Spatial Pythagorean hodographs, quaternions, and rotations in $\mathbb{R}^3$ and $\mathbb{R}^4$

— a study in the evolution of scientific ideas —

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— synopsis —

- motivation — spatial Pythagorean-hodograph curves
- background — Hamilton and the origin of quaternions
- background — fundamentals of quaternion algebra
- basic theory — unit quaternions and rotations in $\mathbb{R}^3$
- historical interlude — vectors versus quaternions
- four dimensions — unit quaternions and rotations in $\mathbb{R}^4$
- any dimensions — Clifford algebra & medial axis transform

— bibliography —

- W. R. Hamilton, *Lectures on Quaternions* (1853), and posthumous *Elements of Quaternions* (1866) — extremely difficult reading

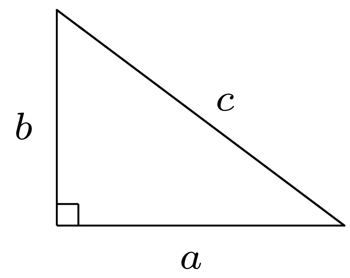
- M. J. Crowe (1967), *A History of Vector Analysis*, Dover, New York — history of struggle between vectors & quaternions

- S. L. Altmann (1986), *Rotations, Quaternions, and Double Groups*, Clarendon Press, Oxford — authoritative modern source

- P. Du Val (1964), *Homographies, Quaternions, and Rotations*, Clarendon Press, Oxford — authoritative modern source

- J. Roe (1993), *Elementary Geometry*, Oxford University Press — gentle introduction to quaternions

- A. J. Hanson (2005), *Visualizing Quaternions*, Morgan Kaufmann — applications to computer graphics



a, b, c = **real numbers**

choose any $a, b \;\to\; c = \sqrt{a^2 + b^2}$

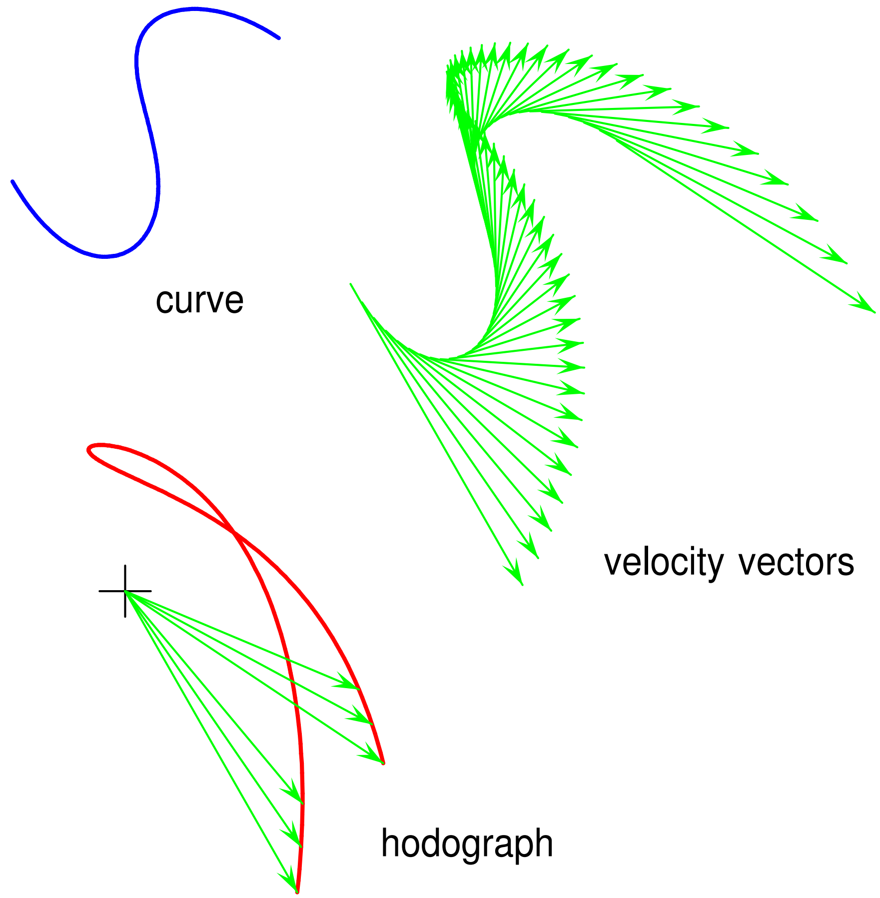
a, b, c = **integers**

$$a^2 + b^2 = c^2 \iff \begin{cases} a = (u^2 - v^2)w \\ b = 2uvw \\ c = (u^2 + v^2)w \end{cases}$$

$a(t), b(t), c(t)$ = **polynomials**

$$a^2(t) + b^2(t) \equiv c^2(t) \iff \begin{cases} a(t) = [\, u^2(t) - v^2(t)\,]\, w(t) \\ b(t) = 2\, u(t) v(t) w(t) \\ c(t) = [\, u^2(t) + v^2(t)\,]\, w(t) \end{cases}$$

hodograph of curve $\mathbf{r}(t)$ = derivative $\mathbf{r}'(t)$



Pythagorean structure — $x'^2(t) + y'^2(t) = \sigma^2(t)$

Pythagorean-hodograph (PH) curves

$\mathbf{r}(\xi) =$ PH curve in $\mathbb{R}^n \iff$ components of hodograph $\mathbf{r}'(\xi)$ are elements of a Pythagorean $(n+1)$–tuple of polynomials

PH curves incorporate special algebraic structures in their hodographs

- rational offset curves $\mathbf{r}_d(\xi) = \mathbf{r}(\xi) + d\,\mathbf{n}(\xi)$
- polynomial parametric speed $\sigma(\xi) = |\mathbf{r}'(\xi)| = \dfrac{\mathrm{d}s}{\mathrm{d}\xi}$
- polynomial arc-length function $s(\xi) = \displaystyle\int_0^\xi |\mathbf{r}'(\xi)|\,\mathrm{d}\xi$
- energy integral $E = \displaystyle\int_0^1 \kappa^2\,\mathrm{d}s$ has closed-form evaluation
- real-time CNC interpolators, rotation-minimizing frames, etc.

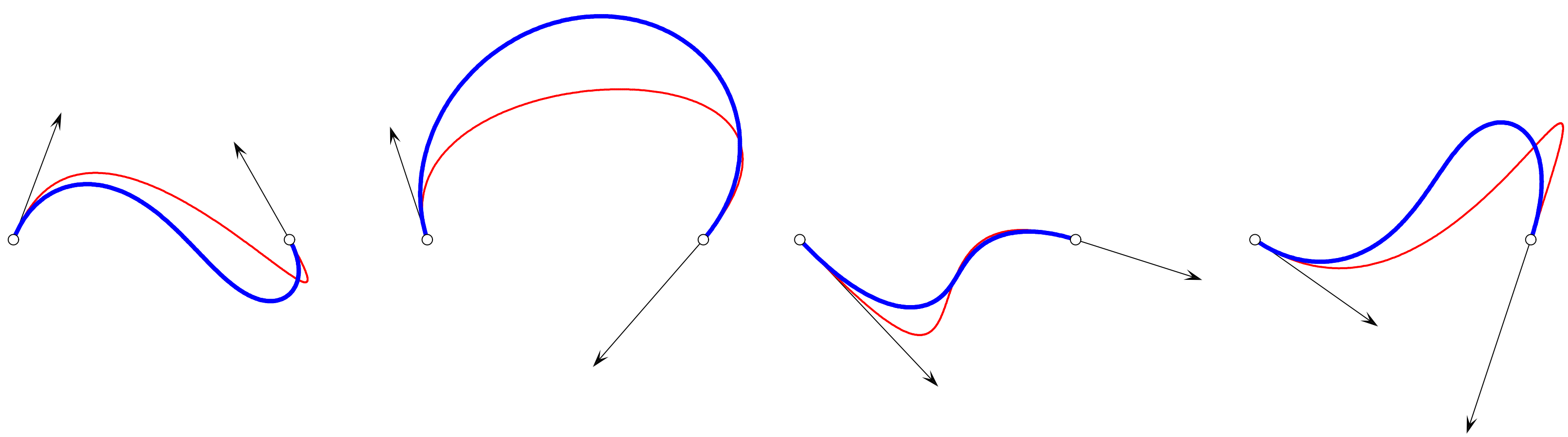
Pythagorean triples — planar PH curves

$$x'^2(t) + y'^2(t) = \sigma^2(t) \qquad \Longleftrightarrow \qquad \begin{cases} x'(t) = u^2(t) - v^2(t) \\ y'(t) = 2\,u(t)v(t) \\ \sigma(t) = u^2(t) + v^2(t) \end{cases}$$

complex model for planar PH curves: $\mathbf{w}(t) = u(t) + \mathrm{i}\,v(t)$

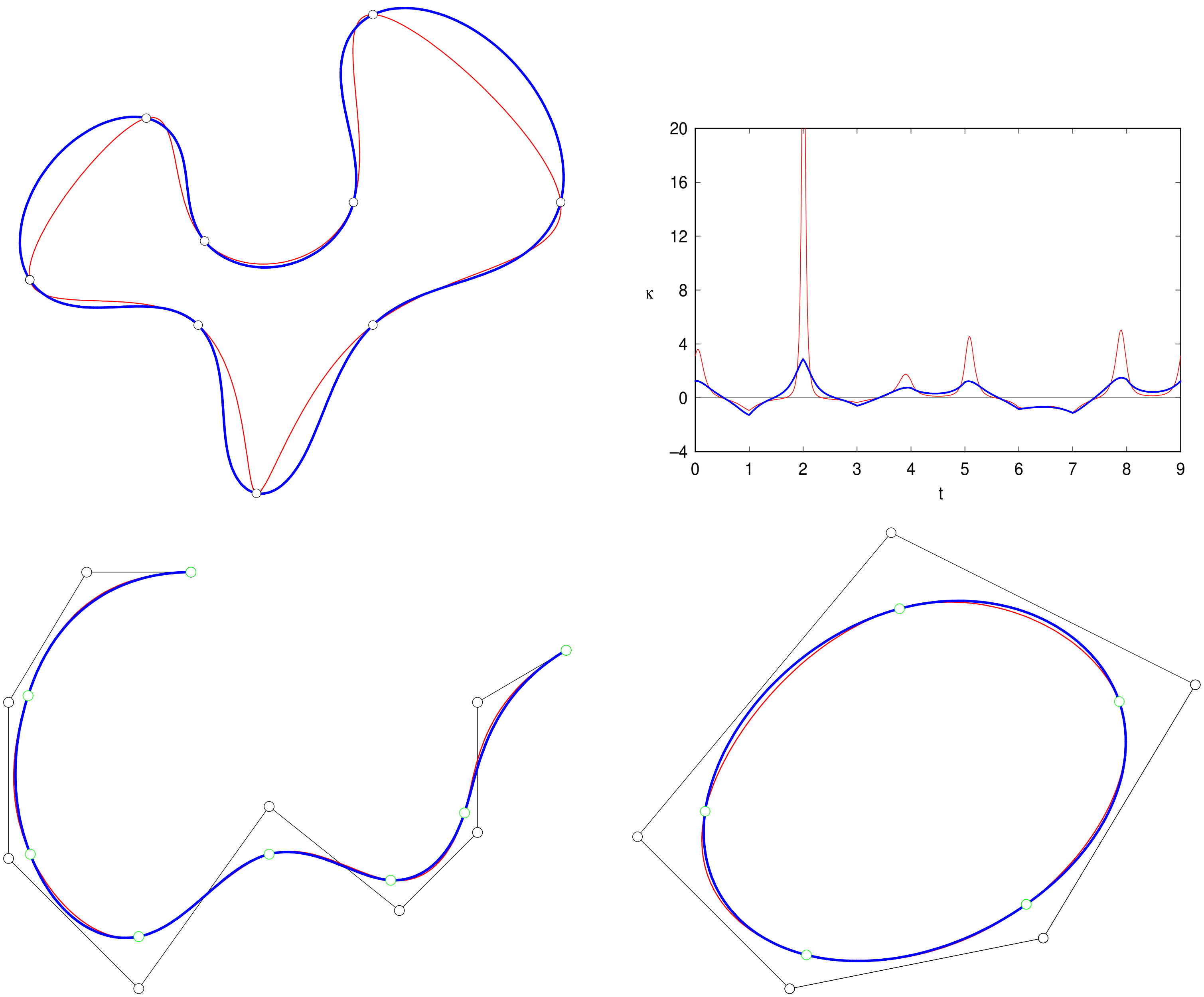
planar Pythagorean hodograph — $\mathbf{r}'(t) = (x'(t), y'(t)) = \mathbf{w}^2(t)$

- planar PH quintics can interpolate arbitrary Hermite data

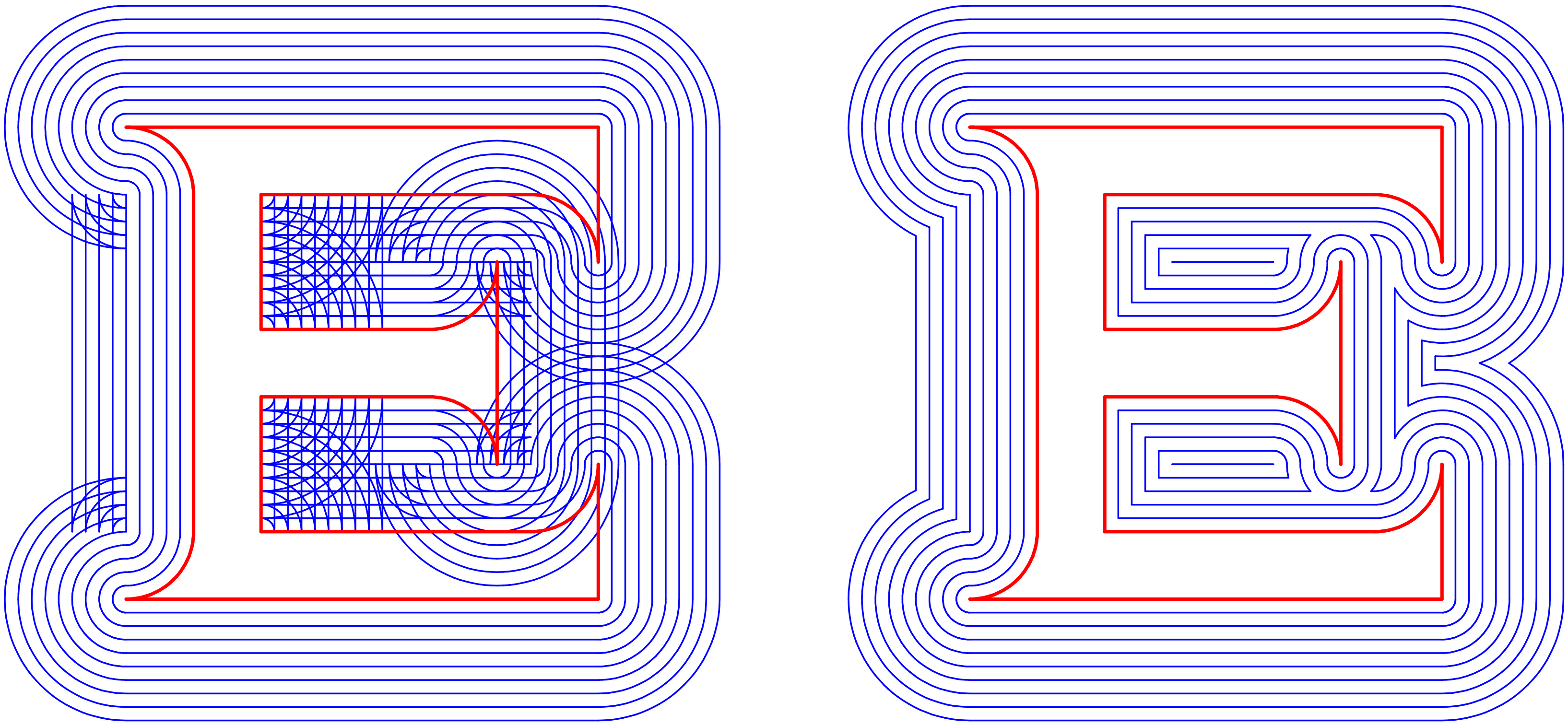


- extend to planar C^2 PH quintic splines — solve "tridiagonal" system of 2^N quadratic equations in N complex unknowns

comparison of PH quintic & “ordinary” cubic splines



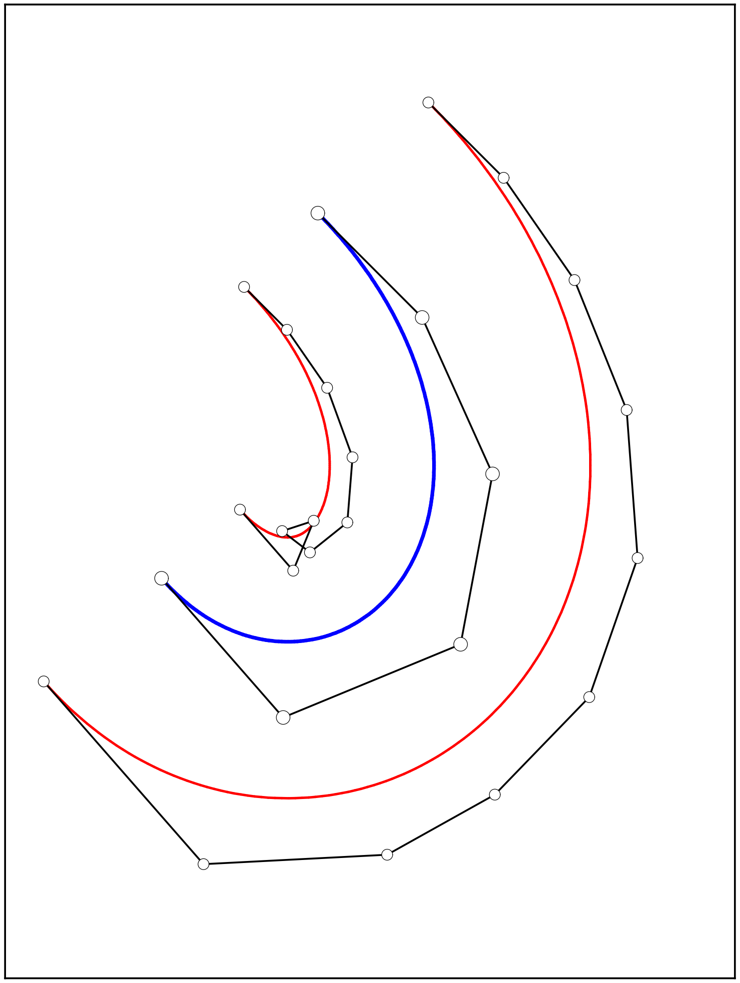
offset curves



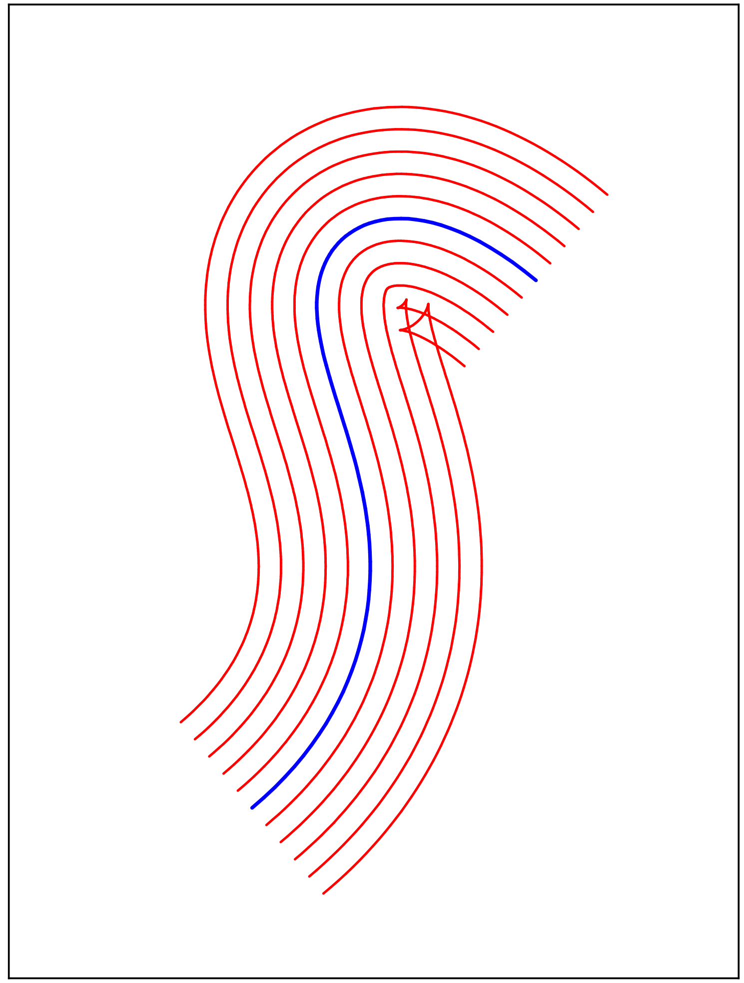
Left: untrimmed offsets obtained by sweeping a normal vector of length d around the original curve (including approrpiate rotations at vertices).

Right: trimmed offsets, obtained by deleting certain segments of the untrimmed offsets, that are not globally distance d from the given curve.

planar PH curves have rational offset curves for use as tool paths



Bezier control polygons of rational offsets



offsets exact at any distance

Pythagorean quartuples — spatial PH curves

$$x'^2(t) + y'^2(t) + z'^2(t) = \sigma^2(t) \quad \Longleftrightarrow \quad \left\{ \begin{array}{l} x'(t) = u^2(t) + v^2(t) - p^2(t) - q^2(t) \\ y'(t) = 2\,[\,u(t)q(t) + v(t)p(t)\,] \\ z'(t) = 2\,[\,v(t)q(t) - u(t)p(t)\,] \\ \sigma(t) = u^2(t) + v^2(t) + p^2(t) + q^2(t) \end{array} \right.$$

quaternion model for spatial PH curves

choose quaternion polynomial $\mathcal{A}(t) \;=\; u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$

spatial Pythagorean hodograph — $\mathbf{r}'(t) \;=\; (x'(t), y'(t), z'(t)) \;=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t)$

- ◦ spatial PH quintics can interpolate first-order arbitrary Hermite data — basic primitive for free-form curve design & toolpath specification

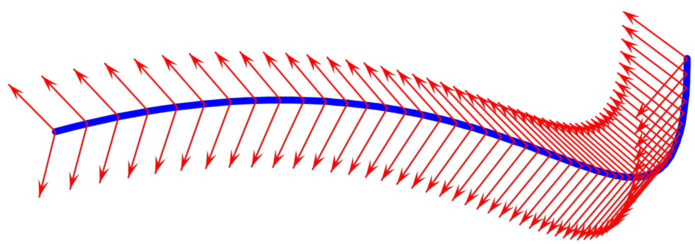
rational rotation–minimizing frame (RRMF) curves

rational adapted frame $(\mathbf{t}, \mathbf{u}, \mathbf{v})$ with angular velocity satisfying $\boldsymbol{\omega} \cdot \mathbf{t} \equiv 0$

RRMF curves are of minimum degree 5 — proper subset of PH quintics

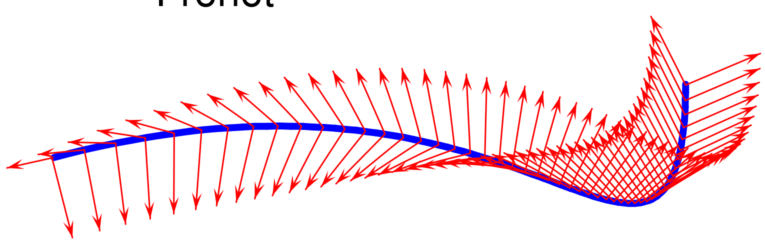
characterized by quadratic (vector) constraint on quaternion coefficients

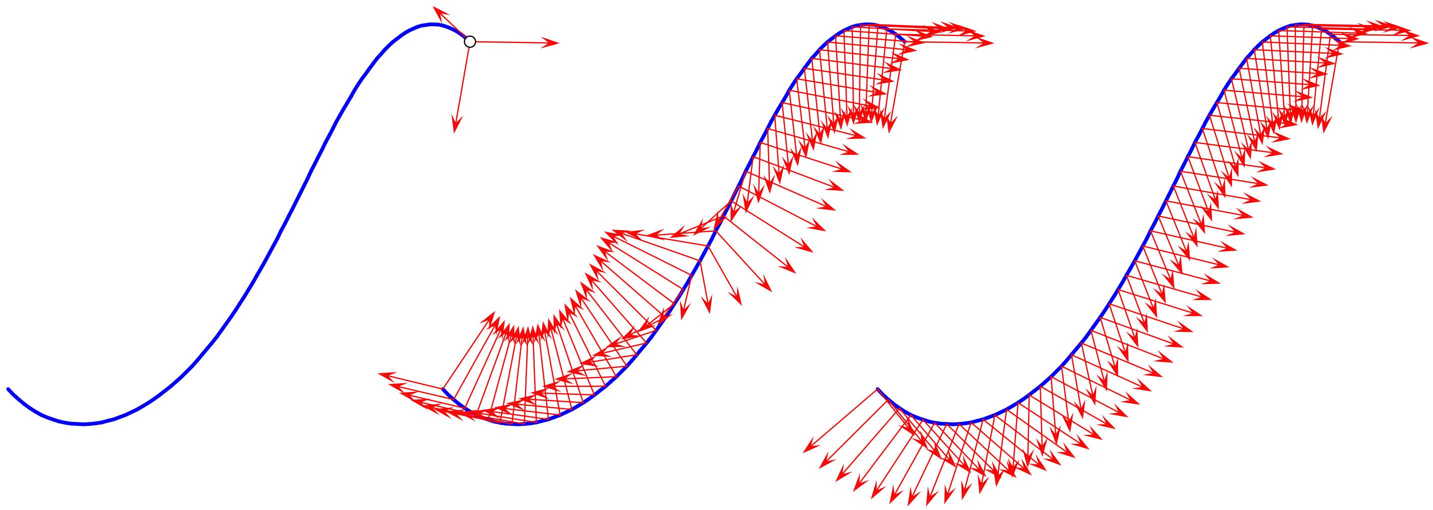
construction by geometric Hermite interpolation algorithm, applications to animation, spatial path planning, robotics, virtual reality, 5-axis machining



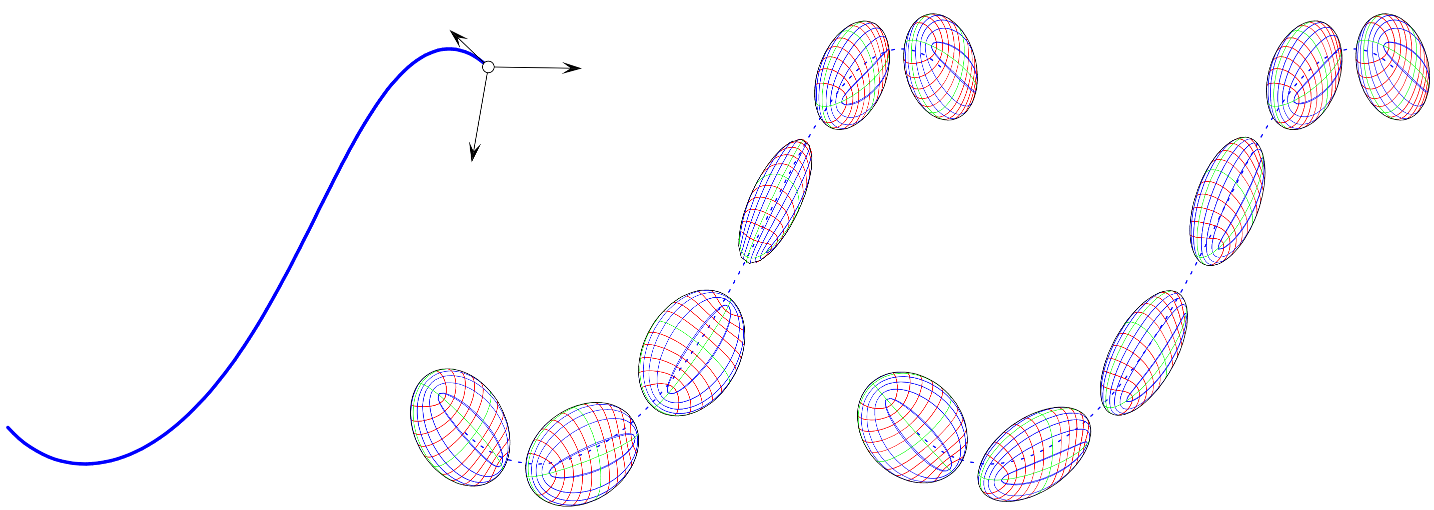
RMF

Frenet

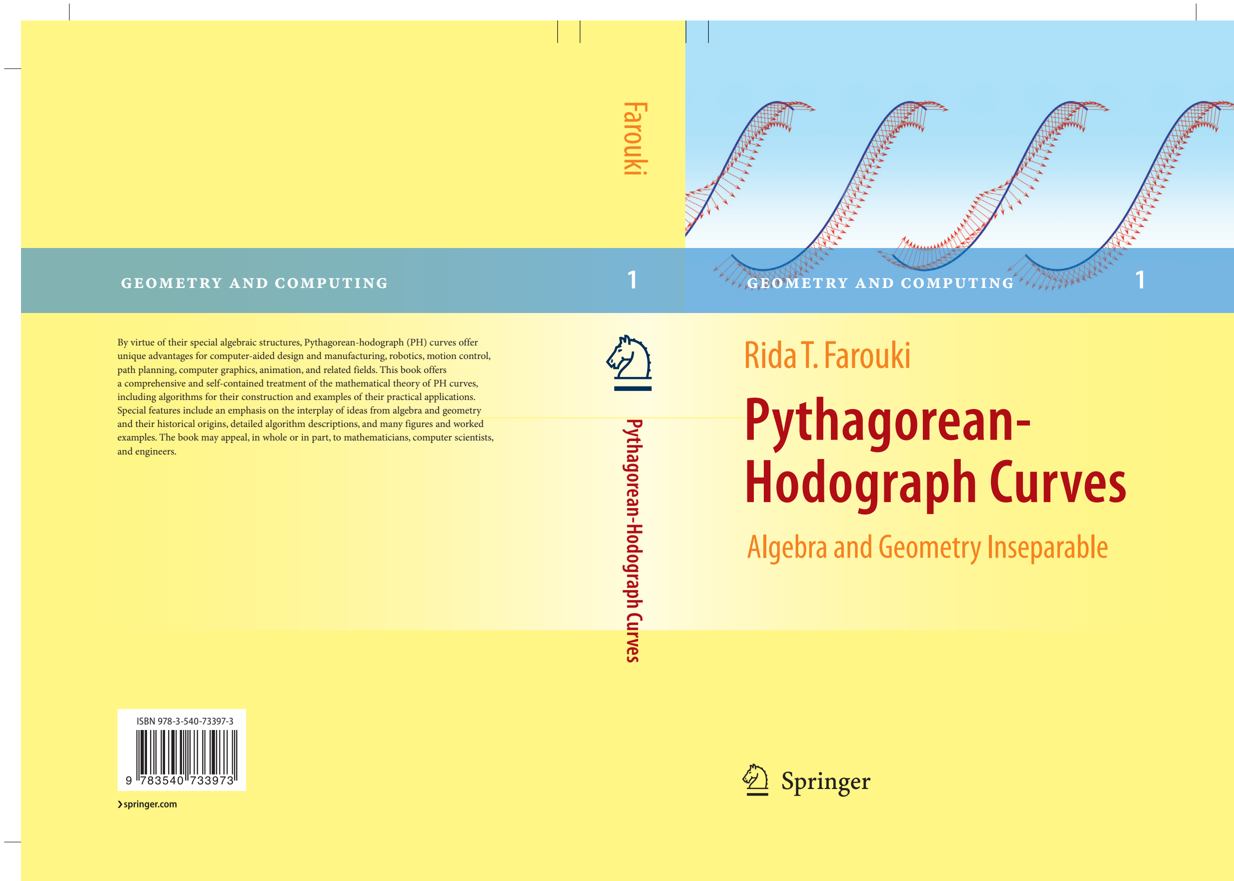




Frenet frame (center) & rotation-minimizing frame (right) on space curve



motion of an ellipsoid oriented by Frenet & rotation-minimizing frames



As long as algebra and geometry were separated, their progress was slow and their uses limited; but once these sciences were united, they lent each other mutual support and advanced rapidly together towards perfection.

Joseph-Louis Lagrange (1736-1813)

Sir William Rowan Hamilton (1805-1865)

- as a child prodigy, acquired varying degrees of proficiency with thirteen different languages
- appointed Professor of Astronomy at Trinity College, Dublin (age 22)
- theoretical prediction of “conical refraction” by biaxial crystals in 1832 — experimentally verified the same year
- Hamiltonian mechanics: systematic derivation of equations of motion for complicated dynamical systems with multiple degrees of freedom — paved way for development of quantum mechanics
- interpretation of complex numbers as “theory of algebraic couples” — search for “theory of algebraic triples” led to discovery of quaternions
- latter career devoted to (failed) effort to establish quaternions as the “new language of science”

there are no three-dimensional numbers

- generalize complex numbers $x + y\,\mathbf{i}$ to 3D numbers $x + y\,\mathbf{i} + z\,\mathbf{j}$
- basis elements 1, $\mathbf{i}$, $\mathbf{j}$ are assumed to be linearly independent
- commutative and associative — exhibit closure under $+$, $-$, $\times$, $\div$
- closure $\Longrightarrow$ must have $\mathbf{i}\,\mathbf{j} = a + b\,\mathbf{i} + c\,\mathbf{j}$ for some $a, b, c \in \mathbb{R}$
- multiply by $\mathbf{i}$ and substitute $\mathbf{i}^2 = -1$, $\mathbf{i}\,\mathbf{j} = a + b\,\mathbf{i} + c\,\mathbf{j}$ then gives

$$\mathbf{j} = \frac{b - ac - (a + bc)\,\mathbf{i}}{1 + c^2}$$

- $\Longrightarrow$ contradicts the assumed linear independence of 1, $\mathbf{i}$, $\mathbf{j}$!

Hurwitz's theorem (1898) on composition algebras

key property — norm of product $=$ product of norms: $|\mathcal{A}\,\mathcal{B}| \;=\; |\mathcal{A}|\,|\mathcal{B}|$

commutative law — $\mathcal{A}\,\mathcal{B} = \mathcal{B}\,\mathcal{A}\,,$ associative law — $(\mathcal{A}\,\mathcal{B})\,\mathcal{C} = \mathcal{A}\,(\mathcal{B}\,\mathcal{C})$

there are four possible composition algebras, of dimension $n = 1$, 2, 4, 8

- $\mathbb{R}$ $(n = 1)$, real numbers — product is commutative & associative
- $\mathbb{C}$ $(n = 2)$, complex numbers — product is commutative & associative
- $\mathbb{H}$ $(n = 4)$, quaternions — product is associative, but not commutative
- $\mathbb{O}$ $(n = 8)$, octonions — product neither commutative nor associative

fundamentals of quaternion algebra

quaternions are four-dimensional numbers of the form

$$\mathcal{A} = a + a_x\,\mathbf{i} + a_y\,\mathbf{j} + a_z\,\mathbf{k} \qquad \text{and} \qquad \mathcal{B} = b + b_x\,\mathbf{i} + b_y\,\mathbf{j} + b_z\,\mathbf{k}$$

that obey the sum and (non-commutative) product rules

$$\mathcal{A} + \mathcal{B} = (a+b) + (a_x+b_x)\,\mathbf{i} + (a_y+b_y)\,\mathbf{j} + (a_z+b_z)\,\mathbf{k}$$

$$\begin{aligned}\mathcal{A}\,\mathcal{B} &= (ab - a_xb_x - a_yb_y - a_zb_z) \\ &+ (ab_x + ba_x + a_yb_z - a_zb_y)\,\mathbf{i} \\ &+ (ab_y + ba_y + a_zb_x - a_xb_z)\,\mathbf{j} \\ &+ (ab_z + ba_z + a_xb_y - a_yb_x)\,\mathbf{k}\end{aligned}$$

basis elements $1, \mathbf{i}, \mathbf{j}, \mathbf{k}$ satisfy $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = \mathbf{i}\,\mathbf{j}\,\mathbf{k} = -1$

equivalently, $\mathbf{i}\,\mathbf{j} = -\,\mathbf{j}\,\mathbf{i} = \mathbf{k}\,,\quad \mathbf{j}\,\mathbf{k} = -\,\mathbf{k}\,\mathbf{j} = \mathbf{i}\,,\quad \mathbf{k}\,\mathbf{i} = -\,\mathbf{i}\,\mathbf{k} = \mathbf{j}$

scalar-vector representation of quaternions

set $\mathcal{A} = (a, \mathbf{a})$ and $\mathcal{B} = (b, \mathbf{b})$ — a, b and $\mathbf{a}$, $\mathbf{b}$ are scalar and vector parts

(a, b and $\mathbf{a}, \mathbf{b}$ also called the real and imaginary parts of $\mathcal{A}, \mathcal{B}$)

$$\mathcal{A} + \mathcal{B} = (\, a + b \,,\, \mathbf{a} + \mathbf{b} \,)$$

$$\mathcal{A}\,\mathcal{B} = (\, ab - \mathbf{a} \cdot \mathbf{b} \,,\, a\,\mathbf{b} + b\,\mathbf{a} + \mathbf{a} \times \mathbf{b})$$

(historical note: Hamilton's quaternions preceded, but were eventually supplanted by, the 3-dimensional vector analysis of Gibbs and Heaviside)

$\mathcal{A}^* = (a, -\mathbf{a})$ is the conjugate of $\mathcal{A}$

modulus : $|\mathcal{A}|^2 = \mathcal{A}^*\mathcal{A} = \mathcal{A}\mathcal{A}^* = a^2 + |\mathbf{a}|^2$

note that $|\mathcal{A}\,\mathcal{B}| = |\mathcal{A}|\,|\mathcal{B}|$ and $(\mathcal{A}\,\mathcal{B})^* = \mathcal{B}^*\mathcal{A}^*$

matrix representation of quaternions

matrix algebra embodies non-commutative nature of quaternion product

quaternion basis elements expressed as complex 2×2 matrices

$$1 \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{i} \to \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{j} \to \begin{bmatrix} 0 & \mathrm{i} \\ \mathrm{i} & 0 \end{bmatrix}, \quad \mathbf{k} \to \begin{bmatrix} \mathrm{i} & 0 \\ 0 & -\mathrm{i} \end{bmatrix}$$

(closely related to Pauli spin matrices $\sigma_x, \sigma_y, \sigma_z$ of quantum mechanics)

general quaternion can be expressed as real skew-symmetric 4×4 matrix

$$\mathcal{A} = a + a_x \mathbf{i} + a_y \mathbf{j} + a_z \mathbf{k} \to \begin{bmatrix} a & -a_x & -a_y & -a_z \\ a_x & a & -a_z & a_y \\ a_y & a_z & a & -a_x \\ a_z & -a_y & a_x & a \end{bmatrix}$$

unit quaternions & spatial rotations

any unit quaternion has the form $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$

describes a spatial rotation by angle θ about unit vector $\mathbf{n}$

for any vector $\mathbf{v}$ the quaternion product

$$\tilde{\mathbf{v}} = \mathcal{U}\,\mathbf{v}\,\mathcal{U}^*$$

yields the vector $\tilde{\mathbf{v}}$ corresponding to a rotation of $\mathbf{v}$ by θ about $\mathbf{n}$

here $\mathbf{v}$ is short-hand for a “pure vector” quaternion $\mathcal{V} = (0, \mathbf{v})$

unit quaternions $\mathcal{U}$ form a (non-commutative) group under multiplication

rotate vector $\mathbf{v}$ by angle θ about unit vector $\mathbf{n}$

decompose $\mathbf{v}$ into components parallel & perpendicular to $\mathbf{n}$

$$\mathbf{v} = \mathbf{v}_{||} + \mathbf{v}_{\perp} = (\mathbf{n}\cdot\mathbf{v})\,\mathbf{n} + (\mathbf{n}\times\mathbf{v})\times\mathbf{n}$$

$\mathbf{v}_{||}$ unchanged, but $\mathbf{v}_{\perp} \to \cos\theta\,(\mathbf{n}\times\mathbf{v})\times\mathbf{n} + \sin\theta\,\mathbf{n}\times\mathbf{v}$

under a rotation of $\mathbf{v}$ by θ about $\mathbf{n}$

in terms of quaternions $\mathcal{V} = (0, \mathbf{v})$ and $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$ we have

$$\mathcal{U}\,\mathcal{V}\,\mathcal{U}^* = (\,0\,,\,(\mathbf{n}\cdot\mathbf{v})\,\mathbf{n} + \sin\theta\,\mathbf{n}\times\mathbf{v} + \cos\theta\,(\mathbf{n}\times\mathbf{v})\times\mathbf{n}\,)$$

matrix form of vector rotation in $\mathbb{R}^3$

can write $\tilde{\mathbf{v}} = \mathbf{M}\,\mathbf{v}$ for 3×3 matrix $\mathbf{M} \in \mathrm{SO}(3)$

$$\begin{bmatrix} \tilde{v}_x \\ \tilde{v}_y \\ \tilde{v}_z \end{bmatrix} = \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} \begin{bmatrix} v_x \\ v_y \\ v_z \end{bmatrix}$$

elements of $\mathbf{M}$ in terms of rotation angle θ and axis $\mathbf{n}$

$$\begin{aligned}
m_{11} &= n_x^2 + (1 - n_x^2)\cos\theta\,, \\
m_{12} &= n_x n_y (1 - \cos\theta) - n_z \sin\theta\,, \\
m_{13} &= n_z n_x (1 - \cos\theta) + n_y \sin\theta\,, \\
m_{21} &= n_x n_y (1 - \cos\theta) + n_z \sin\theta\,, \\
m_{22} &= n_y^2 + (1 - n_y^2)\cos\theta\,, \\
m_{23} &= n_y n_z (1 - \cos\theta) - n_x \sin\theta\,, \\
m_{31} &= n_z n_x (1 - \cos\theta) - n_y \sin\theta\,, \\
m_{32} &= n_y n_z (1 - \cos\theta) + n_x \sin\theta\,, \\
m_{33} &= n_z^2 + (1 - n_z^2)\cos\theta\,.
\end{aligned}$$

concatenation of spatial rotations

rotate θ_1 about $\mathbf{n}_1$ then θ_2 about $\mathbf{n}_2$ $\longrightarrow$ equivalent rotation θ about $\mathbf{n}$

$$\theta \;=\; \pm 2\,\cos^{-1}(\cos\tfrac{1}{2}\theta_1 \cos\tfrac{1}{2}\theta_2 - \mathbf{n}_1\cdot\mathbf{n}_2 \sin\tfrac{1}{2}\theta_1 \sin\tfrac{1}{2}\theta_2)$$

$$\mathbf{n} \;=\; \pm\,\frac{\sin\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_2\,\mathbf{n}_1 \;+\; \cos\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2\,\mathbf{n}_2 \;-\; \sin\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2\,\mathbf{n}_1\times\mathbf{n}_2}{\sqrt{1 \;-\; (\cos\frac{1}{2}\theta_1 \cos\frac{1}{2}\theta_2 \;-\; \mathbf{n}_1\cdot\mathbf{n}_2 \sin\frac{1}{2}\theta_1 \sin\frac{1}{2}\theta_2)^2}}$$

sign ambiguity: equivalence of $-\theta$ about $-\mathbf{n}$ and θ about $\mathbf{n}$

formulae discovered by Olinde Rodrigues (1794-1851)

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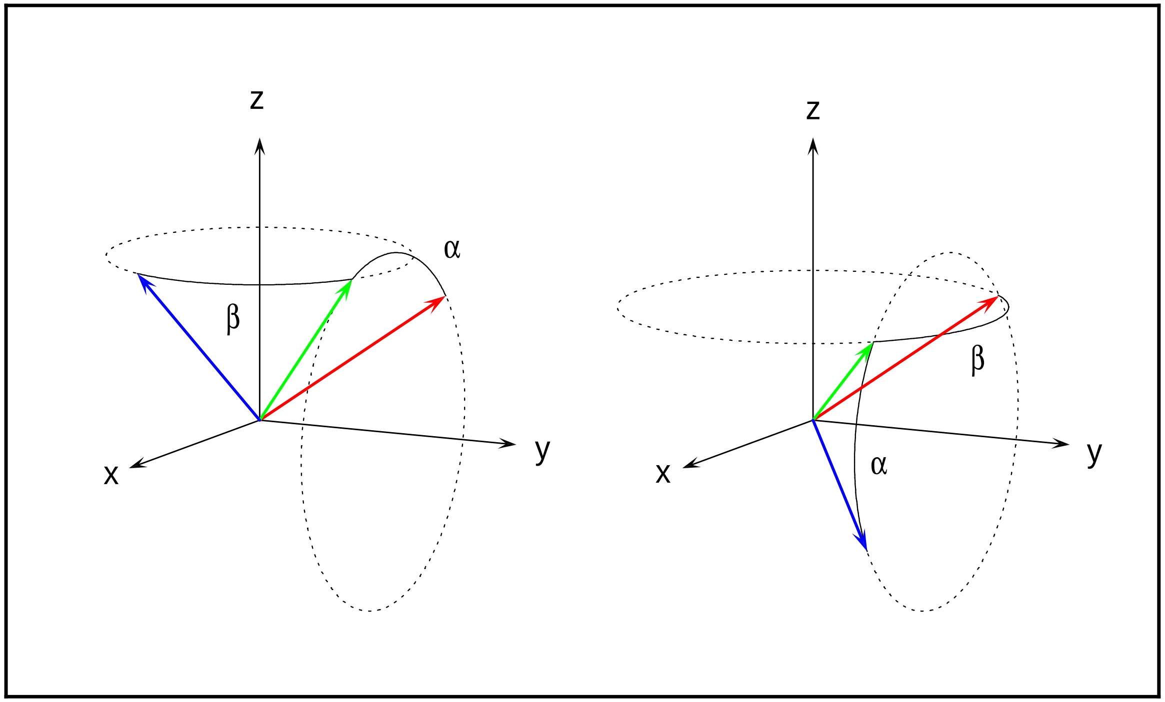
set $\mathcal{U}_1 = (\cos\tfrac{1}{2}\theta_1, \sin\tfrac{1}{2}\theta_1\mathbf{n}_1)$ and $\mathcal{U}_2 = (\cos\tfrac{1}{2}\theta_2, \sin\tfrac{1}{2}\theta_2\mathbf{n}_2)$

$\mathcal{U} = \mathcal{U}_2\mathcal{U}_1 = (\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta\mathbf{n})$ defines angle, axis of compound rotation

spatial rotations do not commute



blue vector is obtained from red vector by the concatenation of two spatial rotations — left: $R_y(\alpha)\,R_z(\beta)$, right: $R_z(\beta)\,R_y(\alpha)$ — the end results differ

define $\mathcal{U}_1 = (\cos\frac{1}{2}\alpha, \sin\frac{1}{2}\alpha\,\mathbf{j})$, $\mathcal{U}_2 = (\cos\frac{1}{2}\beta, \sin\frac{1}{2}\beta\,\mathbf{k})$ — $\mathcal{U}_1\mathcal{U}_2 \neq \mathcal{U}_2\mathcal{U}_1$

. . . the three Russian brothers . . .

. . . Following the collapse of the former Soviet Union, the economy in Russia hit hard times, and jobs were difficult to find. Dmitry, Ivan, and Alexey — the Brothers Karamazov — therefore decided to seek their fortunes by emigrating to America, England, Australia . . .

the "troubled origins" of vector analysis

The algebraically *real* part may receive . . . all values contained on the one *scale* of progression of number from negative to positive infinity; we shall call it therefore the *scalar part*, or simply the *scalar*. On the other hand, the algebraically *imaginary* part, being constructed geometrically by a straight line or radius, which has, in general, for each determined quaternion, a determined length and determined direction in space, may be called the *vector part*, or simply the *vector* . . .

William Rowan Hamilton, *Philosophical Magazine* (1846)

A school of "quaternionists" developed, which was led after Hamilton's death by Peter Tait of Edinburgh and Benjamin Pierce of Harvard. Tait wrote eight books on the quaternions, emphasizing their applications to physics. When Gibbs invented the modern notation for the dot and cross product, Tait condemned it as a "hermaphrodite monstrosity." A war of polemics ensued, with luminaries such as Kelvin and Heaviside writing devastating invective against quaternions. Ultimately the quaternions lost, and acquired a taint of disgrace from which they never fully recovered.

John C. Baez, *The Octonions* (2002)

M. J. Crowe, *A History of Vector Analysis* (1967)

A high level of intensity and a certain fierceness characterized much of the debate, and must have led many readers to follow it with interest.

. . . Gibbs and Heaviside must have appeared to the quaternionists as unwelcome intruders who had burst in upon the developing dialogue between the quaternionists and the scientists of the day to arrive at a moment when success seemed not far distant. Charging forth, these two vectorists, the one brash and sarcastic, the other spouting historical irrelevancies, had promised a bright new day for any who would accept their overtly pragmatic arguments for an algebraically crude and highly arbitrary system. And worst of all, the system they recommended was, not some new system . . . but only a perverted version of the quaternion system. Heretics are always more hated than infidels, and these two heretics had, with little understanding and less acknowledgement, wrenched major portions from the Hamiltonian system and then claimed that these parts surpassed the whole.

the sad demise of quaternions

E. T. Bell, *Men of Mathematics*, Hamilton = "An Irish Tragedy"

Hamilton's *Lectures on Quaternions* (1853) "would take any man a twelve-month to read, and near a lifetime to digest . . ."
– Sir John Herschel, discoverer of the planet Uranus

Hamilton's vision of quaternions as the "universal language" of mathematical and physical sciences was never realized — this role is now occupied by vector analysis, distilled from the quaternion algebra by the physicists James Clerk Maxwell (1831-1879) and Josiah Willard Gibbs (1839-1903), and the engineer Oliver Heaviside (1850-1925)

some "dirty secrets" of vector analysis

- there are two *fundamentally different* types of vector in $\mathbb{R}^3$ — polar vectors and axial vectors

- a polar vector $\mathbf{v} = (v_x, v_y, v_z)$ becomes $(-v_x, -v_y, -v_z)$ under a transformation $(x, y, z) \rightarrow (-x, -y, -z)$ between right–handed and left–handed coordinate systems — also called a true vector

- an axial vector, such as the cross product $\mathbf{a} \times \mathbf{b}$, is unchanged under transformation $(x, y, z) \rightarrow (-x, -y, -z)$ — also called a pseudovector

- similar distinction exists between true scalars and pseudoscalars, e.g., $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$ is a pseudoscalar if $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$ are true (polar) vectors

- vector analysis in $\mathbb{R}^3$ does not have a natural specialization to $\mathbb{R}^2$ or generalization to $\mathbb{R}^n$ for $n \geq 4$

families of spatial rotations

find $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$ that rotates $\mathbf{i} = (1,0,0) \to \mathbf{v} = (\lambda, \mu, \nu)$

$$
\begin{aligned}
n_x^2(1-\cos\theta) + \cos\theta &= \lambda\,, \\
n_x n_y(1-\cos\theta) + n_z\sin\theta &= \mu\,, \\
n_z n_x(1-\cos\theta) - n_y\sin\theta &= \nu\,.
\end{aligned}
$$

$$
\begin{aligned}
n_x &= \frac{\pm\sqrt{\cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\theta}}{\sin\frac{1}{2}\theta}\,, \\
n_y &= \frac{\pm\,\mu\sqrt{\cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\theta} \;-\; \nu\cos\frac{1}{2}\theta}{(1+\lambda)\sin\frac{1}{2}\theta}\,, \\
n_z &= \frac{\pm\,\nu\sqrt{\cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\theta} \;+\; \mu\cos\frac{1}{2}\theta}{(1+\lambda)\sin\frac{1}{2}\theta}\,.
\end{aligned}
$$

general solution, where $\alpha = \cos^{-1}\lambda$ and $\alpha \le \theta \le 2\pi - \alpha$

Parameterizes family of spatial rotations mapping unit vectors $\mathbf{i} \to \mathbf{v}$ by specifying rotation axis $\mathbf{n}$ as a function of rotation angle θ, over restricted domain $\theta \in [\,\alpha, 2\pi - \alpha\,]$ where α is angle between $\mathbf{i}$ and $\mathbf{v}$.

Define unit vectors $\mathbf{e}_\perp$, $\mathbf{e}_0$ orthogonal to and in common plane of $\mathbf{i}$ and $\mathbf{v}$

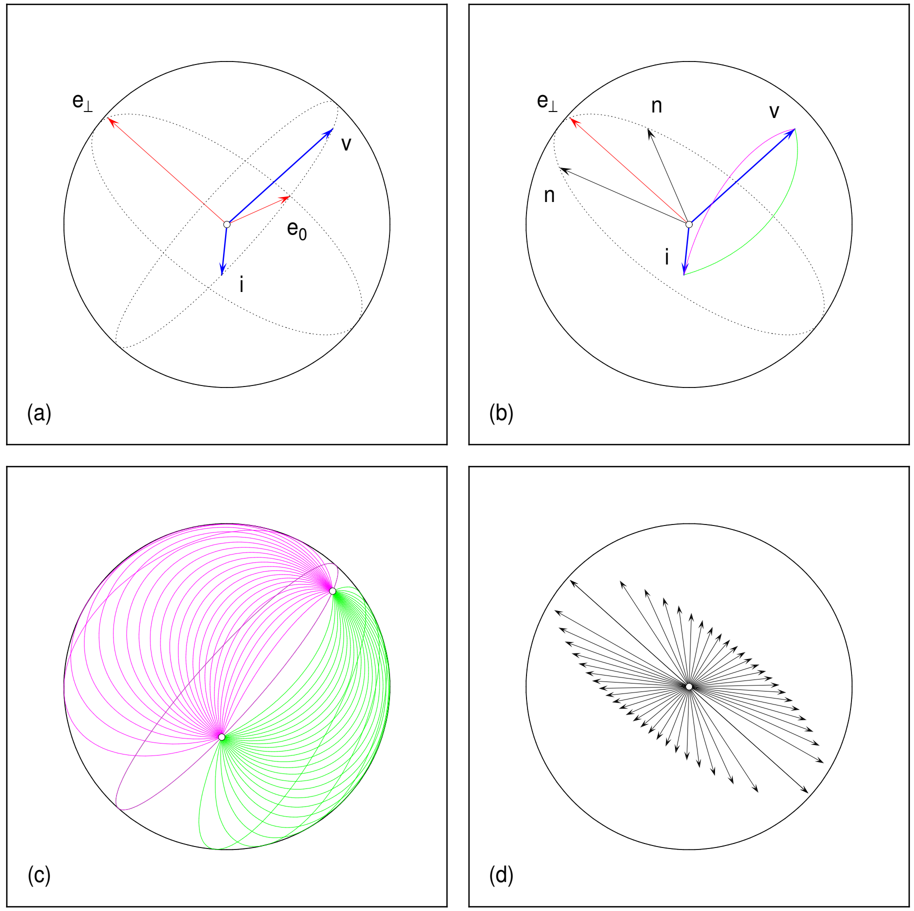
$$\mathbf{e}_\perp = \frac{\mathbf{i} \times \mathbf{v}}{|\,\mathbf{i} \times \mathbf{v}\,|} \qquad \text{and} \qquad \mathbf{e}_0 = \frac{\mathbf{i} + \mathbf{v}}{|\,\mathbf{i} + \mathbf{v}\,|}$$

Rotation axis lies in plane spanned by these vectors, may be written as

$$\mathbf{n}(\theta) = \frac{\sin\frac{1}{2}\alpha\,\cos\frac{1}{2}\theta\;\mathbf{e}_\perp \;\pm\; \sqrt{\cos^2\frac{1}{2}\alpha - \cos^2\frac{1}{2}\theta}\;\mathbf{e}_0}{\cos\frac{1}{2}\alpha\,\sin\frac{1}{2}\theta}\,.$$

for any $\theta \in (\alpha, 2\pi - \alpha)$ there are two axes $\mathbf{n}$ — in the plane of $\mathbf{e}_\perp$, $\mathbf{e}_0$ with equal inclinations to $\mathbf{e}_\perp$ — about which a rotation by angle θ maps $\mathbf{i} \to \mathbf{v}$

- when $\theta = \alpha$ or $2\pi - \alpha$, we have $\mathbf{n} = \mathbf{e}_\perp$ or $-\mathbf{e}_\perp$, and rotation is along great circle between $\mathbf{i}$ and $\mathbf{v}$;
- when $\theta = \pi$, we have $\mathbf{n} = \pm\,\mathbf{e}_0$, so $\mathbf{i}$ executes either a clockwise or anti–clockwise half-rotation about $\mathbf{e}_0$ onto $\mathbf{v}$;



Spatial rotations of unit vectors $\mathbf{i} \to \mathbf{v}$. (a) Vectors $\mathbf{e}_\perp$ (orthogonal to $\mathbf{i}$, $\mathbf{v}$) and $\mathbf{e}_0$ (bisector of $\mathbf{i}, \mathbf{v}$) — the plane Π of $\mathbf{e}_\perp$ and $\mathbf{e}_0$ is orthogonal to that of $\mathbf{i}$ and $\mathbf{v}$. (b) For any rotation angle $\theta \in (\alpha, 2\pi - \alpha)$, where $\alpha = \cos^{-1}(\mathbf{i} \cdot \mathbf{v})$, there are two possible rotations, with axes $\mathbf{n}$ inclined equally to $\mathbf{e}_\perp$ in the plane Π. (c) Sampling of the family of spatial rotations $\mathbf{i} \to \mathbf{v}$, shown as loci on the unit sphere. (d) Axes $\mathbf{n}$ for these rotations, lying in the plane Π.

quaternion model for spatial PH curves

quaternion polynomial $\mathcal{A}(t) = u(t) + v(t)\,\mathbf{i} + p(t)\,\mathbf{j} + q(t)\,\mathbf{k}$

maps to
$$\mathbf{r}'(t) \;=\; \mathcal{A}(t)\,\mathbf{i}\,\mathcal{A}^*(t) \;=\; [\,u^2(t) + v^2(t) - p^2(t) - q^2(t)\,]\,\mathbf{i} + 2\,[\,u(t)q(t) + v(t)p(t)\,]\,\mathbf{j} \;+\; 2\,[\,v(t)q(t) - u(t)p(t)\,]\,\mathbf{k}$$

rotation invariance of spatial PH form: rotate by θ about $\mathbf{n} = (n_x, n_y, n_z)$

define $\mathcal{U} = (\cos\frac{1}{2}\theta, \sin\frac{1}{2}\theta\,\mathbf{n})$ — then $\mathbf{r}'(t) \;\rightarrow\; \tilde{\mathbf{r}}'(t) = \tilde{\mathcal{A}}(t)\,\mathbf{i}\,\tilde{\mathcal{A}}^*(t)$

where $\tilde{\mathcal{A}}(t) \;=\; \mathcal{U}\,\mathcal{A}(t)$ (can interpret as rotation in $\mathbb{R}^4$)

matrix form of $\tilde{\mathcal{A}}(t) = \mathcal{U}\,\mathcal{A}(t)$

$$\begin{bmatrix} \tilde{u} \\ \tilde{v} \\ \tilde{p} \\ \tilde{q} \end{bmatrix} = \begin{bmatrix} \cos\frac{1}{2}\theta & -n_x \sin\frac{1}{2}\theta & -n_y \sin\frac{1}{2}\theta & -n_z \sin\frac{1}{2}\theta \\ n_x \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta & -n_z \sin\frac{1}{2}\theta & n_y \sin\frac{1}{2}\theta \\ n_y \sin\frac{1}{2}\theta & n_z \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta & -n_x \sin\frac{1}{2}\theta \\ n_z \sin\frac{1}{2}\theta & -n_y \sin\frac{1}{2}\theta & n_x \sin\frac{1}{2}\theta & \cos\frac{1}{2}\theta \end{bmatrix} \begin{bmatrix} u \\ v \\ p \\ q \end{bmatrix}$$

matrix $\in \mathrm{SO}(4)$

in general, points have non-closed orbits under rotations in $\mathbb{R}^4$

“strange events” in $\mathbb{R}^4$ defy geometric intuition !

- ◦ an elastic sphere can be turned inside out without tearing the material !
- ◦ a prisoner may escape from a locked room without penetrating its walls !
- ◦ rigid motions change “left-handed objects” into “right-handed objects” !
- ◦ a knot in a length of string can be untied without ever moving its ends !

early 20th century: can existence of a fourth dimension, imperceptible to human senses, explain mysterious psychic and paranormal phenomena?

H. P. Manning, *The Fourth Dimension Simply Explained* (1910)
& *Geometry of Four Dimensions* (1914), Dover (reprint), New York.

$\longrightarrow$ strange phenomena arise from “extra maneuvering freedom” in $\mathbb{R}^4$

elementary geometry of four dimensions

lines, planes, and hyperplanes of $\mathbb{R}^4$ are the sets of points linearly dependent upon two, three, and four points of $\mathbb{R}^4$ in "general position"

alternatively, lines, planes, and hyperplanes are point sets satisfying three, two, and one linear equations in the Cartesian coordinates of $\mathbb{R}^4$

hyperplane = a copy of familiar Euclidean space $\mathbb{R}^3$ — separates $\mathbb{R}^4$ into two disjoint regions (as with a plane in $\mathbb{R}^3$, and a line in $\mathbb{R}^2$)

generic incidence relations for $\mathbb{R}^4$:

- two hyperplanes intersect in a plane
- three hyperplanes intersect in a line
- four hyperplanes intersect in a point

$\Longrightarrow$ two planes intersect in a point

“absolutely orthogonal” planes in $\mathbb{R}^4$

two planes $\Pi_1, \Pi_2 \in \mathbb{R}^4$ with intersection point $\mathbf{p}$ are absolutely orthogonal if every line through $\mathbf{p}$ on Π_1 is orthogonal to every line through $\mathbf{p}$ on Π_2

pairs of “absolutely orthogonal” planes are a strictly four-dimensional phenomenon — they have no analog in $\mathbb{R}^3$

through each point $\mathbf{p}$ of a given plane $\Pi_1 \in \mathbb{R}^4$, there is a unique plane $\Pi_2 \in \mathbb{R}^4$ that is absolutely orthogonal

pairs of absolutely orthogonal planes in $\mathbb{R}^4$ play an important role in characterizing four-dimensional rotations

characterization of rotations in $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^4$

$\mathbb{R}^2$: $(x + \mathrm{i}\, y) \;\to\; \mathrm{e}^{\mathrm{i}\theta}(x + \mathrm{i}\, y)$ — one parameter, rotation angle θ

$\mathbb{R}^3$: $\mathbf{v} \;\to\; \mathcal{U}\,\mathbf{v}\,\mathcal{U}^*$ where $\mathcal{U} = (\cos\tfrac{1}{2}\theta, \sin\tfrac{1}{2}\theta\,\mathbf{n})$
— three parameters, rotation axis $\mathbf{n}$ and angle θ

$\mathbb{R}^4$: $\mathcal{V} \;\to\; \mathcal{U}_1\mathcal{V}\mathcal{U}_2^*$ — two unit quaternions $\mathcal{U}_1, \mathcal{U}_2 \;\Rightarrow$ six parameters

stationary set of rotation in $\mathbb{R}^n$ $=$ set of points that do not move

simple rotation in $\mathbb{R}^n$ — the stationary set is of dimension $n-2$

$\Rightarrow$ in $\mathbb{R}^2$ and $\mathbb{R}^3$, every rotation is simple

simple rotation in $\mathbb{R}^4$ — stationary set is a plane through the origin,
and unique absolutely orthogonal plane rotates on itself

a new possibility in $\mathbb{R}^4$ — *double rotations*

if $\Pi_1, \Pi_2 \in \mathbb{R}^4$ are absolutely orthogonal planes through the origin, each may rotate upon itself about the other, and these rotations commute — i.e., the outcome is independent of their order

the stationary set of such a double rotation is the single common point of Π_1, Π_2 — i.e., the origin

of the six parameters describing a general rotation in $\mathbb{R}^4$, four define the absolutely orthogonal planes Π_1, Π_2 and two specify the rotation angles θ_1, θ_2 associated with them

under a continuous double rotation — with angular speeds ω_1, ω_2 associated with the absolutely orthogonal planes Π_1, Π_2 — points in $\mathbb{R}^4$ have closed orbits if and only if the ratio ω_2/ω_1 is a rational number

Clifford algebra: extensions from $\mathbb{R}^2$, $\mathbb{R}^3$ to $\mathbb{R}^n$ and from Euclidean to Minkowski space

consider "n–dimensional numbers" $\mathbf{x} = x_1\mathbf{e}_1 + \cdots + x_n\mathbf{e}_n$

where $x_1, \ldots, x_n \in \mathbb{R}$ and $(\mathbf{e}_1, \ldots, \mathbf{e}_n)$ = orthonormal basis for $\mathbb{R}^n$

$\mathbf{e}_i\mathbf{e}_i = \sigma_i = \pm 1$ and $\mathbf{e}_j\mathbf{e}_k = -\mathbf{e}_k\mathbf{e}_j$ if $j \neq k$

$$\Rightarrow \quad \mathbf{x}^2 = \sigma_1 x_1^2 + \cdots + \sigma_n x_n^2$$

$\sigma_1, \ldots, \sigma_n$ define signature of Clifford algebra

write $\mathcal{C}\ell_{p,q}$ if $\sigma_1 = \cdots = \sigma_p = +1,\ \sigma_{p+1} = \cdots = \sigma_n = -1$

$\mathcal{C}\ell_{n,1}$ equivalent to Minkowski space $\mathbb{R}^{n,1}$ (special relativity theory) graded algebra — e.g., general element of $\mathcal{C}\ell_3$ is the multivector

$$a_0 + a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 + a_{23}\mathbf{e}_2\mathbf{e}_3 + a_{31}\mathbf{e}_3\mathbf{e}_1 + a_{12}\mathbf{e}_1\mathbf{e}_2 + a_{123}\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$$

- 1 is *grade zero* element (scalar)
- $\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$ are *grade one* elements (vectors)
- $\mathbf{e}_2\mathbf{e}_3$, $\mathbf{e}_3\mathbf{e}_1$, $\mathbf{e}_1\mathbf{e}_2$ are *grade two* elements (bivectors)
- $\mathbf{e}_1\mathbf{e}_2\mathbf{e}_3$ is *highest grade* element (pseudoscalar)

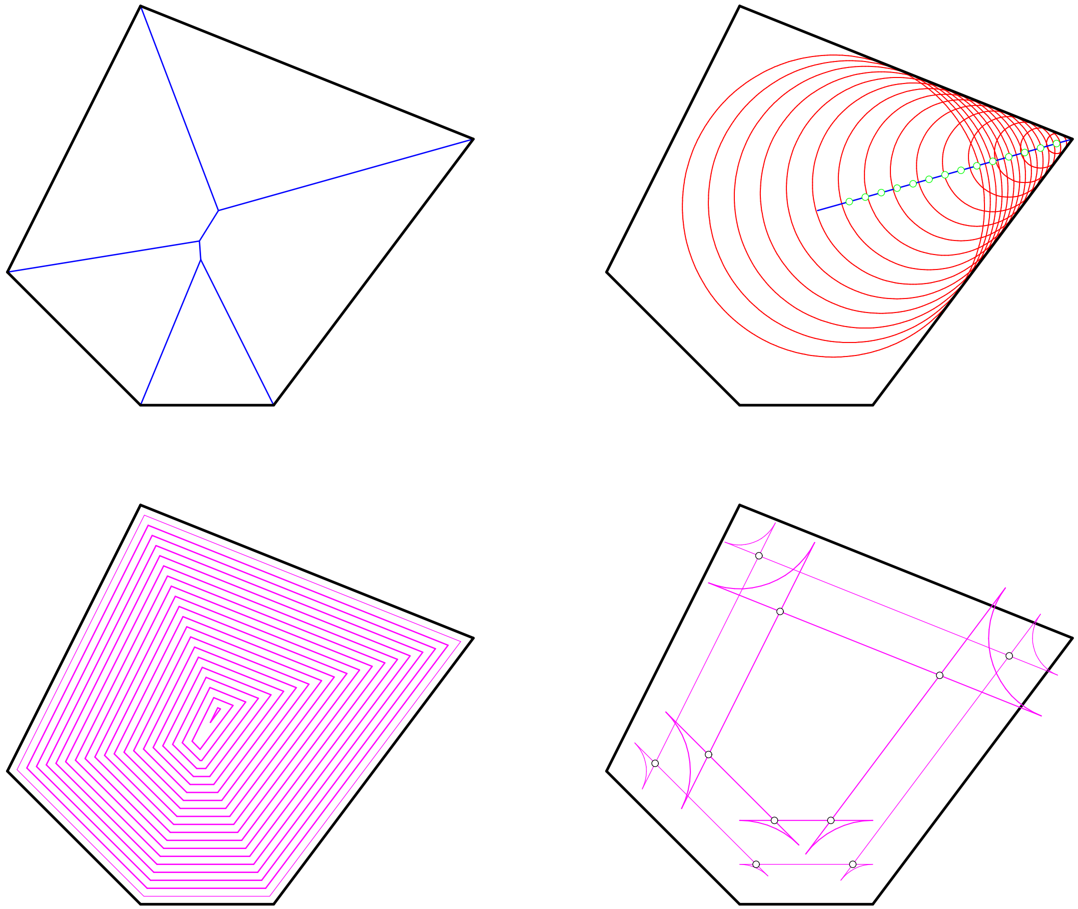
subspace of *even grade* multivectors $=$ sub–algebra $\mathcal{C}\ell_n^+$ of $\mathcal{C}\ell_n$

e.g., complex numbers $\mathbb{C}$ and quaternions $\mathbb{H}$ isomorphic to $\mathcal{C}\ell_2^+$ and $\mathcal{C}\ell_3^+$

$(1, \mathrm{i}) \leftrightarrow (1, \mathbf{e}_1\mathbf{e}_2)$ and $(1, \mathbf{i}, \mathbf{j}, \mathbf{k}) \leftrightarrow (1, \mathbf{e}_2\mathbf{e}_3, \mathbf{e}_1\mathbf{e}_2, \mathbf{e}_3\mathbf{e}_1)$

- inner product $\mathbf{a} \cdot \mathbf{b}$ *reduces grade*
- outer product $\mathbf{a} \wedge \mathbf{b}$ *increases grade*
- geometric product $\mathbf{a}\,\mathbf{b} = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \wedge \mathbf{b}$
- applies to arbitrary multivectors (mixed grade)
- applications — concise description and analysis of reflections and rotations in $\mathbb{R}^n$

medial axis transform of planar domain



medial axis $=$ locus of centers of maximal inscribed disks, touching domain boundary in at least two points; medial axis transform (MAT) $=$ medial axis $+$ superposed function specifying radii of maximal disks

Minkowski Pythagorean-hodograph (MPH) curves in $\mathbb{R}^{2,1}$

medial axis of planar domain $\mathcal{D}$ $=$ locus of centers of maximal disks (touching domain boundary $\partial\mathcal{D}$ in at least two points) inscribed in $\mathcal{D}$

medial axis transform or MAT $(x(t), y(t), r(t))$ $=$ medial axis locus $(x(t), y(t))$ plus function $r(t)$ specifying radii of maximal disks

MAT encodes and characterizes shape of any planar domain $\mathcal{D}$

MAT is a Minkowski Pythagorean-hodograph (MPH) curve in $\mathbb{R}^{2,1}$ if

$$x'^2(t) + y'^2(t) - r'^2(t) = \sigma^2(t)$$

MAT $=$ MPH curve $\Longleftrightarrow$ domain boundary $\partial\mathcal{D}$ can be exactly recovered as a (piecewise) rational curve

interpretation of Minkowski metric

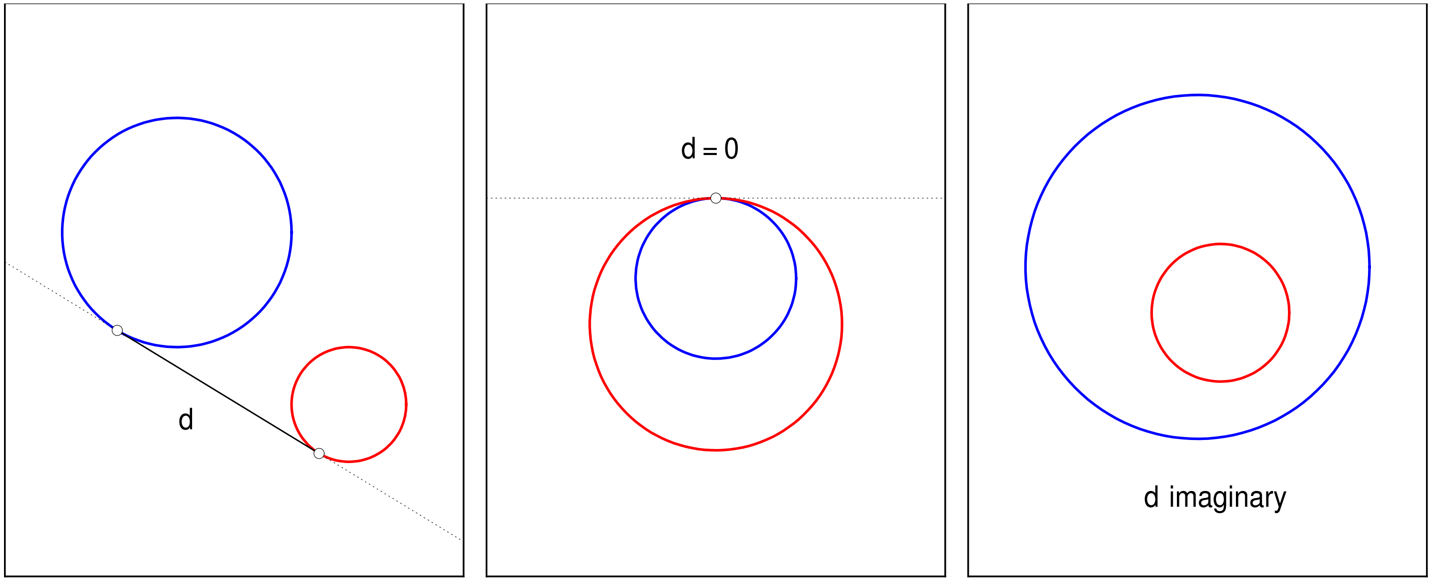
originates in special relativity: distance d between events with space–time coordinates (x_1, y_1, t_1) and (x_2, y_2, t_2) is defined by

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 - c^2(t_2 - t_1)^2$$

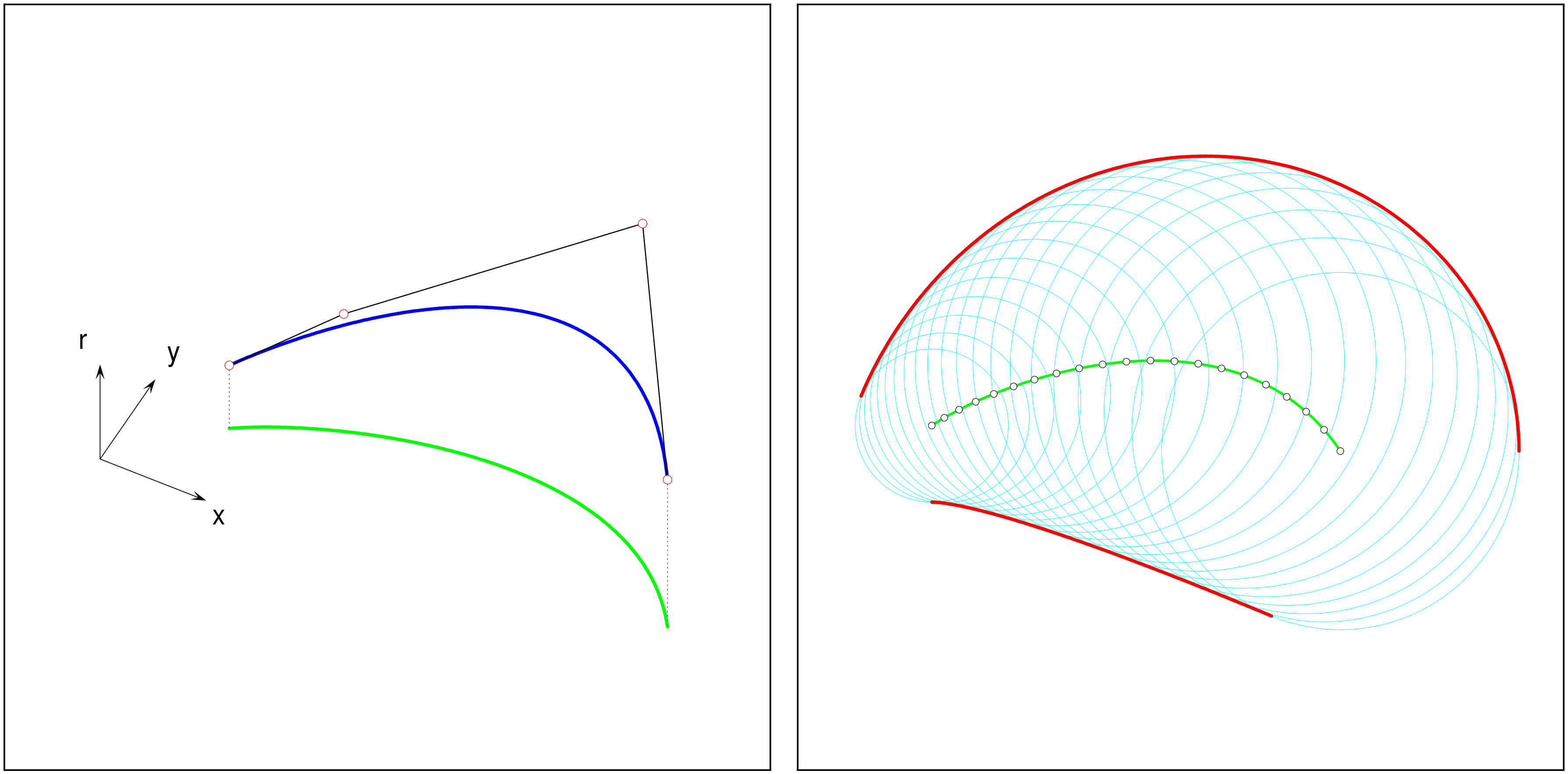
space-like if d real, light-like if $d = 0$, time-like if d imaginary

distance between circles (x_1, y_1, r_1) and (x_2, y_2, r_2) as points in $\mathbb{R}^{(2,1)}$

$$d^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2 - (r_2 - r_1)^2$$



rational boundary reconstructed from MPH curve



closure

- quaternions offer natural language for describing & manipulating spatial rotations

- the quaternion model allows simple and intuitive constructions of spatial Pythagorean-hodograph curves

- the historical legacy of quaternions (origins of vector analysis) has been sadly neglected

- complexity of rotations in $\mathbb{R}^4$ provides cautionary evidence against extending geometric intuition from $\mathbb{R}^2$ and $\mathbb{R}^3$

- Clifford algebra formulations: extension to higher dimensions, and from Euclidean $\rightarrow$ Minkowski space

man's limited insight

Superior beings, when of late they saw
A mortal man unfold all nature's law,
Admired such wisdom in an earthly shape,
And showed a Newton as we show an ape.
Could he, whose rules the rapid comet bind,
Describe or fix one movement of his mind?
Who saw its fires here rise, and there descend,
Explain his own beginning, or his end?
Alas, what wonder! Man's superior part
Unchecked may rise, and climb from art to art:
But when his own great work has but begun,
What reason weaves, by passion is undone.

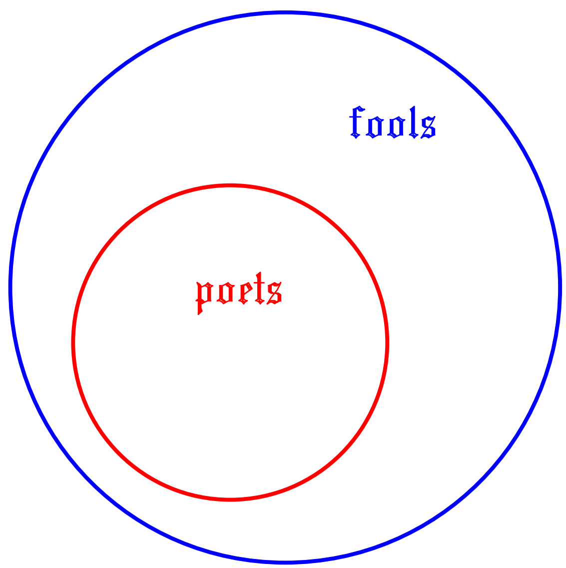
Alexander Pope (1688-1744), *Essay on Man*

Boolean algebra of poets & fools

Sir, I admit your general rule,
That every poet is a fool.

But you yourself may serve to show it,
That every fool is not a poet!

Alexander Pope (1688-1744)



all poets are fools, but not all fools are poets