm \rho_1 &= \pm a_1 R_2 \\
R_2\rho_0 \pm \rho_2 &= \pm a_2 R_1 \\
R_2\rho_1 \pm R_1\rho_2 &= \pm (a_2 - a_1)
\end{aligned}
$$

degenerate cases — limacon of Pascal & cardioid

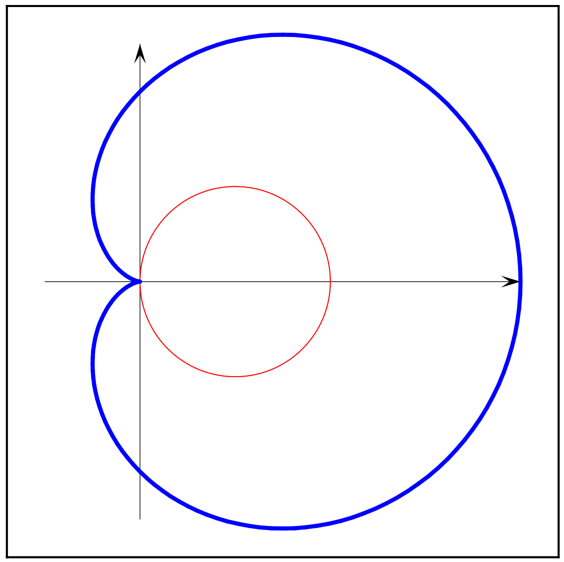
Cartesian oval is an anallagmatic curve
(maps into itself under inversion in a circle)



Cartesian oval: $R_1 \neq 1 \neq R_2$



limacon of Pascal: $R_1 = 1 \neq R_2$

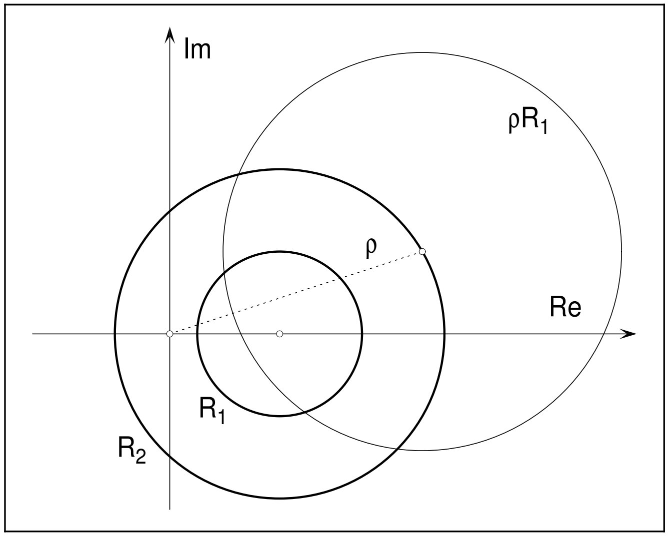


cardioid: $R_1 = 1 = R_2$

Cartesian ovals

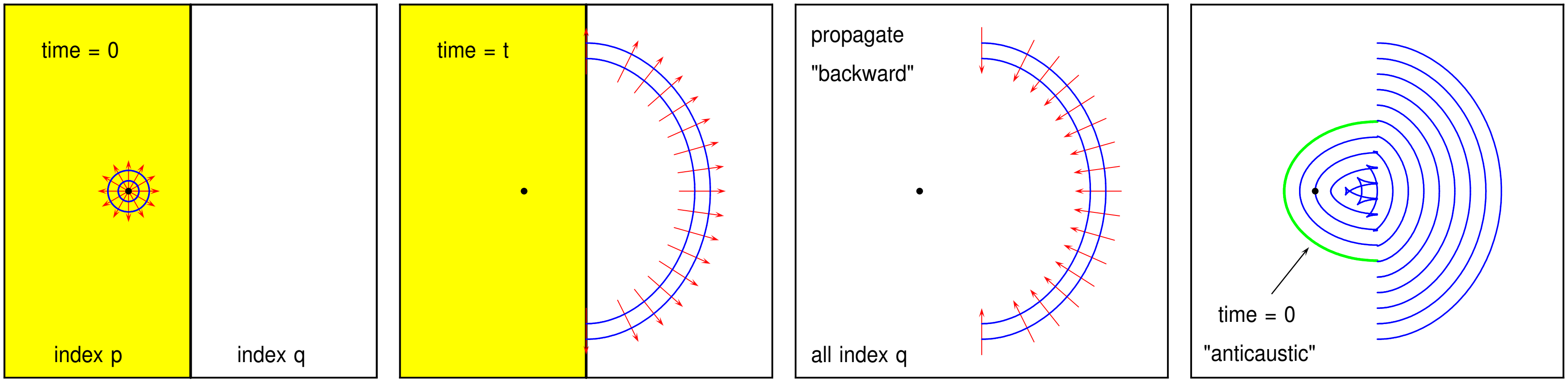
"L'enveloppe d'un cercle variable dont le centre parcourt la circonférence d'un autre cercle donné et dont le rayon varie proportionnellement à la distance de son centre à un point fixe est un couple d'ovales de Descartes."

F. Gomes Teixiera (1905)
Traité des Courbes Spéciales Remarquables Planes et Gauches



Cartesian oval $=$ boundary of Minkowski product of two circles

anticaustic — Jakob Bernoulli (1692)

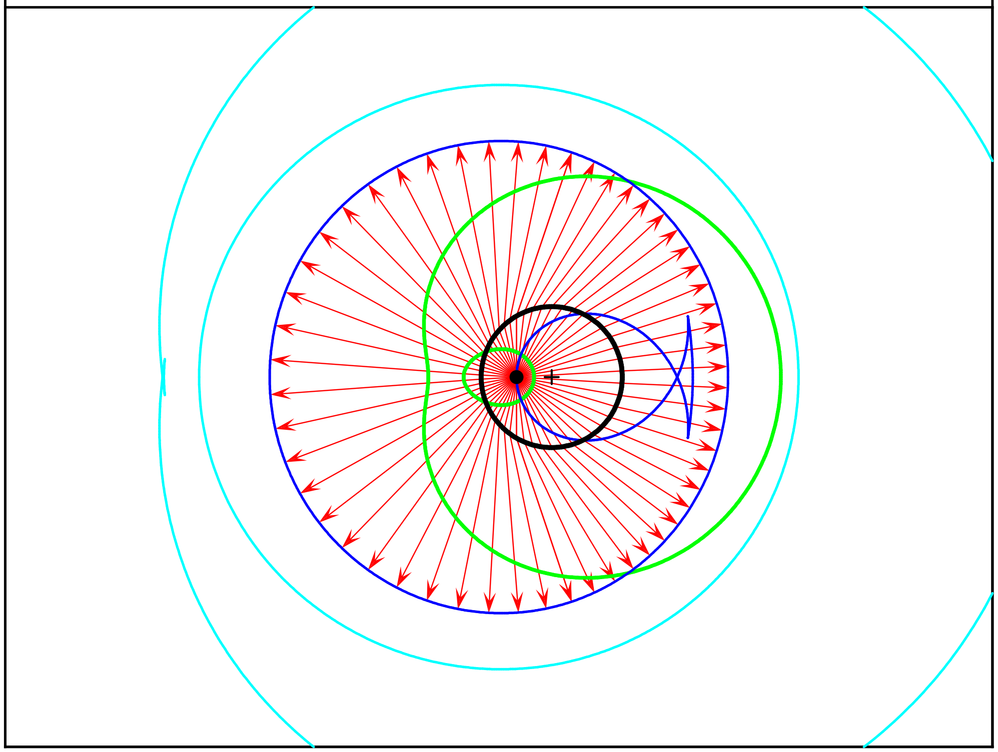


anticaustic = involute of caustic (zero optical path length)

reflection/refraction of spherical waves

surface	mode	anticaustic	wavefront
plane	reflect	point	degree 2
plane	refract	ellipse/hyperbola	degree 8
sphere	reflect	limacon of Pascal	degree 10
sphere	refract	Cartesian oval	degree 14

Farouki & Chastang, Exact equations of "simple" wavefronts, *Optik* 91, 109 (1992)



geometrical optics

"operator language" for optical constructions

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0 = light source, $\mathcal{A}$ = smooth refracting surface, k = refractive index ratio

$\Rightarrow$ anticaustic $\mathcal{S}$ for refraction of spherical waves $= \partial(\mathcal{A} \otimes \mathcal{C})$

where $\mathcal{C}$ = circle with center 1 & radius k^{-1}

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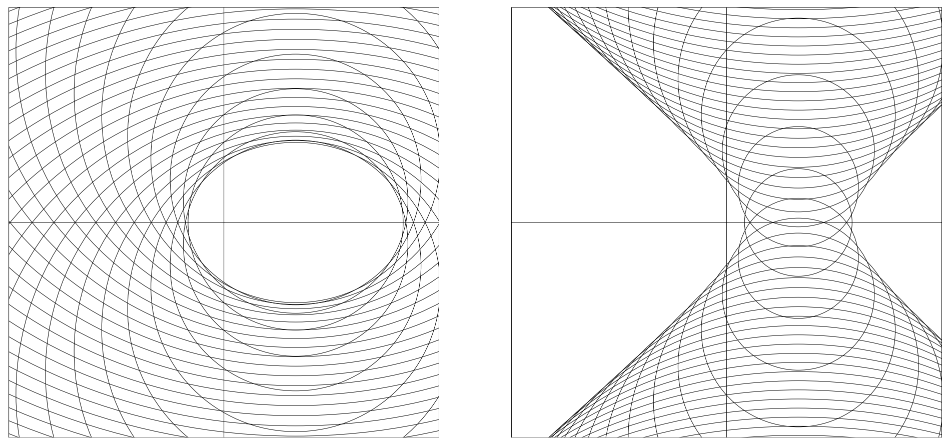
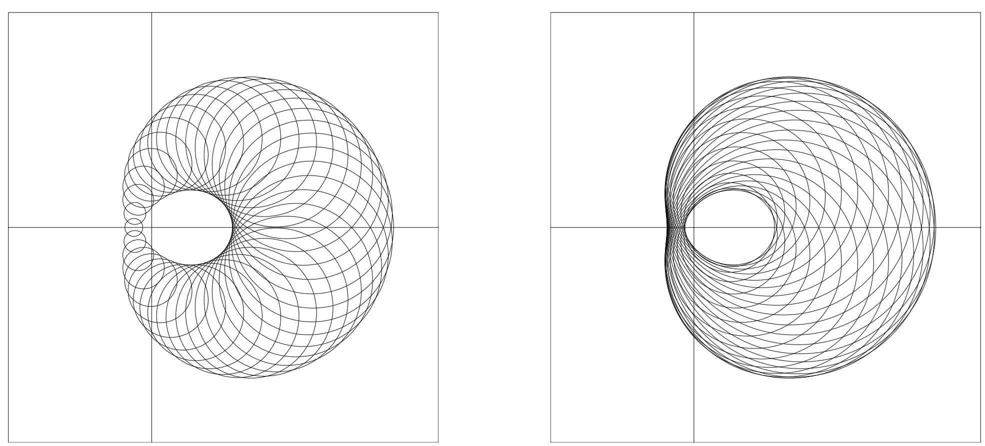
0 = light source, $\mathcal{L}$ = line with $\mathrm{Re}(\mathbf{z}) = 1$, $\mathcal{S}$ = desired anticaustic

$\Rightarrow$ mirror $\mathcal{M}$ yielding anticaustic $\mathcal{S}$ by reflection $= \frac{1}{2}\partial(\mathcal{S} \otimes \mathcal{L})$

simple Minkowski product examples

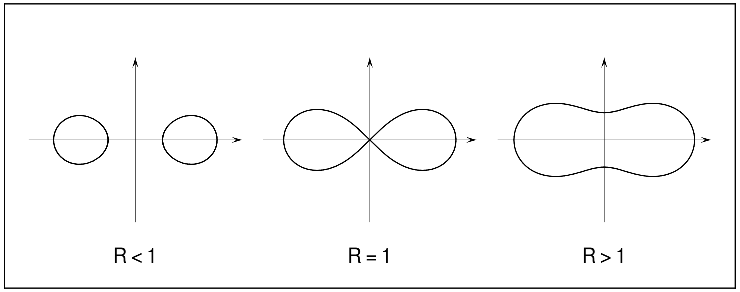


line $\otimes$ circle — ellipse or hyperbola

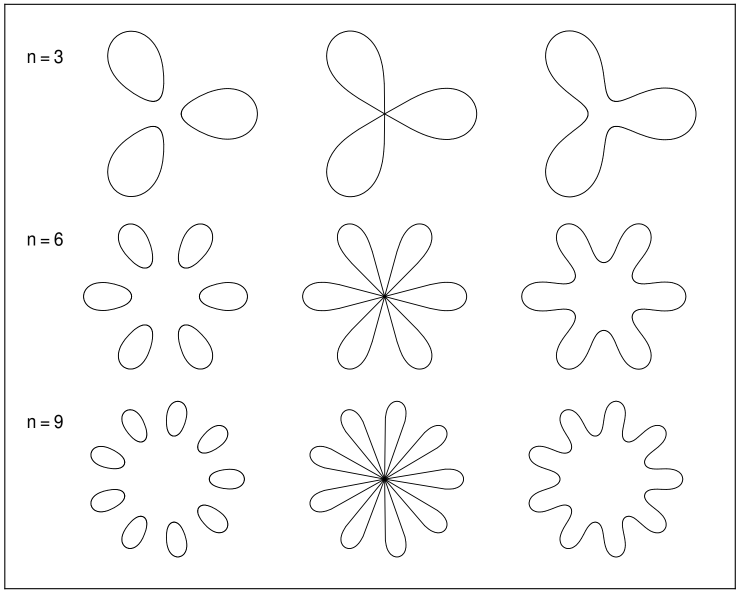


circle $\otimes$ circle — Cartesian oval ($R_1, R_2 \neq 1$ here)

Minkowski roots – ovals of Cassini

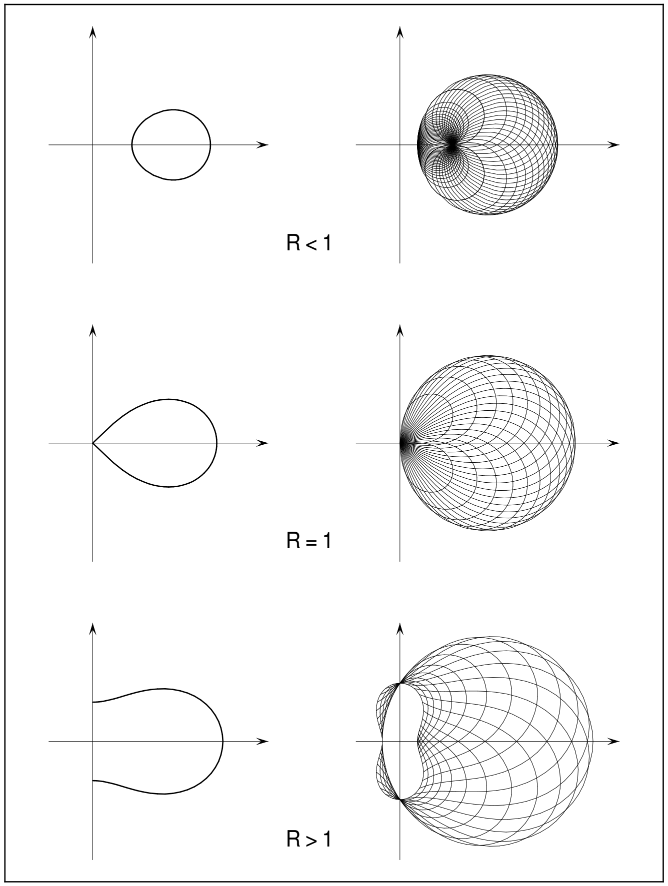


"ordinary": $r_1 r_2 = R$ or $r^4 - 2r^2 \cos\theta + 1 = R^2$



n^{th} order: $r_1 \cdots r_n = R$ or $r^{2n} - 2r^n \cos n\theta + 1 = R^2$

$\boxed{\bigotimes^{1/2}\ \textbf{circle}}$



circle containing origin is *not* logarithmically convex
— require *composite* curve as Minkowski root

catalog of Minkowski operations

set operation	set boundary
line $\otimes$ line	parabola
line $\otimes$ circle	ellipse or hyperbola
circle $\otimes$ circle	Cartesian oval
$\otimes^{1/2}$ disk	ovals of Cassini
$\otimes^{1/n}$ disk	n^{th} order ovals of Cassini
line $\otimes$ curve	negative pedal of curve wrt origin
circle $\otimes$ curve	anticaustic for refraction by curve
circle $\otimes \cdots \otimes$ circle	generalized Cartesian oval
disk $\otimes$ $\mathcal{A}$ = disk	$\partial\mathcal{A}$ = inner loop of Cartesian oval

. . . **the three Russian brothers** . . .

. . . Following the collapse of the former Soviet Union, the economy in Russia hit hard times, and jobs were difficult to find. Dmitry, Ivan, and Alexey — the Brothers Karamazov — therefore decided to seek their fortunes by emigrating to America, England, Australia . . .

Minkowski product algorithm

$\mathbf{z} \to \log \mathbf{z}$: Minkowski product $\to$ Minkowski sum

for curves $\boldsymbol{\gamma}(t)$, $\boldsymbol{\delta}(u)$ write $\boldsymbol{\gamma}(t) \otimes \boldsymbol{\delta}(u) \,=\, \exp\left(\log \boldsymbol{\gamma}(t) \oplus \log \boldsymbol{\delta}(u)\right)$
and then invoke Minkowski sum algorithm

problems $\Rightarrow$ work directly with $\boldsymbol{\gamma}(t)$ and $\boldsymbol{\delta}(u)$

1. $\log(\mathbf{z})$ defined on multi-sheet Riemann surface
2. $\exp(\mathbf{z})$ exaggerates any approximation errors
3. $\log \boldsymbol{\gamma}(t)$ & $\log \boldsymbol{\delta}(u)$ are transcendental curves

logarithmic curvature theory: for curve $\boldsymbol{\gamma}(t)$ define $\kappa_{\log}(t)$
$=$ ordinary curvature of image, $\log \boldsymbol{\gamma}(t)$, under $\mathbf{z} \to \log \mathbf{z}$

hence . . . *logarithmic lines, inflections, convexity, Gauss map, etc.*

ordinary & logarithmic curvature of $\gamma(t)$

$$r(t) = |\boldsymbol{\gamma}(t)|, \quad \theta(t) = \arg \boldsymbol{\gamma}(t), \quad \psi(t) = \arg \boldsymbol{\gamma}'(t)$$

$$\kappa = \frac{\mathrm{d}\psi}{\mathrm{d}s} \quad \text{invariant under } \textit{translation}, \text{ but not } \textit{scaling}$$

$$\kappa_{\log} = r\,\frac{\mathrm{d}}{\mathrm{d}s}(\psi - \theta) \quad \text{invariant under } \textit{scaling}, \text{ but not } \textit{translation}$$

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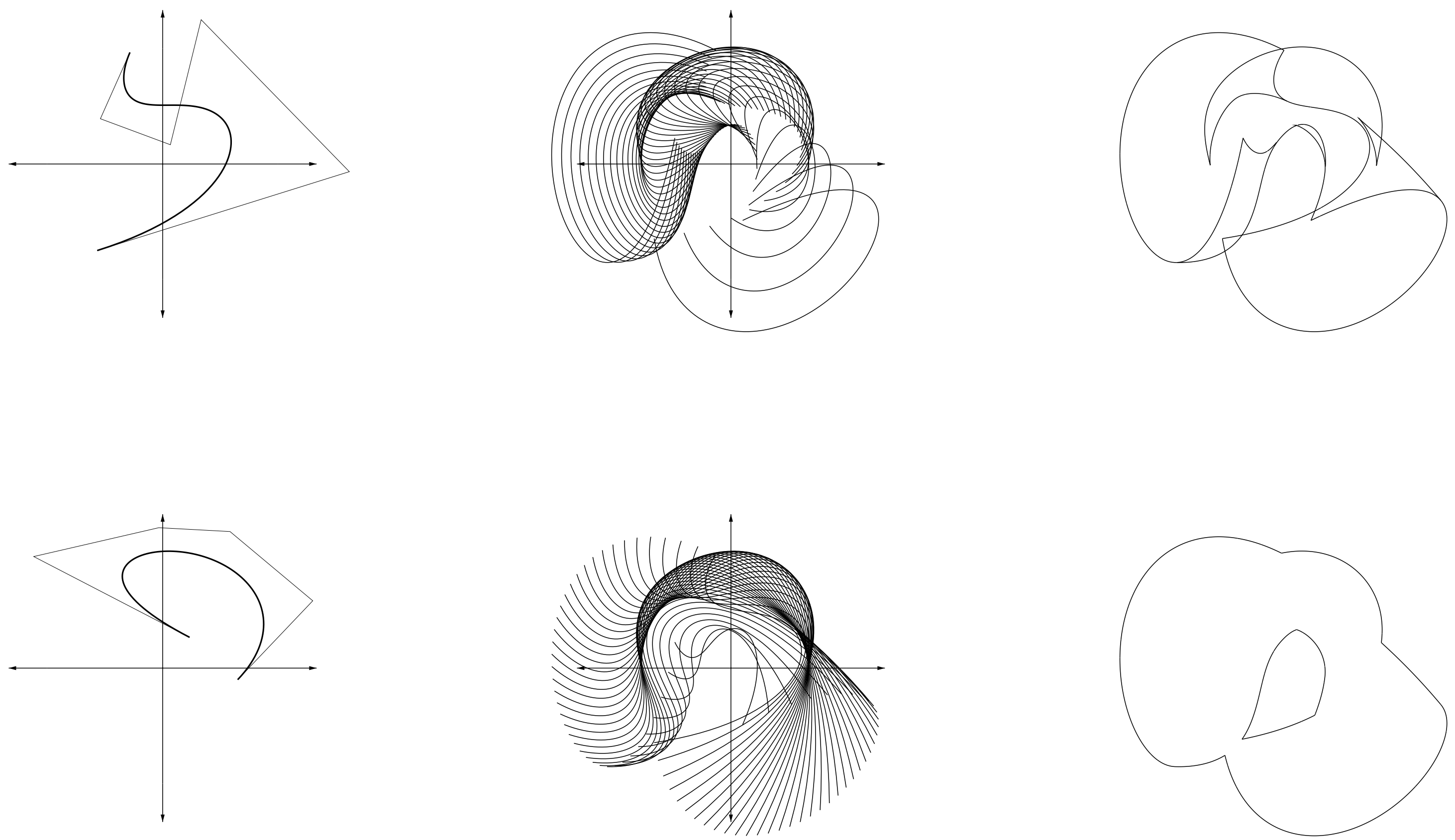
1. compute *logarithmic Gauss maps* of $\boldsymbol{\gamma}(t)$ & $\boldsymbol{\delta}(u)$
2. subdivide $\boldsymbol{\gamma}(t)$ & $\boldsymbol{\delta}(u)$ into corresponding *log-convex segments*
3. simultaneously trace corresponding segments and generate candidate edges for Minkowski product boundary
4. test edges for status (interior/boundary) w.r.t. Minkowski product
5. establish orientation & ordering of retained boundary edges

Minkowski product example

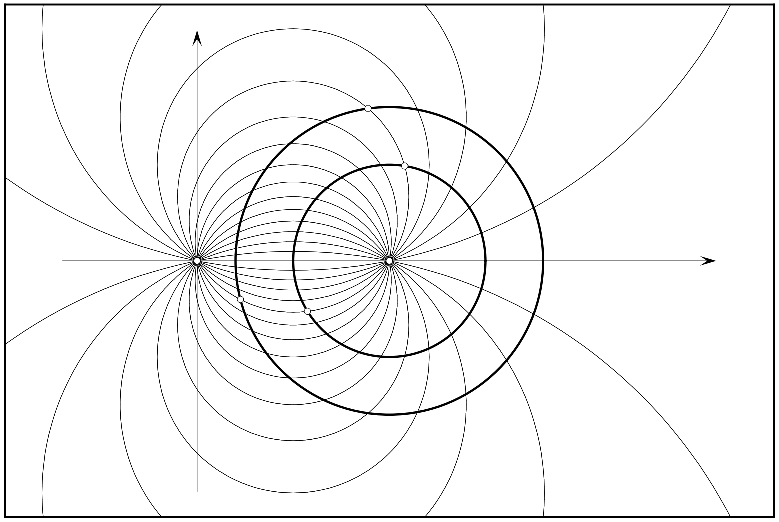


left: quintic Bézier curve operands; center: products of one operand with points of other; right: untrimmed & trimmed Minkowski product boundary

Minkowski product of N circles

match logarithmic Gauss maps : $$\frac{\sin\theta_1}{R_1+\cos\theta_1} = \cdots = \frac{\sin\theta_N}{R_N+\cos\theta_N}$$

geometrical interpretation: intersections of operands with circles of coaxal system (common points 0 & 1)



proof — inversion in operand circles

$\partial(\mathcal{C}_1 \otimes \cdots \otimes \mathcal{C}_N)$ = "$N^{\rm th}$ order Cartesian oval"

multipolar representation with respect to poles at $0, a_1, a_2, \ldots, a_N$?

implicitly-defined complex sets

$$\mathcal{A} \,\textcircled{\scriptsize f}\, \mathcal{B} \;=\; \{\; \mathbf{f}(\mathbf{a},\mathbf{b}) \mid \mathbf{a} \in \mathcal{A},\, \mathbf{b} \in \mathcal{B} \;\}$$

example: $\mathbf{f}(\mathbf{a},\mathbf{b}) = \mathbf{ab} + \mathbf{b}^2$ and $\mathcal{A}, \mathcal{B}$ $=$ disks $|\mathbf{z}| \le 1$, $|\mathbf{z}-1| \le 1$

subdistributivity $\Rightarrow$ $\mathcal{A} \,\textcircled{\scriptsize f}\, \mathcal{B} \subseteq (\mathcal{A} \oplus \mathcal{B}) \otimes \mathcal{B} \subseteq (\mathcal{A} \otimes \mathcal{B}) \oplus (\mathcal{B} \otimes \mathcal{B})$

set $\mathbf{a}(\lambda) = \mathrm{e}^{\mathrm{i}\lambda}$ and $\mathbf{b}(t) = 1 + \mathrm{e}^{\mathrm{i}t}$ for $0 \le \lambda, t \le 2\pi$ in $\mathbf{f}(\mathbf{a},\mathbf{b})$

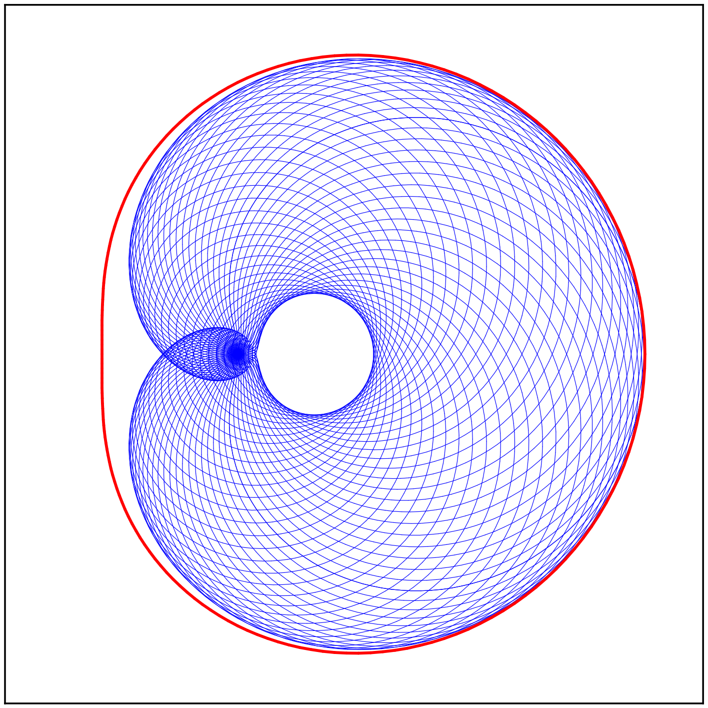
$\rightarrow$ family of limacons $\quad \mathbf{r}(\lambda,t) \;=\; \mathrm{e}^{\mathrm{i}2t} + \mathrm{e}^{\mathrm{i}(t+\lambda)} + 2\mathrm{e}^{\mathrm{i}t} + \mathrm{e}^{\mathrm{i}\lambda} + 1$

---

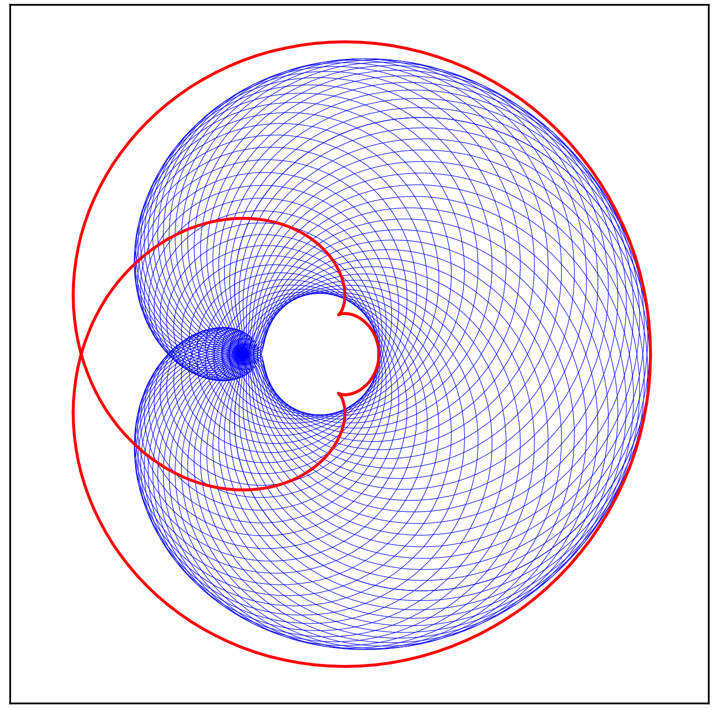
generalize Minkowski sum & product algorithms to $\mathcal{A} \,\textcircled{\scriptsize f}\, \mathcal{B}$:

matching condition $\quad \arg\dfrac{\mathrm{d}\mathbf{a}}{\mathrm{d}\lambda} - \arg\dfrac{\mathrm{d}\mathbf{b}}{\mathrm{d}t} = k\pi + \arg\left(\dfrac{\mathrm{d}\mathbf{b}}{\mathrm{d}\mathbf{a}}\right)_{\mathbf{f}=\mathrm{const.}}$

implicitly–defined set bounded by Minkowski combinations

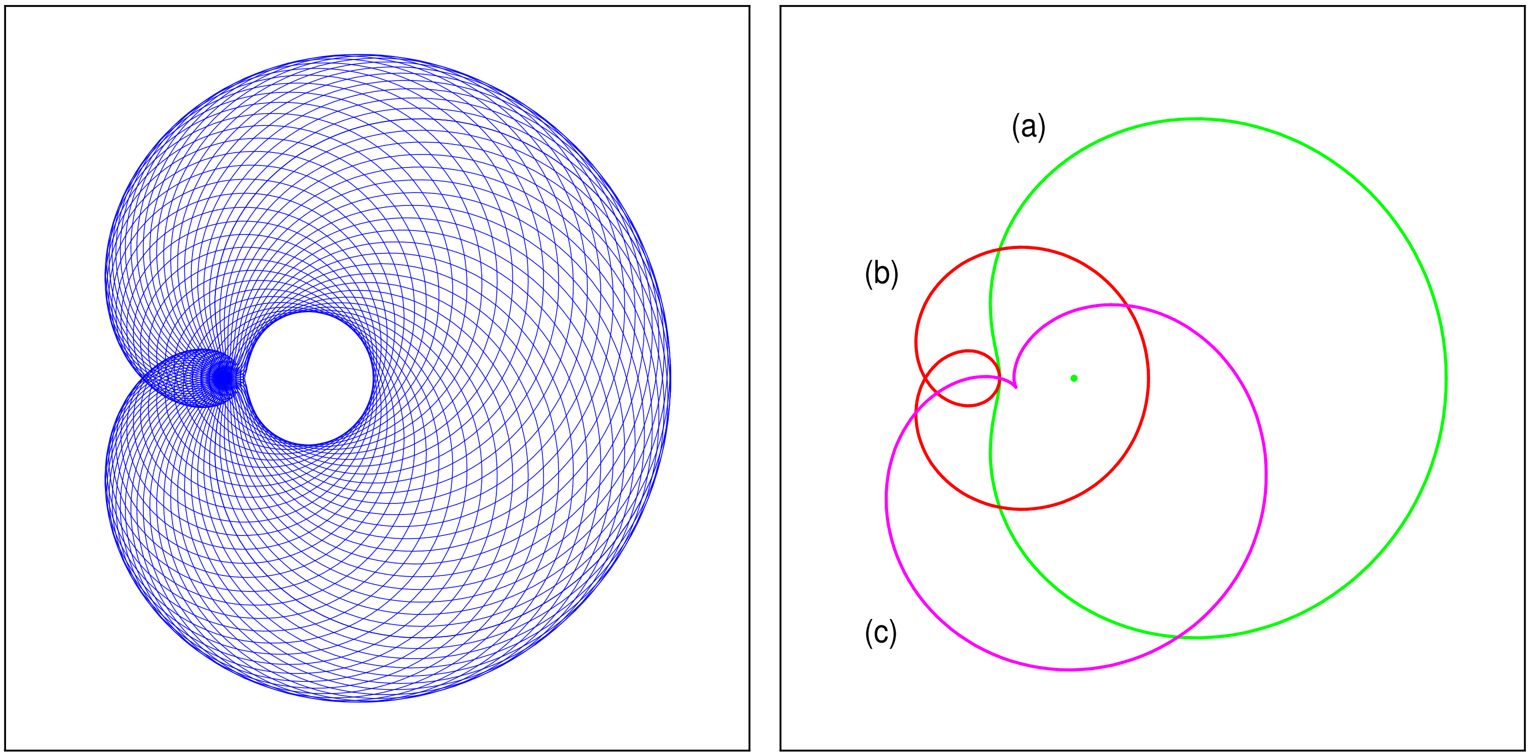


limacon of Pascal

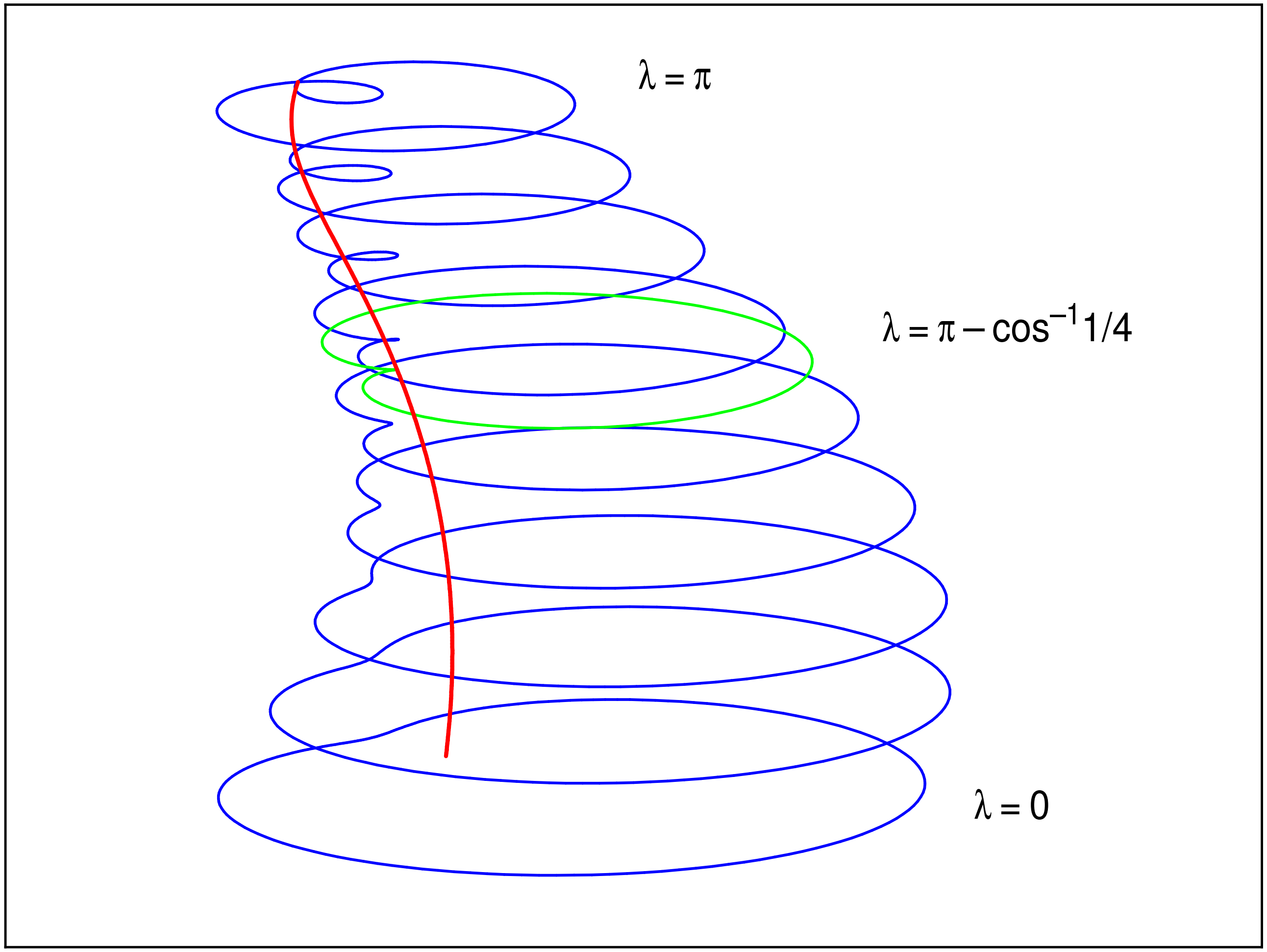


offset to cardioid

implicitly–defined set as one–parameter family of limacons



(a): acnodal (b): crunodal (c): cuspidal

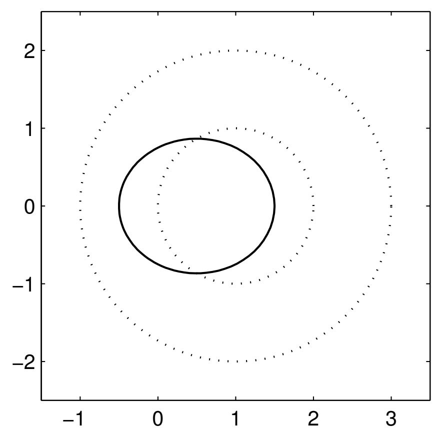


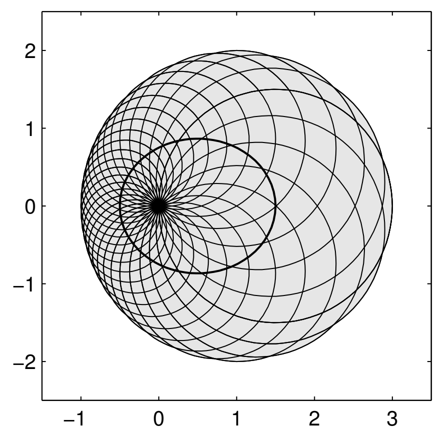
singular curve of surface $r(\lambda,t)$ generated by implicitly–defined set

$$\boxed{\textbf{solution of linear equation } \mathcal{A} \otimes \mathcal{X} = \mathcal{B}}$$

$\mathcal{A}$, $\mathcal{B}$ = circular disks with radii a, b

$$\boxed{\textbf{solution exists} \iff a \leq b}$$





solution $=$ region within inner loop of a Cartesian oval !

generalization to polynomial equations, linear systems?

stability of linear dynamic system

Laplace transform of linear n^{th} order system:

$$a_n \frac{\mathrm{d}^n y}{\mathrm{d}t^n} + \cdots + a_1 \frac{\mathrm{d}y}{\mathrm{d}t} + a_0 y = 0$$

characteristic polynomial $\mathbf{p}(\mathbf{s}) = a_n \mathbf{s}^n + \cdots + a_1 \mathbf{s} + a_0$

stability $\Longleftrightarrow$ roots $\mathbf{z}_1, \ldots, \mathbf{z}_n$ satisfy $\mathrm{Re}(\mathbf{z}_k) < 0$

$$\Delta_n = \begin{vmatrix} a_1 & a_3 & a_5 & a_7 & a_9 & \cdot & \cdot & \cdot \\ a_0 & a_2 & a_4 & a_6 & a_8 & \cdot & \cdot & \cdot \\ 0 & a_1 & a_3 & a_5 & a_7 & \cdot & \cdot & \cdot \\ 0 & a_0 & a_2 & a_4 & a_6 & \cdot & \cdot & \cdot \\ 0 & 0 & a_1 & a_3 & a_5 & \cdot & \cdot & \cdot \\ 0 & 0 & a_0 & a_2 & a_4 & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{vmatrix}$$

classical Routh-Hurwitz criterion: $\Delta_n, \Delta_{n-1}, \ldots, \Delta_1 > 0$

(can generalize to *complex* coefficients $\mathbf{a}_0, \ldots, \mathbf{a}_n$)

Kharitonov conditions

desire "robust stability" of system with uncertain parameters

$$\mathbf{p}(\mathbf{s}) = a_n\mathbf{s}^n + \cdots + a_1\mathbf{s} + a_0 \quad \text{where } a_k \in [\,\underline{a}_k, \overline{a}_k\,]$$

$$\begin{aligned}
\mathbf{p}_1(\mathbf{s}) &= \underline{a}_0 + \underline{a}_1\mathbf{s} + \overline{a}_2\mathbf{s}^2 + \overline{a}_3\mathbf{s}^3 + \cdots \\
\mathbf{p}_2(\mathbf{s}) &= \underline{a}_0 + \overline{a}_1\mathbf{s} + \overline{a}_2\mathbf{s}^2 + \underline{a}_3\mathbf{s}^3 + \cdots \\
\mathbf{p}_3(\mathbf{s}) &= \overline{a}_0 + \underline{a}_1\mathbf{s} + \underline{a}_2\mathbf{s}^2 + \overline{a}_3\mathbf{s}^3 + \cdots \\
\mathbf{p}_4(\mathbf{s}) &= \overline{a}_0 + \overline{a}_1\mathbf{s} + \underline{a}_2\mathbf{s}^2 + \underline{a}_3\mathbf{s}^3 + \cdots
\end{aligned}$$

Kharitonov polynomials $\mathbf{p}_1(\mathbf{s}), \ldots, \mathbf{p}_4(\mathbf{s})$ stable $\Longleftrightarrow$ $\mathbf{p}(\mathbf{s})$ "robustly stable"

Kharitonov, *Differential'nye Uraveniya* **14**, 1483 (1978)

value set : $\mathcal{V}(\mathbf{p}(\mathbf{s})) =$ values assumed by $\mathbf{p}(\mathbf{s})$ at fixed $\mathbf{s}$ as coeffs a_k vary over intervals $[\,\underline{a}_k, \overline{a}_k\,] =$ rectangle with corners $\mathbf{p}_1(\mathbf{s}), \ldots, \mathbf{p}_4(\mathbf{s})$

(complex coeffs — *eight* Kharitonov polynomials)

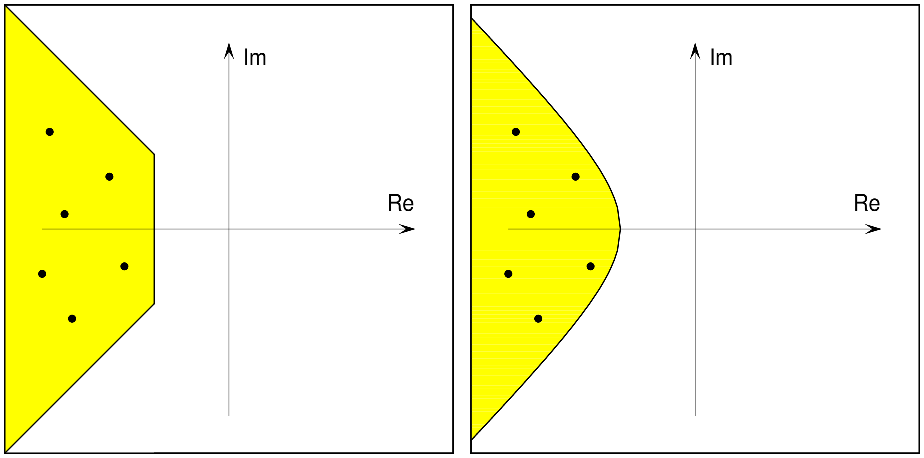
Γ-stability of system

roots $\mathbf{z}_1, \ldots, \mathbf{z}_n$ of characteristic polynomial with coeffs $\mathbf{a}_k \in \mathcal{A}_k$

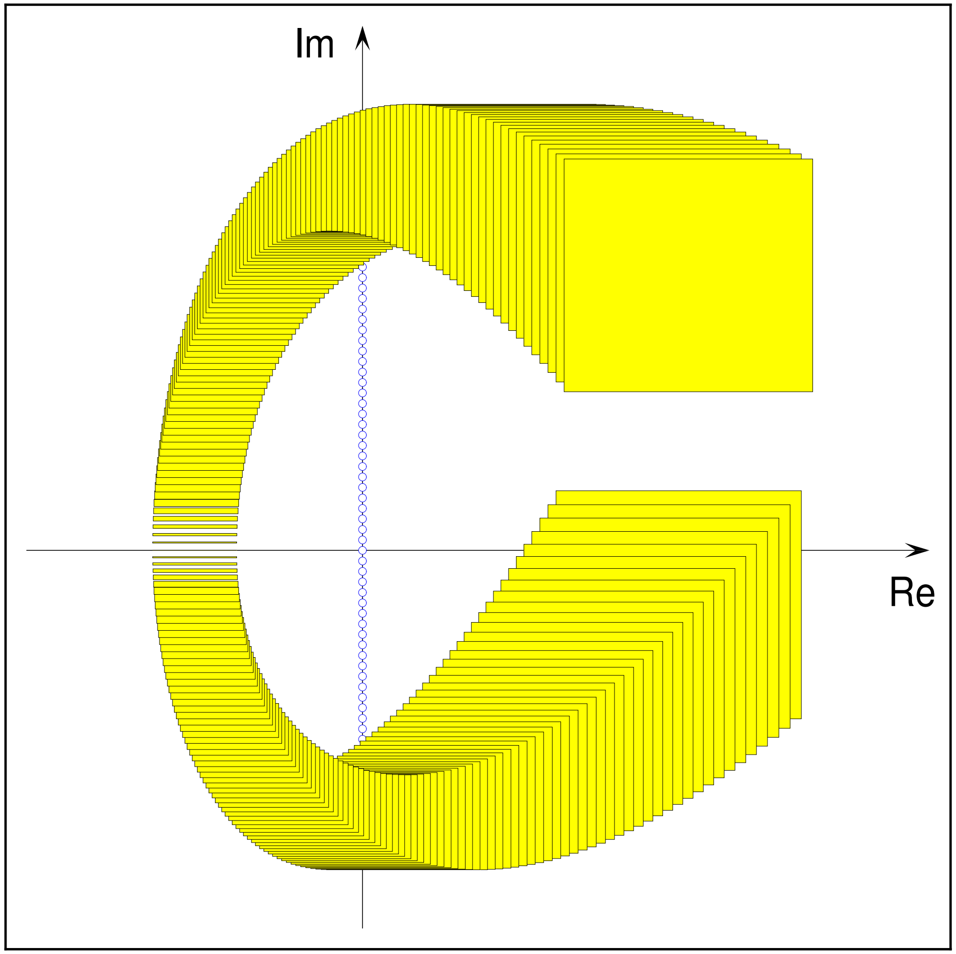
$$\mathbf{p}(\mathbf{s}) \;=\; \mathbf{a}_n \mathbf{s}^n \,+\, \cdots \,+\, \mathbf{a}_1 \mathbf{s} \,+\, \mathbf{a}_0$$

Hurwitz stability $\mathrm{Re}(\mathbf{z}_k) < 0$ may be inadequate;
also desire good damping and fast response

for any subset Γ of left half-plane, $\mathbf{p}(\mathbf{s})$ is Γ-stable if $\mathbf{z}_1, \ldots, \mathbf{z}_n \in \Gamma$



$\mathbf{p}(\mathbf{s})$ "robustly" Γ-stable $\iff$ one case Γ-stable, and value set satisfies $0 \notin \mathcal{V}(\mathbf{p}(\mathbf{s}))$ for all $\mathbf{s} \in \partial\Gamma$ (zero exclusion principle)



variation of value–set along the imaginary axis
for a cubic polynomial with interval coefficients

example problem

consider Γ-stability of quadratic $\quad \mathbf{p}(\mathbf{s}) \,=\, \mathbf{a}_2\mathbf{s}^2 + \mathbf{a}_1\mathbf{s} + \mathbf{a}_0$

coefficients disks $\mathcal{A}_2, \mathcal{A}_1, \mathcal{A}_0$ have centers $\mathbf{c}_2 = 1$,
$\mathbf{c}_1 = p + q$, $\mathbf{c}_0 = pq$ and radii $R_2 = R_1 = R_0 = 0.25$

stability region Γ boundary: $\boldsymbol{\gamma}(t) = (-\cosh t, \sinh t)$, $-\infty < t < +\infty$

value set $\mathcal{V}(t)$ for $\mathbf{p}(\mathbf{s})$ along boundary $\boldsymbol{\gamma}(t)$
= family of disks with center curve & radius function

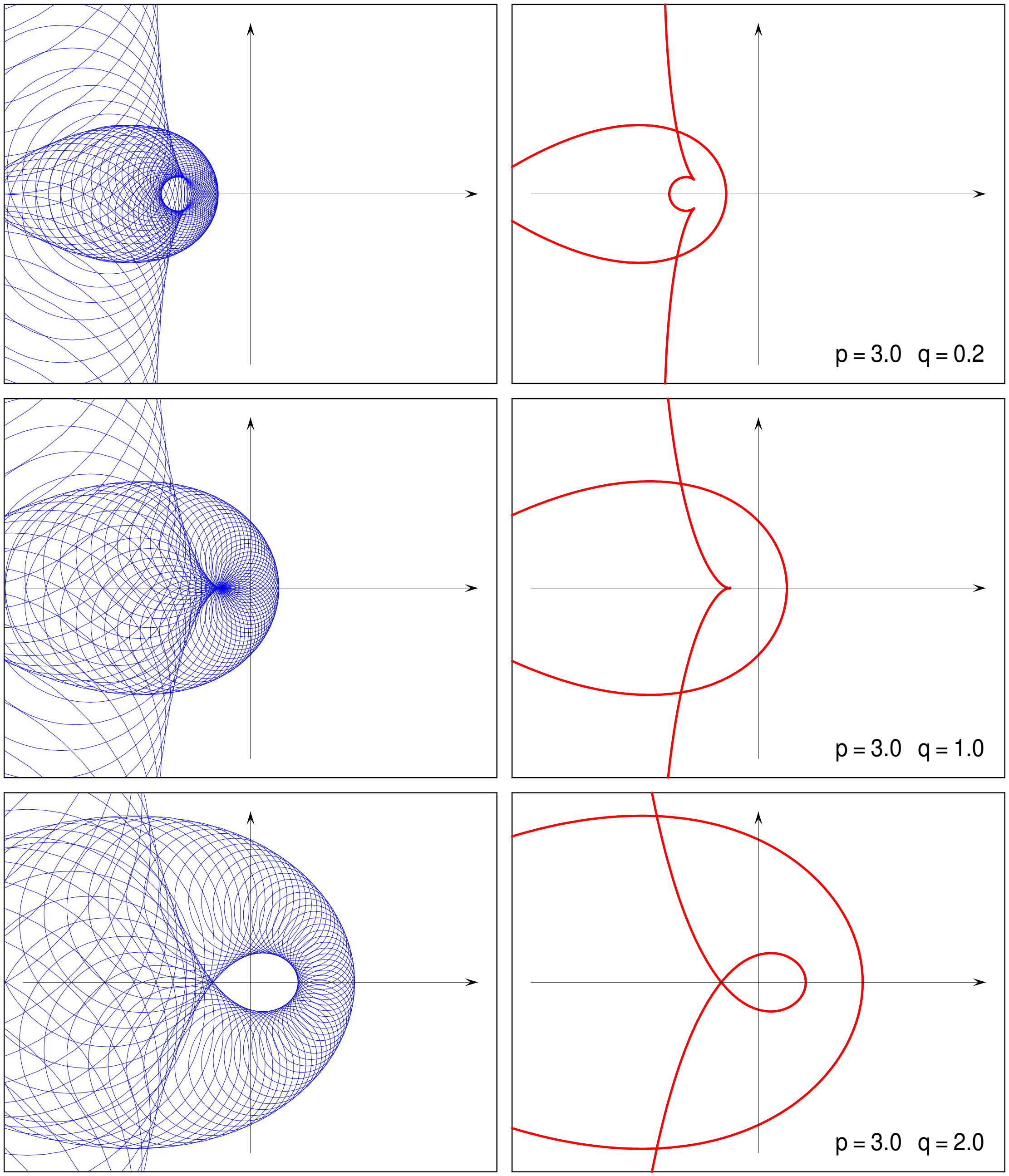
$$\mathbf{c}(t) = 1 + pq - (p+q)\cosh t \,+\, \mathrm{i}\,[\,(p+q) - 2\cosh t\,]\,\sinh t$$

$$R(t) = R_0(1 + \sqrt{\cosh 2t} + \cosh 2t)$$

---

stability condition: $0 \notin \mathcal{V}(t)$ for $-\infty < t < +\infty$
$\iff$ 2 *real* polynomials have no real roots

(true for *any* “complex disk polynomial”)


p = 3.0 q = 0.2
p = 3.0 q = 1.0
p = 3.0 q = 2.0

closure

- basic functions: Minkowski sums, products, roots,
  implicitly-defined complex sets, solution of equations

- lack of closure for finitely-describable sets
  $\rightarrow$ *rich geometrical structures & applications*

- 2D shape generation and analysis operators

- generalization of interval arithmetic to complex sets

- curves in bipolar & multipolar coordinates —
  generalize classical Cassini and Cartesian ovals

- operator language for direct & inverse
  problems of wavefront reflection & refraction

- robust stability of dynamic/control systems —
  extend Routh-Hurwitz & Kharitonov conditions