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Signal Processing 86 (2006) 639–697

SIGNAL PROCESSING

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Review

Cyclostationarity: Half a century of research

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Received 22 December 2004; accepted 29 June 2005
Available online 6 September 2005

Abstract

In this paper, a concise survey of the literature on cyclostationarity is presented and includes an extensive bibliography. The literature in all languages, in which a substantial amount of research has been published, is included. Seminal contributions are identified as such. Citations are classified into 22 categories and listed in chronological order. Both stochastic and nonstochastic approaches for signal analysis are treated. In the former, which is the classical one, signals are modelled as realizations of stochastic processes. In the latter, signals are modelled as single functions of time and statistical functions are defined through infinite-time averages instead of ensemble averages. Applications of cyclostationarity in communications, signal processing, and many other research areas are considered.

Keywords: Cyclostationarity; Almost-cyclostationary time series; Almost-cyclostationary processes; Bibliography

Contents

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0165-1684/$ - see front matter
doi:10.1016/j.sigpro.2005.06.016

1. Introduction

Many processes encountered in nature arise from periodic phenomena. These processes, although not periodic functions of time, give rise to random data whose statistical characteristics vary periodically with time and are called *cyclostationary processes* [2.5]. For example, in telecommunications, telemetry, radar, and sonar applications, periodicity is due to modulation, sampling, multiplexing, and coding operations. In mechanics it is due, for example, to gear rotation. In radio astronomy, periodicity results from revolution and rotation of planets and on pulsation of stars. In econometrics, it is due to seasonality; and in atmospheric science it is due to rotation and revolution of the earth. The relevance of the theory of cyclostationarity to all these fields of study and more was first proposed in [2.5].

Wide-sense cyclostationary stochastic processes have autocorrelation functions that vary periodically with time. This function, under mild regularity conditions, can be expanded in a Fourier series whose coefficients, referred to as *cyclic autocorrelation functions*, depend on the lag parameter; the frequencies, called *cycle frequencies*, are all multiples of the reciprocal of the period of cyclostationarity [2.5]. Cyclostationary processes have also been referred to as *periodically correlated processes* [2.18,3.5]. More generally, if the frequencies of the (generalized) Fourier series expansion of the autocorrelation function are not commensurate, that is, if the autocorrelation function is an almost-periodic function of time, then the process is said to be *almost-cyclostationary* [3.26] or, equivalently, *almost-periodically correlated* [3.5]. The almost-periodicity property of the autocorrelation function is manifested in the frequency domain as correlation among the spectral components of the process that are separated by amounts equal to the cycle frequencies. In contrast to this, wide-sense stationary processes have autocorrelation functions that are independent of time, depending on only the lag parameter, and all distinct spectral components are uncorrelated.

As an alternative, the presence of periodicity in the underlying data-generating mechanism of a phenomenon can be described without modelling the available data as a sample path of a stochastic process but, rather, by modelling it as a single function of time [4.31]. Within this nonstochastic framework, a time-series is said to exhibit second-order cyclostationarity (in the wide sense), as first defined in [2.8], if there exists a stable quadratic time-invariant transformation of the time-series that gives rise to finite-strength additive sinewave components.

In this paper, a concise survey of the literature (in all languages in which a substantial amount of research has been published) on cyclostationarity is presented and includes an extensive bibliography and list of issued patents. Citations are classified into 22 categories and listed, for each category, in chronological order. In Section 2, general treatments and tutorials on the theory of cyclostationarity are cited. General properties of processes and time-series are presented in Section 3. In Section 4, the problem of estimating statistical functions is addressed. Models for manufactured and natural signals are considered in Sections 5 and 6, respectively. Communications systems and related problems are treated in Sections 7–11. Specifically, the analysis and design of communications systems is addressed in Section 7, the problem of synchronization is addressed in Section 8, the estimation of signal parameters and waveforms is addressed in Section 9, the identification and equalization of channels is addressed in Section 10, and the signal detection and classification problems and the problem of source separation are addressed in Section 11. Periodic autoregressive (AR) and autoregressive moving-average (ARMA) modelling and prediction are treated in Section 12. In Section 13, theory and applications of higher-order cyclostationarity are presented. We address applications to circuits, systems, and control in Section 14, to acoustics and mechanics in Section 15, to econometrics in Section 16, and to biology in Section 17. Applications to the problems of level crossing and queueing are addressed in Sections 18 and 19, respectively. Cyclostationary random fields are treated in Section 20. In Section 21, some classes of nonstationary signals that extend the class of almost-cyclostationary signals are considered. Finally, some miscellaneous references are listed [22.1–22.8]. Further references only indirectly related to cyclostationarity are [23.1–23.15].

To assist readers in "going to the source", seminal contributions—if known— are identified within the literature published in English. In some cases, identified sources may have been preceded in the literature of another language, most likely Russian. For the most part, the subject of this survey developed independently in the literature published in English.

2. General treatments

General treatments on cyclostationarity are in [2.1–2.18]. The first extensive treatments of the theory of cyclostationary processes can be found in the pioneering works of Hurd [2.1] and Gardner [2.2]. In [4.13,2.5,2.11], the theory of second-order cyclostationary processes is developed mainly with reference to continuous-time stochastic processes, but discrete-time is the focus in [4.13]. Discrete-time processes are treated more generally in [2.17] in a manner largely analogous to that in [2.5]. The statistical characterization of cyclostationary time-series in the nonstochastic framework is introduced and treated in depth [2.6,2.8,2.12,2.14] with reference to continuous-time signals and in [2.15] for both continuous- and discrete-time signals. The case of complex signals is introduced and treated in depth in [2.8,2.9]. Finally, a rigorous treatment of periodically correlated processes within the framework of harmonizable processes is given in [2.18].

The theory of higher-order cyclostationarity in the nonstochastic framework is introduced in [13.4,13.5], and treated in depth in [13.9,13.13, 13.14,13.17]. An analogous treatment for stochastic processes is given in [13.10].

3. General properties and structure of stochastic processes and time-series

3.1. Introduction

General properties of cyclostationary processes (see [3.1–3.91]) are derived in terms of the Fourier series expansion of the autocorrelation function. The frequencies, called *cycle frequencies*, are multiples of the reciprocal of the period of cyclostationarity and the coefficients, referred to as *cyclic autocorrelation functions*, are continuous functions of the lag parameter. Cyclostationary processes are characterized in the frequency domain by the *cyclic spectra*, which are the Fourier transforms of the cyclic autocorrelation functions. The cyclic spectrum at a given cycle frequency represents the density of correlation between two spectral components of the process that are separated by an amount equal to the cycle frequency. Almost-cyclostationary processes have autocorrelation functions that can be expressed in a (generalized) Fourier series whose frequencies are possibly incommensurate.

The first contributions to the analysis of the general properties of cyclostationary stochastic processes are in the Russian literature [3.1–3.3, 3.5,3.7–3.9]. The problem of spectral analysis is mainly treated in [3.10,3.11,3.32,3.35,3.51,3.53, 3.61,3.68,3.69,3.80,3.88]. The stationarizing effects of random shifts are examined in [3.20,3.26,3.57] and the important impact of random shifts on cyclo-ergodic properties is exposed in [2.15]. The problem of filtering is considered in [3.9,3.13,3.67].

The mathematical link between the stochastic and nonstochastic approaches, which was first established in [2.5,2.12], is rigorously treated in [3.87]. Wavelet analysis of cyclostationary processes is addressed in [3.70,3.81].

For further-related references, see the general treatments in [2.1,2.2,2.4–2.8,2.11–2.13,2.15,2.18], and also [4.13,4.31,4.41,13.2,13.3,13.6,16.3,18.3, 21.9,21.14].

3.2. Stochastic processes

3.2.1. Continuous-time processes

Let us consider a continuous-time real-valued stochastic process $\{x(t,\omega), t \in \mathbb{R}, \omega \in \Omega\}$, with abbreviated notation $x(t)$ when this does not create ambiguity, defined on a probability space $(\Omega, \mathscr{F}, P)$, where Ω is the sample space, equipped with the σ-field $\mathscr{F}$, and P is a probability measure defined on the elements of $\mathscr{F}$.

The process $x(t)$ is said to be *Nth-order cyclostationary in the strict sense* [2.2,2.5] if its Nth-order distribution function

$$\begin{aligned} F_{x(t+\tau_1)\cdots x(t+\tau_{N-1})x(t)}&(\xi_1,\ldots,\xi_{N-1},\xi_N) \\ &\triangleq P\{x(t+\tau_1)\leqslant\xi_1,\ldots,x(t+\tau_{N-1}) \\ &\quad\leqslant\xi_{N-1}, x(t)\leqslant\xi_N\} \end{aligned} \tag{3.1}$$

is periodic in t with some period, say T_0:

$$\begin{aligned} F_{x(t+\tau_1+T_0)\cdots x(t+\tau_{N-1}+T_0)x(t+T_0)}&(\xi_1,\ldots,\xi_{N-1},\xi_N) \\ &= F_{x(t+\tau_1)\cdots x(t+\tau_{N-1})x(t)}(\xi_1,\ldots,\xi_{N-1},\xi_N) \\ &\quad \forall t\in\mathbb{R} \quad \forall(\tau_1,\ldots,\tau_{N-1})\in\mathbb{R}^{N-1} \\ &\quad \forall(\xi_1,\ldots,\xi_N)\in\mathbb{R}^N. \end{aligned} \tag{3.2}$$

The process $x(t)$ is said to be *second-order cyclostationary in the wide sense* [2.2,2.5] if its mean $\mathrm{E}\{x(t)\}$ and autocorrelation function

$$\mathscr{R}_x(t,\tau) \triangleq \mathrm{E}\{x(t+\tau)x(t)\} \tag{3.3}$$

are periodic with some period, say T_0:

$$\mathrm{E}\{x(t+T_0)\} = \mathrm{E}\{x(t)\}, \tag{3.4}$$

$$\mathscr{R}_x(t+T_0,\tau) = \mathscr{R}_x(t,\tau) \tag{3.5}$$

for all t and τ. Therefore, by assuming that the Fourier series expansion of $\mathscr{R}_x(t,\tau)$ is convergent to $\mathscr{R}_x(t,\tau)$, we can write

$$\mathscr{R}_x(t,\tau) = \sum_{n=-\infty}^{+\infty} \mathscr{R}_x^{n/T_0}(\tau)\mathrm{e}^{\mathrm{j}2\pi(n/T_0)t}, \tag{3.6}$$

where the Fourier coefficients

$$\mathscr{R}_x^{n/T_0}(\tau) \triangleq \frac{1}{T_0}\int_{-T_0/2}^{T_0/2} \mathscr{R}_x(t,\tau)\mathrm{e}^{-\mathrm{j}2\pi(n/T_0)t}\,\mathrm{d}t \tag{3.7}$$

are referred to as *cyclic autocorrelation functions* and the frequencies $\{n/T_0\}_{n\in\mathbb{Z}}$ are called *cycle frequencies*. Wide-sense cyclostationary processes have also been called *periodically correlated* processes (see e.g., [2.1,3.3,3.59]). As first shown in [3.23], $x(t)$ and its frequency-shifted version $x(t)\mathrm{e}^{\mathrm{j}2\pi nt/T_0}$ are correlated. The wide-sense stationary processes are the special case of cyclostationary processes for which $\mathscr{R}_x^{n/T_0}(\tau)\not\equiv 0$ only for $n=0$. It can be shown that if $x(t)$ is cyclostationary with period T_0, then the stochastic process $x(t+\theta)$, where θ is a random variable that is uniformly distributed in $[0,T_0)$ and is statistically independent of $x(t)$, is wide sense stationary [2.5,3.20,3.26,3.57].

A more general class of stochastic processes is obtained if the autocorrelation function $\mathscr{R}_x(t,\tau)$ is almost periodic in t for each τ.

A function $z(t)$ is said to be *almost periodic* if it is the limit of a uniformly convergent sequence of trigonometric polynomials in t [23.2,23.3,23.4, Paragraphs 24–25,23.6,23.7, Part 5, 23.11]:

$$z(t) = \sum_{\alpha\in A} z_\alpha \mathrm{e}^{\mathrm{j}2\pi\alpha t}, \tag{3.8}$$

where A is a countable set, the frequencies $\alpha\in A$ are possibly incommensurate, and the coefficients z_α are given by

$$\begin{aligned} z_\alpha &\triangleq \langle z(t)\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\rangle_t \\ &= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} z(t)\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\,\mathrm{d}t. \end{aligned} \tag{3.9}$$

Such functions are called *almost-periodic* in the sense of Bohr [23.3, Paragraphs 84–92] or, equivalently, uniformly almost periodic in t in the sense of Besicovitch [23.2, Chapter 1].

The process $x(t)$ is said to be *Nth-order almost-cyclostationary in the strict sense* [2.2,2.5] if its Nth-order distribution function (3.1) is almost-periodic in t. With reference to the autocorrelation properties, a continuous-time real-valued stochastic process $x(t)$ is said to be *almost-cyclostationary* (*ACS*) *in the wide sense* [2.2,2.5,3.26] if its autocorrelation function $\mathscr{R}_x(t,\tau)$ is an almost periodic function of t (with frequencies not depending on τ). Thus, it is the limit of a uniformly convergent sequence of trigonometric polynomials in t:

$$\mathscr{R}_x(t,\tau) = \sum_{\alpha\in A}\mathscr{R}_x^\alpha(\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha t}, \tag{3.10}$$

where

$$\mathscr{R}_x^\alpha(\tau) \triangleq \lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \mathscr{R}_x(t,\tau)\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\,\mathrm{d}t \tag{3.11}$$

is the cyclic autocorrelation function at cycle frequency α. As first shown in [3.26], if $x(t)$ is an ACS process, then the process $x(t)$ and its frequency-shifted version $x(t)\mathrm{e}^{\mathrm{j}2\pi\alpha t}$ are correlated when $\alpha \in A$. Wide-sense almost-cyclostationary processes have also been called *almost-periodically correlated* processes (see e.g., [2.1,3.5,3.59]). The wide-sense cyclostationary processes are obtained as a special case of the ACS processes when $A \equiv \{n/T_0\}_{n\in\mathbb{Z}}$ for some T_0.

Let A_τ be the set

$$A_\tau \triangleq \{\alpha \in \mathbb{R} : \mathscr{R}_x^\alpha(\tau) \neq 0\}. \tag{3.12}$$

In [3.59], it is shown that the ACS processes are characterized by the following conditions:

(1) The set
$$A \triangleq \bigcup_{\tau\in\mathbb{R}} A_\tau \tag{3.13}$$
is countable.
(2) The autocorrelation function $\mathscr{R}_x(t,\tau)$ is uniformly continuous in t and τ.
(3) The time-averaged autocorrelation function $\mathscr{R}_x^0(\tau) \triangleq \langle \mathscr{R}_x(t,\tau)\rangle_t$ is continuous for $\tau = 0$ (and, hence, for every τ).
(4) The process is mean-square continuous, that is
$$\sup_{t\in\mathbb{R}} \mathrm{E}\{|x(t+\tau) - x(t)|^2\} \to 0 \quad \text{as } \tau \to 0. \tag{3.14}$$

More generally, a stochastic process $x(t)$ is said to *exhibit cyclostationarity* at cycle frequency α if $\mathscr{R}_x^\alpha(\tau) \not\equiv 0$ [2.5]. In such a case, the autocorrelation function can be expressed as

$$\mathscr{R}_x(t,\tau) = \sum_{\alpha\in A} \mathscr{R}_x^\alpha(\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha t} + r_x(t,\tau), \tag{3.15}$$

where the function

$$\sum_{\alpha\in A} \mathscr{R}_x^\alpha(\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha t}$$

is not necessarily continuous in t and the term $r_x(t,\tau)$ does not contain any finite-strength additive sinewave component:

$$\langle r_x(t,\tau)\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\rangle_t = 0 \quad \forall\alpha \in \mathbb{R}. \tag{3.16}$$

In the special case where $\lim_{|t|\to\infty} r_x(t,\tau) = 0$, $x(t)$ is said to be *asymptotically almost cyclostationary* [3.26].

Let us now consider the second-order (wide-sense) characterization of ACS processes in the frequency domain. Let

$$X(f) \triangleq \int_{\mathbb{R}} x(t)\mathrm{e}^{-\mathrm{j}2\pi f t}\,\mathrm{d}t \tag{3.17}$$

be the stochastic process obtained from Fourier transformation of the ACS process $x(t)$, where the Fourier transform is assumed to exist, at least in the sense of distributions (generalized functions) [23.10], with probability 1. By using (3.10), in the sense of distributions we obtain

$$\mathrm{E}\{X(f_1)X^*(f_2)\} = \sum_{\alpha\in A} \mathscr{S}_x^\alpha(f_1)\delta(f_2 - f_1 + \alpha), \tag{3.18}$$

where $\delta(\cdot)$ is the Dirac delta, superscript $*$ is complex conjugation, and

$$\mathscr{S}_x^\alpha(f) \triangleq \int_{\mathbb{R}} \mathscr{R}_x^\alpha(\tau)\mathrm{e}^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \tag{3.19}$$

is referred to as the *cyclic spectrum* at cycle frequency α. Therefore, for an ACS process, correlation exists between spectral components that are separated by amounts equal to the cycle frequencies. The support in the (f_1,f_2) plane of the spectral correlation function $\mathrm{E}\{X(f_1)X^*(f_2)\}$ consists of parallel lines with unity slope. The density of spectral correlation on this support is described by the cyclic spectra $\mathscr{S}_x^\alpha(f)$, $\alpha \in A$, which can be expressed as [2.5]

$$\mathscr{S}_x^\alpha(f) = \lim_{\Delta f\to 0}\lim_{T\to\infty} \frac{1}{T}\int_{-T/2}^{T/2} \mathrm{E}\{\Delta f X_{1/\Delta f}(t,f) \times X_{1/\Delta f}^*(t,f-\alpha)\}\,\mathrm{d}t, \tag{3.20}$$

where the order of the two limits cannot be reversed, and

$$X_Z(t,f) \triangleq \int_{t-Z/2}^{t+Z/2} x(s)\mathrm{e}^{-\mathrm{j}2\pi f s}\,\mathrm{d}s. \tag{3.21}$$

Therefore, the cyclic spectrum $\mathscr{S}_x^{\alpha}(f)$ is also called the *spectral correlation density function*. It represents the time-averaged statistical correlation (with zero lag) of two spectral components at frequencies f and $f-\alpha$, as the bandwidth approaches zero. For $\alpha=0$, the cyclic spectrum reduces to the power spectrum or spectral density function $\mathscr{S}_x^{0}(f)$ and (3.19) reduces to the *Wiener–Khinchin Relation*. Consequently, when (3.19) and (3.20) was discovered in [2.5] it was dubbed the *Cyclic Wiener–Khinchin Relation*.

In contrast, for wide-sense stationary processes the autocorrelation function does not depend on t,

$$\mathrm{E}\{x(t+\tau)x(t)\} = \mathscr{R}_x^{0}(\tau) \tag{3.22}$$

and, equivalently, no correlation exists between distinct spectral components,

$$\mathrm{E}\{X(f_1)X^*(f_2)\} = \mathscr{S}_x^{0}(f_1)\delta(f_2-f_1). \tag{3.23}$$

A covariance (or autocorrelation) function $\mathrm{E}\{x(t_1)x^*(t_2)\}$ is said to be *harmonizable* if it can be expressed as

$$\mathrm{E}\{x(t_1)x^*(t_2)\} = \int_{\mathbb{R}^2} \mathrm{e}^{\mathrm{j}2\pi(f_1t_1-f_2t_2)}\,\mathrm{d}\gamma(f_1,f_2), \tag{3.24}$$

where $\gamma(f_1,f_2)$ is a covariance of bounded variation on $\mathbb{R}\times\mathbb{R}$ and the integral is a Fourier–Stieltjes transform [23.5]. Moreover, a second-order stochastic process $x(t)$ is said to be harmonizable if there exists a second-order stochastic process $\chi(f)$ with covariance function $\mathrm{E}\{\chi(f_1)\chi^*(f_2)\}=\gamma(f_1,f_2)$ of bounded variation on $\mathbb{R}\times\mathbb{R}$ such that

$$x(t) = \int_{\mathbb{R}} \mathrm{e}^{\mathrm{j}2\pi ft}\,\mathrm{d}\chi(f) \tag{3.25}$$

with probability one. In [23.5], it is shown that a necessary condition for a stochastic process to be harmonizable is that it be second-order continuous. Moreover, it is shown that a stochastic process is harmonizable if and only if its covariance is harmonizable. If a stochastic process is harmonizable, in the sense of distributions [23.10], we have $\mathrm{d}\chi(f)=X(f)\,\mathrm{d}f$ and $\mathrm{d}\gamma(f_1,f_2)=\mathrm{E}\{\mathrm{d}\chi(f_1)\,\mathrm{d}\chi^*(f_2)\}=\mathrm{E}\{X(f_1)X^*(f_2)\}\,\mathrm{d}f_1\,\mathrm{d}f_2$, and $\chi(f)$ is the indefinite integral of $X(f)$ or the integrated Fourier transform of $x(t)$ [23.4]. Therefore, if the stochastic process is harmonizable and ACS, then it follows that [2.1,2.18,3.59]

$$\mathrm{d}\gamma(f_1,f_2) = \sum_{\alpha\in A} \mathscr{S}_x^{\alpha}(f_1)\delta(f_2-f_1+\alpha)\,\mathrm{d}f_1\,\mathrm{d}f_2. \tag{3.26}$$

Finally, note that symmetric definitions of cyclic autocorrelation function and cyclic spectrum have been widely used (see, e.g., [2.5,2.8,2.11]). They are linked to the asymmetric definitions (3.11) and (3.20) by the relationships

$$\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\mathrm{E}\{x(t+\tau/2)x(t-\tau/2)\} \times\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\,\mathrm{d}t = \mathscr{R}_x^{\alpha}(\tau)\mathrm{e}^{-\mathrm{j}\pi\alpha\tau} \tag{3.27}$$

and

$$\lim_{\Delta f\to 0}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\Delta f\,\mathrm{E}\{X_{1/\Delta f}(t,f+\alpha/2) \times X^*_{1/\Delta f}(t,f-\alpha/2)\}\,\mathrm{d}t = \mathscr{S}_x^{\alpha}(f+\alpha/2), \tag{3.28}$$

respectively.

3.2.2. Discrete-time processes

Let us consider a discrete-time real-valued stochastic process $\{x(n,\omega), n\in\mathbb{Z}, \omega\in\Omega\}$, with abbreviated notation $x(n)$ when this does not create ambiguity. The stochastic process $x(n)$ is said to be *second-order almost-cyclostationary in the wide sense* [2.2,2.5,4.13] if its autocorrelation function

$$\widetilde{\mathscr{R}}_x(n,m) \triangleq \mathrm{E}\{x(n+m)x(n)\} \tag{3.29}$$

is an almost-periodic function of the discrete-time parameter n. Thus, it can be expressed as

$$\widetilde{\mathscr{R}}_x(n,m) = \sum_{\widetilde{\alpha}\in\widetilde{A}} \widetilde{\mathscr{R}}_x^{\widetilde{\alpha}}(m)\mathrm{e}^{\mathrm{j}2\pi\widetilde{\alpha}n}, \tag{3.30}$$

where

$$\widetilde{\mathscr{R}}_x^{\widetilde{\alpha}}(m) \triangleq \lim_{N\to\infty}\frac{1}{2N+1}\sum_{n=-N}^{N}\widetilde{\mathscr{R}}_x(n,m)\mathrm{e}^{-\mathrm{j}2\pi\widetilde{\alpha}n} \tag{3.31}$$

is the *cyclic autocorrelation function* at *cycle frequency* $\widetilde{\alpha}$ and

$$\widetilde{A} \triangleq \{\widetilde{\alpha}\in[-\tfrac{1}{2},\tfrac{1}{2}) : \widetilde{\mathscr{R}}_x^{\widetilde{\alpha}}(m)\not\equiv 0\} \tag{3.32}$$

is a countable set. Note that the cyclic autocorrelation function $\widetilde{\mathscr{R}}_x^{\widetilde{\alpha}}(m)$ is periodic in $\widetilde{\alpha}$ with period 1. Thus, the sum in (3.30) can be equivalently extended to the set $\widetilde{A}_1 \triangleq \{\widetilde{\alpha} \in [0,1) : \widetilde{\mathscr{R}}_x^{\widetilde{\alpha}}(m) \not\equiv 0\}$.

In general, the set $\widetilde{A}$ (or $\widetilde{A}_1$) contains possibly incommensurate cycle frequencies $\widetilde{\alpha}$. In the special case where $\widetilde{A}_1 \equiv \{0, 1/N_0, \ldots, (N_0-1)/N_0\}$ for some integer N_0, the autocorrelation function $\widetilde{\mathscr{R}}_x(n,m)$ is periodic in n with period N_0 and the process $x(n)$ is said to be *cyclostationary in the wide sense*. If $N_0 = 1$ then $x(n)$ is *wide-sense stationary*.

Let

$$\widetilde{X}(v) \triangleq \sum_{n\in\mathbb{Z}} x(n) e^{-j2\pi v n} \tag{3.33}$$

be the stochastic process obtained from Fourier transformation (in a generalized sense) of the ACS process $x(n)$. By using (3.30), in the sense of distributions it can be shown that [3.59]

$$\mathrm{E}\{\widetilde{X}(v_1)\widetilde{X}^*(v_2)\} = \sum_{\widetilde{\alpha}\in\widetilde{A}} \widetilde{\mathscr{S}}_x^{\widetilde{\alpha}}(v_1) \sum_{\ell\in\mathbb{Z}} \delta(v_2 - v_1 + \widetilde{\alpha} - \ell), \tag{3.34}$$

where

$$\widetilde{\mathscr{S}}_x^{\widetilde{\alpha}}(v) \triangleq \sum_{m\in\mathbb{Z}} \widetilde{\mathscr{R}}_x^{\widetilde{\alpha}}(m) e^{-j2\pi v m} \tag{3.35}$$

is the cyclic spectrum at cycle frequency $\widetilde{\alpha}$. The cyclic spectrum $\widetilde{\mathscr{S}}_x^{\widetilde{\alpha}}(v)$ is periodic in both v and $\widetilde{\alpha}$ with period 1.

In [3.3,4.41], it is shown that discrete-time cyclostationary processes are harmonizable.

3.3. Time series

3.3.1. Continuous-time time series

A statistical analysis framework for time series that is an alternative to the classical stochastic-process framework is the *fraction-of-time* (FOT) probability framework first introduced for ACS time series in [2.5,2.8,2.12]; see also [2.15]. In the FOT probability framework, signals are modelled as single functions of time (time series) rather than sample paths of stochastic processes. This approach turns out to be more appropriate when an ensemble of realizations does not exist and would have to be artificially introduced just to create a mathematical model, that is, the stochastic process. Such a model can be unnecessarily abstract when there is only a single time series at hand.

In the FOT probability approach, probabilistic parameters are defined through infinite-time averages of a single time series (and functions of this time series) rather than through expected values or ensemble averages of a stochastic process. Moreover, starting from this single time series, a (possibly time varying) probability distribution function can be constructed and this leads to an expectation operation and all the associated familiar probabilistic concepts and parameters, such as stationarity, cyclostationarity, nonstationarity, independence, mean, variance, moments, cumulants, etc. For comprehensive treatments of the FOT probability framework see [2.8,2.12,2.15] and, for more mathematical rigor on foundations and existence proofs, see [23.15]. The extension of the Wold isomorphism to cyclostationary sequences was first introduced in [2.5], treated more in depth in [2.12], and finally—with mathematical rigor—in [3.87].

The time-variant FOT probability framework is based on the decomposition of functions of a time series into their (possibly zero) almost-periodic components and residual terms. Starting from such a decomposition, the expectation operator is defined. Specifically, for any finite-average-power time series $x(t)$, let us consider the decomposition

$$x(t) \triangleq x_{\mathrm{ap}}(t) + x_{\mathrm{r}}(t), \tag{3.36}$$

where $x_{\mathrm{ap}}(t)$ is an almost-periodic function and $x_{\mathrm{r}}(t)$ a residual term not containing finite-strength additive sinewave components; that is,

$$\langle x_{\mathrm{r}}(t) e^{-j2\pi\alpha t}\rangle_t \equiv 0 \quad \forall \alpha \in \mathbb{R}. \tag{3.37}$$

The *almost-periodic component extraction operator* $\mathrm{E}^{\{\alpha\}}\{\cdot\}$ is defined to be the operator that extracts all the finite-strength additive sinewave components of its argument, that is,

$$\mathrm{E}^{\{\alpha\}}\{x(t)\} \triangleq x_{\mathrm{ap}}(t). \tag{3.38}$$

Let $x(t)$, $t \in \mathbb{R}$, be a real-valued continuous-time time series (a single function of time) and let us assume that the set Γ_1 of frequencies (for every ξ) of the almost-periodic component of the function of t $\quad \mathbf{1}_{\{x(t) \leqslant \xi\}}$ is countable, where

$$\mathbf{1}_{\{x(t) \leqslant \xi\}} \triangleq \begin{cases} 1, & t : x(t) \leqslant \xi, \\ 0, & t : x(t) > \xi \end{cases} \tag{3.39}$$

is the indicator function of the set $\{t \in \mathbb{R} : x(t) \leqslant \xi\}$. In [2.12] it is shown that the function of ξ

$$F_{x(t)}^{\{\alpha\}}(\xi) \triangleq \mathrm{E}^{\{\alpha\}}\{\mathbf{1}_{\{x(t) \leqslant \xi\}}\} \tag{3.40}$$

for any t is a valid cumulative distribution function except for the right-continuity property (in the discontinuity points). That is, it has values in $[0, 1]$, is non decreasing, $F_{x(t)}^{\{\alpha\}}(-\infty) = 0$, and $F_{x(t)}^{\{\alpha\}}(+\infty) = 1$. Moreover, its derivative with respect to ξ, denoted by $f_{x(t)}^{\{\alpha\}}(\xi)$, is a valid probability density function and, for any well-behaved function $g(\cdot)$, it follows that

$$\mathrm{E}^{\{\alpha\}}\{g(x(t))\} = \int_{\mathbb{R}} g(\xi) f_{x(t)}^{\{\alpha\}}(\xi)\,\mathrm{d}\xi \tag{3.41}$$

which reveals that $\mathrm{E}^{\{\alpha\}}\{\cdot\}$ is the expectation operator with respect to the distribution function $F_{x(t)}^{\{\alpha\}}(\xi)$ for the time series $x(t)$. The result (3.41), first introduced in [2.8], is referred to as the *fundamental theorem of temporal expectation*, by analogy with the corresponding fundamental theorem of expectation from probability theory.

For an almost-periodic signal $x(t)$, we have $x(t) \equiv x_{\mathrm{ap}}(t)$ and, hence,

$$\mathrm{E}^{\{\alpha\}}\{x(t)\} = x(t). \tag{3.42}$$

That is, the almost-periodic functions are the deterministic signals in the FOT probability framework. All the other signals are the random signals. Note that *the term "random" here is not intended to be synonymous with "stochastic"*. In fact, the adjective stochastic is adopted, as usual, when an ensemble of realizations or sample paths exists, whereas the adjective random is used in reference to a single function of time.

Analogously, a second-order characterization for the real-valued time series $x(t)$ can be obtained by using the almost-periodic component extraction operator as the expectation operator. Specifically, let us assume that the set Γ_2 of frequencies (for every ξ_1, ξ_2 and τ) of the almost-periodic component of the function of t $\quad \mathbf{1}_{\{x(t+\tau) \leqslant \xi_1\}}\mathbf{1}_{\{x(t) \leqslant \xi_2\}}$ is countable. Then, the function of ξ_1 and ξ_2

$$F_{x(t+\tau)x(t)}^{\{\alpha\}}(\xi_1, \xi_2) \triangleq \mathrm{E}^{\{\alpha\}}\{\mathbf{1}_{\{x(t+\tau) \leqslant \xi_1\}}\mathbf{1}_{\{x(t) \leqslant \xi_2\}}\} \tag{3.43}$$

is a valid second-order joint cumulative distribution function for every fixed t and τ, except for the right-continuity property (in the discontinuity points) with respect to ξ_1 and ξ_2. Moreover, the second-order derivative, with respect to ξ_1 and ξ_2, of $F_{x(t+\tau)x(t)}^{\{\alpha\}}(\xi_1, \xi_2)$, denoted by $f_{x(t+\tau)x(t)}^{\{\alpha\}}(\xi_1, \xi_2)$, is a valid second-order joint probability density function [2.12]. Furthermore, it can be shown that the function

$$R_x(t, \tau) \triangleq \mathrm{E}^{\{\alpha\}}\{x(t+\tau)x(t)\} \tag{3.44}$$

is a valid autocorrelation function and can be characterized by

$$\begin{aligned} R_x(t, \tau) &= \int_{\mathbb{R}^2} \xi_1 \xi_2 f_{x(t+\tau)x(t)}^{\{\alpha\}}(\xi_1, \xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 \\ &= \sum_{\alpha \in A} R_x^{\alpha}(\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha t}, \end{aligned} \tag{3.45}$$

where A is a countable set and

$$R_x^{\alpha}(\tau) \triangleq \langle x(t+\tau)x(t)\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\rangle_t \tag{3.46}$$

is the (nonstochastic) *cyclic autocorrelation function* at cycle frequency α ($\alpha \in \mathbb{R}$).

The classification of the kind of nonstationarity of a time series is made on the basis of the elements contained in the set A. In general, the set A can contain incommensurate cycle frequencies α and, in such a case, the time series is said to be *wide-sense almost cyclostationary*. In the special case where $A \equiv \{k/T_0\}_{k\in\mathbb{Z}}$ the time series $x(t)$ is said to be *wide-sense cyclostationary*. If the set A contains only the element $\alpha = 0$, then the time series $x(t)$ is said to be *wide-sense stationary*.

Finally, note that the periodic component with period T_0 of an almost-periodic time series (or lag product time series) $z(t)$ can be extracted by exploiting the synchronized averaging identity introduced in [2.5,2.8]:

$$\sum_{k=-\infty}^{+\infty} z_{k/T_0} \mathrm{e}^{\mathrm{j}2\pi(k/T_0)t} = \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} z(t-nT_0), \tag{3.47}$$

where the Fourier coefficients z_{k/T_0} are defined according to (3.9).

ACS time series are characterized in the spectral domain by the (nonstochastic) *cyclic spectrum* or *spectral correlation density function* at cycle frequency α:

$$S_x^{\alpha}(f) \triangleq \lim_{\Delta f\to 0} \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \Delta f \times X_{1/\Delta f}(t,f) X^*_{1/\Delta f}(t,f-\alpha)\,\mathrm{d}t, \tag{3.48}$$

where $X_{1/\Delta f}(t,f)$ is defined according to (3.21) and the order of the two limits cannot be reversed. The *Cyclic Wiener Relation* introduced in [2.8]

$$S_x^{\alpha}(f) = \int_{\mathbb{R}} R_x^{\alpha}(\tau) \mathrm{e}^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \tag{3.49}$$

links the cyclic autocorrelation function to the cyclic spectrum. This relation generalizes that for $\alpha = 0$, which was first dubbed the *Wiener Relation* in [2.5] to distinguish it from the *Khinchin Relation* (3.19), which is frequently called the *Wiener–Khinchin Relation*.

3.3.2. Discrete-time time series

The characterization of discrete-time time series (sequences) is similar to that of continuous-time time series. We consider here only the wide-sense second-order characterization.

Let $x(n)$, $n \in \mathbb{Z}$, be a real-valued discrete-time time series. Let us assume that the set $\widetilde{\Gamma}_2$ of frequencies (for every ξ_1, ξ_2 and m) of the almost-periodic component of the function of n $\mathbf{1}_{\{x(n+m)\leqslant\xi_1\}}\mathbf{1}_{\{x(n)\leqslant\xi_2\}}$ is countable. Then, the function of ξ_1 and ξ_2

$$F_{x(n+m)x(n)}^{\{\widetilde{\alpha}\}}(\xi_1,\xi_2) \triangleq \mathrm{E}^{\{\widetilde{\alpha}\}}\{\mathbf{1}_{\{x(n+m)\leqslant\xi_1\}}\mathbf{1}_{\{x(n)\leqslant\xi_2\}}\} \tag{3.50}$$

is a valid second-order joint cumulative distribution function for every fixed n and m, except for the right-continuity property (in the discontinuity points) with respect to ξ_1 and ξ_2. In (3.50), $\mathrm{E}^{\{\widetilde{\alpha}\}}\{\cdot\}$ denotes the discrete-time almost-periodic component extraction operator, which is defined analogously to its continuous-time counterpart. The second-order derivative with respect to ξ_1 and ξ_2 of $F_{x(n+m)x(n)}^{\{\widetilde{\alpha}\}}(\xi_1,\xi_2)$, denoted by $f_{x(n+m)x(n)}^{\{\widetilde{\alpha}\}}(\xi_1,\xi_2)$, is a valid second-order joint probability density function [2.12]. Furthermore, it can be shown that the function

$$\widetilde{R}_x(n,m) \triangleq \mathrm{E}^{\{\widetilde{\alpha}\}}\{x(n+m)x(n)\} \tag{3.51}$$

is a valid autocorrelation function and can be characterized by

$$\widetilde{R}_x(n,m) = \int_{\mathbb{R}^2} \xi_1\xi_2 f_{x(n+m)x(n)}^{\{\widetilde{\alpha}\}}(\xi_1,\xi_2)\,\mathrm{d}\xi_1\,\mathrm{d}\xi_2 = \sum_{\widetilde{\alpha}\in\widetilde{A}} \widetilde{R}_x^{\widetilde{\alpha}}(m)\mathrm{e}^{\mathrm{j}2\pi\widetilde{\alpha}n}, \tag{3.52}$$

where

$$\widetilde{A} \triangleq \{\widetilde{\alpha} \in [-\tfrac{1}{2},\tfrac{1}{2}) : \widetilde{R}_x^{\widetilde{\alpha}}(m) \not\equiv 0\} \tag{3.53}$$

is a countable set and

$$\widetilde{R}_x^{\widetilde{\alpha}}(m) \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} x(n+m)x(n)\mathrm{e}^{-\mathrm{j}2\pi\widetilde{\alpha}n} \tag{3.54}$$

is the (nonstochastic) discrete-time *cyclic autocorrelation function* at cycle frequency $\widetilde{\alpha}$.

Obviously, as in the stochastic framework, the cyclic autocorrelation function $\widetilde{R}_x^{\widetilde{\alpha}}(m)$ is periodic in $\widetilde{\alpha}$ with period 1. Thus, the sum in (3.52) can be equivalently extended to the set $\widetilde{A}_1 \triangleq \{\widetilde{\alpha} \in [0,1) : \widetilde{R}_x^{\widetilde{\alpha}}(m) \not\equiv 0\}$. Moreover, as in the continuous-time case, the classification of the kind of nonstationarity of discrete-time time series is made on the basis of the elements contained in set $\widetilde{A}$. That is, in general, set $\widetilde{A}$ can contain incommensurate cycle frequencies $\widetilde{\alpha}$ and, in such a case, the time series is said to be *wide-sense almost cyclostationary*. In the special case where $\widetilde{A}_1 \equiv \{0, 1/N_0, \ldots, (N_0 - 1)/N_0\}$ for some integer N_0, the time series $x(n)$ is said to be *wide-sense cyclostationary*. If the set $\widetilde{A}$ contains only the element $\widetilde{\alpha} = 0$, then the time series $x(n)$ is said to be *wide-sense stationary*.

Discrete-time ACS time series are characterized in the spectral domain by the (nonstochastic) *cyclic spectrum* or *spectral correlation density function* at cycle frequency $\widetilde{\alpha}$:

$$\widetilde{S}_x^{\widetilde{\alpha}}(v) \triangleq \lim_{\Delta v \to 0} \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \Delta v \times X_{\lfloor 1/\Delta v \rfloor}(n, v) X^*_{\lfloor 1/\Delta v \rfloor}(n, v - \widetilde{\alpha}), \tag{3.55}$$

where

$$X_{2M+1}(n, v) \triangleq \sum_{k=n-M}^{n+M} x(k) \mathrm{e}^{-\mathrm{j}2\pi v k} \tag{3.56}$$

and $\lfloor \cdot \rfloor$ denotes the closest odd integer. The *Cyclic Wiener relation* [2.5]

$$\widetilde{S}_x^{\widetilde{\alpha}}(v) = \sum_{m \in \mathbb{Z}} \widetilde{R}_x^{\widetilde{\alpha}}(m) \mathrm{e}^{-\mathrm{j}2\pi v m} \tag{3.57}$$

links the cyclic autocorrelation function to the cyclic spectrum.

3.4. Link between the stochastic and fraction-of-time approaches

In the time-variant nonstochastic framework (for ACS time series), the almost-periodic functions play the same role as that played by the deterministic functions in the stochastic-process framework, and the expectation operator is the almost-periodic component extraction operator. Therefore, in the case of processes exhibiting suitable ergodicity properties, results derived in the FOT probability approach for time series can be interpreted in the classical stochastic process framework by substituting, in place of the almost-periodic component extraction operator $\mathrm{E}^{\{\alpha\}}\{\cdot\}$, the statistical expectation operator $\mathrm{E}\{\cdot\}$. In fact, by assuming that $\{\xi(t, \omega), t \in \mathbb{R}, \omega \in \Omega\}$ is a stochastic process satisfying appropriate ergodicity properties (see Section 4), the stochastic and FOT autocorrelation functions are identical for almost all sample paths $x(t)$,

$$\mathscr{R}_\xi(t, \tau) \triangleq \mathrm{E}\{\xi(t+\tau)\xi(t)\} = \mathrm{E}^{\{\alpha\}}\{x(t+\tau)x(t)\} \triangleq R_x(t, \tau) \tag{3.58}$$

and, hence, so too are their cyclic components and frequency-domain counterparts,

$$\mathscr{R}_\xi^\alpha(\tau) = R_x^\alpha(\tau), \tag{3.59}$$

$$\mathscr{S}_\xi^\alpha(f) = S_x^\alpha(f). \tag{3.60}$$

Analogous equivalences hold in the discrete-time case.

The link between the two approaches is treated in depth in [2.5,2.8,2.9,2.11,2.12,2.15,3.87,4.31, 23.15]. In the following, most results are presented in the FOT probability framework.

3.5. Complex processes and time series

The case of complex-valued processes and time series, first treated in depth in [2.9], is extensively treated in [2.6,2.8,3.43,13.13,13.14,13.20,13.25]. Several results with reference to higher-order statistics are reported in Section 13.

Let $x(t)$ be a zero-mean complex-valued continuous-time time series. Its wide-sense characterization can be made in terms of the two second-order moments

$$R_{xx^*}(t, \tau) \triangleq \mathrm{E}^{\{\alpha\}}\{x(t+\tau)x^*(t)\} = \sum_{\alpha \in A_{xx^*}} R_{xx^*}^\alpha(\tau) \mathrm{e}^{\mathrm{j}2\pi\alpha t}, \tag{3.61}$$

$$R_{xx}(t, \tau) \triangleq \mathrm{E}^{\{\alpha\}}\{x(t+\tau)x(t)\} = \sum_{\beta \in A_{xx}} R_{xx}^\beta(\tau) \mathrm{e}^{\mathrm{j}2\pi\beta t} \tag{3.62}$$

which are called the *autocorrelation function* and *conjugate autocorrelation function*, respectively. The Fourier coefficients

$$R_{xx^*}^\alpha(\tau) \triangleq \langle x(t+\tau)x^*(t)\mathrm{e}^{-\mathrm{j}2\pi\alpha t} \rangle_t, \tag{3.63}$$

$$R_{xx}^\beta(\tau) \triangleq \langle x(t+\tau)x(t)\mathrm{e}^{-\mathrm{j}2\pi\beta t} \rangle_t \tag{3.64}$$

are referred to as the *cyclic autocorrelation function* at *cycle frequency* α and the *conjugate cyclic autocorrelation function* at *conjugate cycle frequency* β, respectively. The Fourier transforms of the cyclic autocorrelation function and conjugate cyclic autocorrelation function

$$S^{\alpha}_{xx^*}(f) \triangleq \int_{\mathbb{R}} R^{\alpha}_{xx^*}(\tau) \mathrm{e}^{-\mathrm{j}2\pi f\tau}\, \mathrm{d}\tau, \tag{3.65}$$

$$S^{\beta}_{xx}(f) \triangleq \int_{\mathbb{R}} R^{\beta}_{xx}(f) \mathrm{e}^{-\mathrm{j}2\pi f\tau}\, \mathrm{d}\tau \tag{3.66}$$

are called the *cyclic spectrum* and the *conjugate cyclic spectrum*, respectively.

Let us denote by the superscript $(*)$ an optional complex conjugation. In the following, when it does not create ambiguity, both functions defined in (3.63) and (3.64) are simultaneously represented by

$$R^{\alpha}_{xx^{(*)}}(\tau) \triangleq \left\langle x(t+\tau)x^{(*)}(t)\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\right\rangle_t, \quad \alpha \in A_{xx^{(*)}}. \tag{3.67}$$

Analogously, both functions defined in (3.65) and (3.66) are simultaneously represented by

$$S^{\alpha}_{xx^{(*)}}(f) \triangleq \int_{\mathbb{R}} R^{\alpha}_{xx^{(*)}}(\tau) \mathrm{e}^{-\mathrm{j}2\pi f\tau}\, \mathrm{d}\tau. \tag{3.68}$$

It should be noted that, in [2.6,2.8,2.9], for example, a different conjugation notation in the subscripts is adopted to denote the autocorrelation and the conjugate autocorrelation function. Specifically, $R_{xx}(t,\tau) = \mathrm{E}^{\{\alpha\}}\{x(t+\tau/2)x^*(t-\tau/2)\}$ and $R_{xx^*}(t,\tau) = \mathrm{E}^{\{\alpha\}}\{x(t+\tau/2)x(t-\tau/2)\}$. Moreover, an analogous notation is adopted in these references and others for the (conjugate) cyclic autocorrelation function and the (conjugate) cyclic spectrum.

3.6. Linear filtering

The linear almost-periodically time-variant filtering of ACS signals introduced in [2.2] and followed by [2.5] for cyclostationary processes and generalized in [2.8,2.9] to ACS signals, is also considered in follow-on work in [7.60,9.51,13.9, 13.14,13.20] as well as the early Russian work in [3.9]. Properties of linear periodically time-varying systems are analyzed in [23.8,23.12,23.13,23.14]. More general time-variant linear filtering is addressed in [21.10,21.11].

3.6.1. Structure of linear almost-periodically time-variant systems

A linear time-variant system with input $x(t)$, output $y(t)$, impulse-response function $h(t,u)$, and input–output relation

$$y(t) = \int_{\mathbb{R}} h(t,u)x(u)\,\mathrm{d}u \tag{3.69}$$

is said to be linear almost-periodically time-variant (LAPTV) if the impulse-response function admits the Fourier series expansion

$$h(t,u) = \sum_{\sigma \in G} h_\sigma(t-u)\mathrm{e}^{\mathrm{j}2\pi\sigma u}, \tag{3.70}$$

where G is a countable set.

By substituting (3.70) into (3.69) we see that the output $y(t)$ can be expressed in the two equivalent forms [2.8,9.51]:

$$y(t) = \sum_{\sigma \in G} h_\sigma(t) \otimes [x(t)\mathrm{e}^{\mathrm{j}2\pi\sigma t}] \tag{3.71a}$$

$$= \sum_{\sigma \in G} [g_\sigma(t) \otimes x(t)]\mathrm{e}^{\mathrm{j}2\pi\sigma t}, \tag{3.71b}$$

where

$$g_\sigma(t) \triangleq h_\sigma(t)\mathrm{e}^{-\mathrm{j}2\pi\sigma t}. \tag{3.72}$$

From (3.71a) it follows that a LAPTV systems performs a linear time-invariant filtering of frequency-shifted version of the input signal. For this reason LAPTV filtering is also referred to as *frequency-shift* (FRESH) filtering [7.31]. Equivalently, from (3.71b) it follows that a LAPTV systems performs a frequency shift of linear time-invariant filtered versions of the input.

In the special case for which $G \equiv \{k/T_0\}_{k\in\mathbb{Z}}$ for some period T_0, the system is said to be linear periodically time-variant (LPTV). If G contains only the element $\sigma = 0$, then the system is linear time-invariant (LTI).

3.6.2. Input/output relations in terms of cyclic statistics

Let $x_i(t)$, $i = 1,2$, $t \in \mathbb{R}$, be two possibly-complex ACS continuous-time time series with second-order (conjugate) cross-correlation function

$$R_{x_1 x_2^{(*)}}(t,\tau) \triangleq \mathrm{E}^{\{\alpha\}}\{x_1(t+\tau)x_2^{(*)}(t)\} = \sum_{\alpha \in A_{12}} R^{\alpha}_{x_1 x_2^{(*)}}(\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha t}, \tag{3.73}$$

where

$$R^{\alpha}_{x_1 x_2^{(*)}}(\tau) \triangleq \langle x_1(t+\tau)x_2^{(*)}(t)\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\rangle_t \tag{3.74}$$

is the (*conjugate*) *cyclic cross-correlation* between x_1 and x_2 at cycle frequency α and

$$A_{12} \triangleq \{\alpha \in \mathbb{R} : R^{\alpha}_{x_1 x_2^{(*)}}(\tau) \not\equiv 0\} \tag{3.75}$$

is a countable set. If the set A_{12} contains at least one nonzero element, then the time series $x_1(t)$ and $x_2(t)$ are said to be jointly ACS. Note that, in general, set A_{12} depends on whether $(*)$ is conjugation or not and can be different from the sets A_{11} and A_{22} (both defined according to (3.75)).

Let us consider now two linear LAPTV systems whose impulse-response functions admit the Fourier series expansions

$$h_i(t,u) = \sum_{\sigma_i \in G_i} h_{\sigma_i}(t-u)\mathrm{e}^{\mathrm{j}2\pi\sigma_i u}, \quad i=1,2. \tag{3.76}$$

The (conjugate) cross-correlation of the outputs

$$y_i(t) = \int_{\mathbb{R}} h_i(t,u)x_i(u)\,\mathrm{d}u, \quad i=1,2 \tag{3.77}$$

is given by

$$\begin{aligned} R_{y_1 y_2^{(*)}}(t,\tau) &\triangleq \mathrm{E}^{\{\alpha\}}\{y_1(t+\tau)y_2^{(*)}(t)\} \\ &= \sum_{\alpha \in A_{12}} \sum_{\sigma_1 \in G_1} \sum_{\sigma_2 \in G_2} [R^{\alpha}_{x_1 x_2^{(*)}}(\tau)\mathrm{e}^{\mathrm{j}2\pi\sigma_1\tau}] \\ &\quad \underset{\tau}{\otimes} r^{\alpha+\sigma_1+(-)\sigma_2}_{\sigma_1\sigma_2(*)}(\tau)\mathrm{e}^{\mathrm{j}2\pi(\alpha+\sigma_1+(-)\sigma_2)t}, \end{aligned} \tag{3.78}$$

where $\otimes_\tau$ denotes convolution with respect to τ, $(-)$ is an optional minus sign that is linked to $(*)$, and

$$r^{\gamma}_{\sigma_1\sigma_2(*)}(\tau) \triangleq \int_{\mathbb{R}} h_{\sigma_1}(\tau+s)h^{(*)}_{\sigma_2}(s)\mathrm{e}^{-\mathrm{j}2\pi\gamma s}\,\mathrm{d}s. \tag{3.79}$$

Thus,

$$\begin{aligned} R^{\beta}_{y_1 y_2^{(*)}}(\tau) &\triangleq \langle y_1(t+\tau)y_2^{(*)}(t)\mathrm{e}^{-\mathrm{j}2\pi\beta t}\rangle_t \\ &= \sum_{\sigma_1 \in G_1} \sum_{\sigma_2 \in G_2} [R^{\beta-\sigma_1-(-)\sigma_2}_{x_1 x_2^{(*)}}(\tau)\mathrm{e}^{\mathrm{j}2\pi\sigma_1\tau}] \\ &\quad \underset{\tau}{\otimes} r^{\beta}_{\sigma_1\sigma_2(*)}(\tau), \end{aligned} \tag{3.80}$$

$$\begin{aligned} S^{\beta}_{y_1 y_2^{(*)}}(f) &\triangleq \int_{\mathbb{R}} R^{\beta}_{y_1 y_2^{(*)}}(\tau)\mathrm{e}^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \\ &= \sum_{\sigma_1 \in G_1} \sum_{\sigma_2 \in G_2} S^{\beta-\sigma_1-(-)\sigma_2}_{x_1 x_2^{(*)}}(f-\sigma_1) \\ &\quad \times H_{\sigma_1}(f)H^{(*)}_{\sigma_2}((-)(\beta-f)), \end{aligned} \tag{3.81}$$

where

$$H_{\sigma_i}(f) \triangleq \int_{\mathbb{R}} h_{\sigma_i}(\tau)\mathrm{e}^{-\mathrm{j}2\pi f\tau}\,\mathrm{d}\tau \tag{3.82}$$

and, in the sums in (3.80) and (3.81), only those $\sigma_1 \in G_1$ and $\sigma_2 \in G_2$ such that $\beta-\sigma_1-(-)\sigma_2 \in A_{12}$ give nonzero contribution.

Eqs. (3.80) and (3.81) can be specialized to several cases of interest. For example, if $x_1 = x_2 = x$, $h_1 = h_2 = h$, $y_1 = y_2 = y$, and $(*)$ is conjugation, then we obtain the input–output relations for LAPTV systems in terms of cyclic autocorrelation functions and cyclic spectra:

$$R^{\beta}_{yy^*}(\tau) = \sum_{\sigma_1 \in G} \sum_{\sigma_2 \in G} [R^{\beta-\sigma_1+\sigma_2}_{xx^*}(\tau)\mathrm{e}^{\mathrm{j}2\pi\sigma_1\tau}] \underset{\tau}{\otimes} r^{\beta}_{12}(\tau), \tag{3.83}$$

$$S^{\beta}_{yy^*}(f) = \sum_{\sigma_1 \in G} \sum_{\sigma_2 \in G} S^{\beta-\sigma_1+\sigma_2}_{xx^*}(f-\sigma_1) \times H_{\sigma_1}(f)H^*_{\sigma_2}(f-\beta), \tag{3.84}$$

where

$$r^{\beta}_{12}(\tau) \triangleq \int_{\mathbb{R}} h_{\sigma_1}(\tau+s)h^*_{\sigma_2}(s)\mathrm{e}^{-\mathrm{j}2\pi\beta s}\,\mathrm{d}s. \tag{3.85}$$

For further examples and applications, see Sections 7 and 10.

3.7. Product modulation

Let $x(t)$ and $c(t)$ be two ACS signals with (conjugate) autocorrelation functions

$$R_{xx^{(*)}}(t,\tau) = \sum_{\alpha_x \in A_{xx^{(*)}}} R^{\alpha_x}_{xx^{(*)}}(\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha_x t}, \tag{3.86}$$

$$R_{cc^{(*)}}(t,\tau) = \sum_{\alpha_c \in A_{cc^{(*)}}} R^{\alpha_c}_{cc^{(*)}}(\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha_c t}. \tag{3.87}$$

If $x(t)$ and $c(t)$ are statistically independent in the FOT probability sense, then their joint probability density function factors into the product of the marginal probability densities [2.8,2.9,2.12] and the (conjugate) autocorrelation functions of the product waveform

$$y(t) = c(t)x(t) \tag{3.88}$$

also factor,

$$R_{yy^{(*)}}(t,\tau) = R_{cc^{(*)}}(t,\tau)R_{xx^{(*)}}(t,\tau). \tag{3.89}$$

Therefore, the (conjugate) cyclic autocorrelation function and the (conjugate) cyclic spectrum of $y(t)$ are [2.5,2.8]:

$$R^{\alpha}_{yy^{(*)}}(\tau) = \sum_{\alpha_x \in A_{xx^{(*)}}} R^{\alpha_x}_{xx^{(*)}}(\tau)R^{\alpha-\alpha_x}_{cc^{(*)}}(\tau), \tag{3.90}$$

$$S^{\alpha}_{yy^{(*)}}(f) = \sum_{\alpha_x \in A_{xx^{(*)}}} \int_{\mathbb{R}} S^{\alpha_x}_{xx^{(*)}}(\lambda)S^{\alpha-\alpha_x}_{cc^{(*)}}(f-\lambda)\,\mathrm{d}\lambda, \tag{3.91}$$

where, in the sums, only those (conjugate) cycle frequencies α_x such that $\alpha - \alpha_x \in A_{cc^{(*)}}$ give non-zero contribution.

Observe that, if $c(t)$ is an almost-periodic function

$$c(t) = \sum_{\gamma \in G} c_\gamma \mathrm{e}^{\mathrm{j}2\pi\gamma t}, \tag{3.92}$$

then

$$\begin{aligned} R_{cc^{(*)}}(t,\tau) &= c(t+\tau)c^{(*)}(t) \\ &= \sum_{\gamma_1 \in G}\sum_{\gamma_2 \in G} c_{\gamma_1}c^{(*)}_{\gamma_2}\mathrm{e}^{(-)j2\pi\gamma_2\tau} \\ &\quad \times \mathrm{e}^{\mathrm{j}2\pi(\gamma_1+(-)\gamma_2)t} \end{aligned} \tag{3.93}$$

and

$$R^{\alpha}_{cc^{(*)}}(\tau) = \sum_{\gamma \in G} c_\gamma c^{(*)}_{(-)(\alpha-\gamma)}\mathrm{e}^{\mathrm{j}2\pi(\alpha-\gamma)\tau}, \tag{3.94}$$

$$S^{\alpha}_{cc^{(*)}}(f) = \sum_{\gamma \in G} c_\gamma c^{(*)}_{(-)(\alpha-\gamma)}\delta(f+\gamma-\alpha). \tag{3.95}$$

3.8. Supports of cyclic spectra of band limited signals

By specializing (3.81) to the case $x_1 = x_2 = x$, $h_1 = h_2 = h$ (LTI), and $y_1 = y_2 = y$, we obtain

$$S^{\alpha}_{yy^{(*)}}(f) = S^{\alpha}_{xx^{(*)}}(f)H(f)H^{(*)}((-)(\alpha-f)). \tag{3.96}$$

From (3.96) it follows that the support in the (α,f) plane of the (conjugate) cyclic spectrum of $y(t)$ is such that

$$\begin{aligned} \mathrm{supp}[S^{\alpha}_{yy^{(*)}}(f)] &\triangleq \{(\alpha,f) \in \mathbb{R}\times\mathbb{R} : S^{\alpha}_{yy^{(*)}}(f) \neq 0\} \\ &\subseteq \{(\alpha,f) \in \mathbb{R}\times\mathbb{R} : H(f) \\ &\qquad \times H^{(*)}((-)(\alpha-f)) \neq 0\}. \end{aligned} \tag{3.97}$$

Let $x(t)$ be a strictly band-limited low-pass signal with monolateral bandwidth B; that is, $S^0_{xx^*}(f) \equiv 0$ for $f \notin (-B,B)$. Then,

$$x(t) \equiv x(t) \otimes h_{\mathrm{LPF}}(t), \tag{3.98}$$

where $h_{\mathrm{LPF}}(t)$ is the impulse-response function of the ideal low-pass filter with harmonic-response function:

$$H_{\mathrm{LPF}}(f) = \mathrm{rect}\left(\frac{f}{2B}\right) \triangleq \begin{cases} 1, & |f| \leqslant B, \\ 0, & |f| > B. \end{cases} \tag{3.99}$$

Accounting for (3.97), we have (see Fig. 1)

$$\begin{aligned} \mathrm{supp}[S^{\alpha}_{xx^{(*)}}(f)] \subseteq \{&(\alpha,f) \in \mathbb{R}\times\mathbb{R} : H_{\mathrm{LPF}}(f) \\ &\times H_{\mathrm{LPF}}(f-\alpha) \neq 0\}, \end{aligned} \tag{3.100}$$

where the fact that rect(f) is real and even is used.

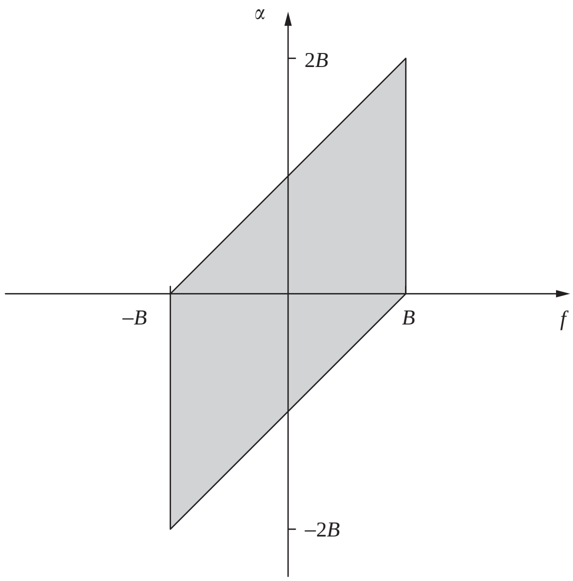

Fig. 1. Cyclic-spectrum support of a low-pass signal.

Let $x(t)$ be a strictly band-limited high-pass signal; that is, $S^0_{xx^*}(f) \equiv 0$ for $f \in (-b, b)$. Then,

$$x(t) \equiv x(t) \otimes h_{\mathrm{HPF}}(t), \tag{3.101}$$

where $h_{\mathrm{HPF}}(t)$ is the impulse-response function of the ideal high-pass filter with harmonic-response function

$$H_{\mathrm{HPF}}(f) = 1 - \mathrm{rect}\left(\frac{f}{2b}\right). \tag{3.102}$$

From (3.97), we have (see Fig. 2)

$$\mathrm{supp}[S^{\alpha}_{xx^{(*)}}(f)] \subseteq \{(\alpha, f) \in \mathbb{R} \times \mathbb{R} : H_{\mathrm{HPF}}(f) \times H_{\mathrm{HPF}}(f-\alpha) \neq 0\}. \tag{3.103}$$

Finally, let $x(t)$ be a strictly band-limited band-pass signal; that is, $S^0_{xx^*}(f) \equiv 0$ for $f \notin (-B, -b) \cup (b, B)$, where $0 < b < B$. Then,

$$x(t) \equiv x(t) \otimes h_{\mathrm{BPF}}(t), \tag{3.104}$$

where $h_{\mathrm{BPF}}(t)$ is the impulse-response function of the ideal band-pass filter with harmonic-response function

$$H_{\mathrm{BPF}}(f) = H_{\mathrm{LPF}}(f) H_{\mathrm{HPF}}(f), \tag{3.105}$$

where $H_{\mathrm{LPF}}(f)$ and $H_{\mathrm{HPF}}(f)$ are given by (3.99) and (3.102), respectively. Therefore, accounting for (3.97), we have (see Fig. 3)

$$\begin{aligned}\mathrm{supp}[S^{\alpha}_{xx^{(*)}}(f)] &\subseteq \{(\alpha, f) \in \mathbb{R} \times \mathbb{R} : H_{\mathrm{BPF}}(f) H_{\mathrm{BPF}}(f-\alpha) \neq 0\} \\ &= \{(\alpha, f) \in \mathbb{R} \times \mathbb{R} : H_{\mathrm{LPF}}(f) H_{\mathrm{LPF}}(f-\alpha) \neq 0\} \\ &\quad \cap \{(\alpha, f) \in \mathbb{R} \times \mathbb{R} : H_{\mathrm{HPF}}(f) \\ &\quad \times H_{\mathrm{HPF}}(f-\alpha) \neq 0\}.\end{aligned} \tag{3.106}$$

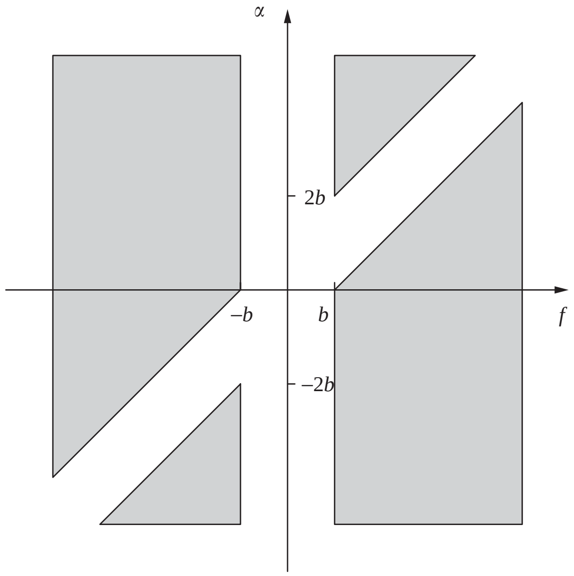

Fig. 2. Cyclic-spectrum support of a high-pass signal.

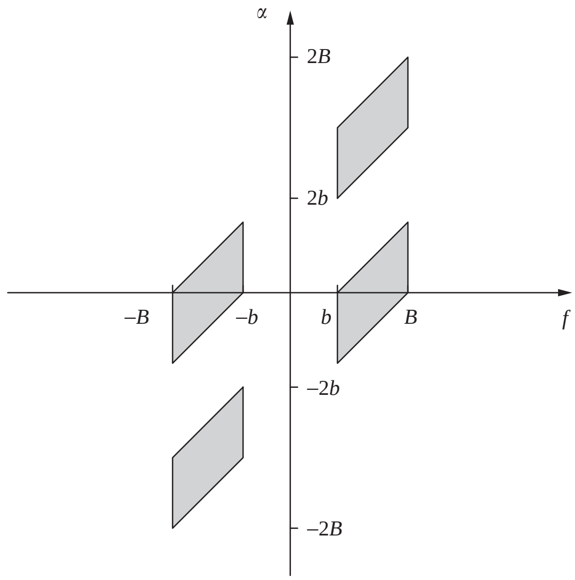

Fig. 3. Cyclic-spectrum support of a band-pass signal.

It is noted that supports for symmetric definitions of cyclic spectra (3.28) are reported in [2.6,2.8].

3.9. Sampling and aliasing

Let $x(n)$ be the sequence obtained by uniformly sampling, with period $T_{\mathrm{s}} = 1/f_{\mathrm{s}}$, the continuous-time signal $x_{\mathrm{a}}(t)$:

$$x(n) \triangleq x_{\mathrm{a}}(t)|_{t=nT_{\mathrm{s}}}. \tag{3.107}$$

The (conjugate) autocorrelation function of the discrete-time signal $x(n)$ can be shown to be the sampled version of the (conjugate) autocorrelation function of the continuous-time signal $x_{\mathrm{a}}(t)$:

$$\begin{aligned}&\mathrm{E}^{\widetilde{\{\alpha\}}}\{x(n+m)x^{(*)}(n)\} \\ &\quad = \mathrm{E}^{\{\alpha\}}\{x_{\mathrm{a}}(t+\tau)x_{\mathrm{a}}(t)\}|_{t=nT_{\mathrm{s}}, \tau=mT_{\mathrm{s}}}.\end{aligned} \tag{3.108}$$

However, the (conjugate) cyclic autocorrelation functions of $x(n)$ are not sampled versions of the (conjugate) cyclic autocorrelation functions of $x_a(t)$ because of the presence of aliasing in the cycle-frequency domain. Consequently, for the (conjugate) cyclic spectra, aliasing in both the spectral-frequency and cycle-frequency domains occurs. Specifically, the (conjugate) cyclic autocorrelation functions and the (conjugate) cyclic spectra of $x(n)$ can be expressed in terms of the (conjugate) cyclic autocorrelation functions and the (conjugate) cyclic spectra of $x_a(t)$ by the relations [2.5,2.8,13.20,13.23]:

$$\widetilde{R}^{\widetilde{\alpha}}_{xx^{(*)}}(m) = \sum_{p\in\mathbb{Z}} R^{\alpha-pf_s}_{x_a x_a^{(*)}}(\tau)|_{\tau=mT_s,\alpha=\widetilde{\alpha}f_s}, \tag{3.109}$$

$$\widetilde{S}^{\widetilde{\alpha}}_{xx^{(*)}}(\nu) = \frac{1}{T_s}\sum_{p\in\mathbb{Z}}\sum_{q\in\mathbb{Z}} S^{\alpha-pf_s}_{x_a x_a^{(*)}}(f-qf_s)|_{f=\nu f_s,\alpha=\widetilde{\alpha}f_s}. \tag{3.110}$$

Let $x(t)$ be a strictly band-limited low-pass signal with monolateral bandwidth B. From (3.100), it follows that

$$\begin{aligned}&\mathrm{supp}[S^{\alpha}_{xx^{(*)}}(f)]\\ &\quad\subseteq \{(\alpha,f)\in\mathbb{R}\times\mathbb{R}: |f|\leqslant B, |\alpha-f|\leqslant B\}\\ &\quad\subseteq \{(\alpha,f)\in\mathbb{R}\times\mathbb{R}: |f|\leqslant B, |\alpha|\leqslant 2B\}.\end{aligned} \tag{3.111}$$

Thus, the support of each replica in (3.110) is contained in the set

$$\{(\alpha,f)\in\mathbb{R}\times\mathbb{R}: |f-qf_s|\leqslant B, |\alpha-pf_s|\leqslant 2B\}$$

and, consequently, a sufficient condition for assuring that the replicas in (3.110) do not overlap is

$$f_s \geqslant 4B. \tag{3.112}$$

In such a case, only the replica with $p=0$ gives nonzero contribution in the base support region in (3.109) and (3.110). Thus, the (conjugate) cyclic autocorrelation functions and the cyclic spectra of the continuous-time signal $x_a(t)$ are amplitude- and/or time- or frequency-scaled versions of the (conjugate) cyclic autocorrelation functions and cyclic spectra of the discrete-time signal $x(n)$ [13.20]:

$$R^{\alpha}_{x_a x_a^{(*)}}(\tau)|_{\tau=mT_s} = \begin{cases}\widetilde{R}^{\widetilde{\alpha}}_{xx^{(*)}}(m)|_{\widetilde{\alpha}=\alpha/f_s}, & |\alpha|\leqslant \dfrac{f_s}{2},\\ 0 & \text{otherwise},\end{cases} \tag{3.113}$$

$$S^{\alpha}_{x_a x_a^{(*)}}(f) = \begin{cases} T_s\widetilde{S}^{\widetilde{\alpha}}_{xx^{(*)}}(\nu)|_{\nu=f/f_s,\widetilde{\alpha}=\alpha/f_s}, & |\alpha|\leqslant\frac{f_s}{2}, |f|\leqslant\frac{f_s}{2},\\ 0 & \text{otherwise}.\end{cases} \tag{3.114}$$

The discrete-time signal obtained by sampling, with period T_s, a cyclostationary continuous-time signal with cyclostationarity period $T_0 = KT_s$, K integer, is a discrete-time cyclostationary signal with cyclostationarity period K. If, however, $T_0 = KT_s + \varepsilon$, with $0<\varepsilon<T_s$ and ε incommensurate with T_s, then the discrete-time signal is almost-cyclostationary [13.23]. In [3.44], common pitfalls arising in the application of the stationary signal theory to time sampled cyclostationary signals are examined.

3.10. Representations by stationary components

3.10.1. Continuous-time processes and time series

A continuous-time wide-sense cyclostationary signal (process or time-series) $x(t)$ can be expressed in terms of singularly and jointly wide-sense stationary signals with non-overlapping spectral bands [2.5,3.22,3.23,3.77,4.41]. That is, if T_0 is the period of cyclostationarity, and we define

$$\widetilde{x}_k(t) \triangleq [x(t)e^{-j2\pi(k/T_0)t}]\otimes h_0(t), \tag{3.115}$$

where $h_0(t)\triangleq(1/T_0)\,\mathrm{sinc}(t/T_0)$ is the impulse-response function of an ideal low-pass filter with monolateral bandwidth $1/(2T_0)$, then $x(t)$ can be expressed by the *harmonic series representation*:

$$x(t) = \sum_{k=-\infty}^{+\infty}\widetilde{x}_k(t)e^{j2\pi(k/T_0)t}, \tag{3.116}$$

where

$$\mathrm{E}^{\{\alpha\}}\{\widetilde{x}_k(t+\tau)\widetilde{x}_h^*(t)\} = \int_{-1/(2T_0)}^{1/(2T_0)} S_{xx^*}^{(h-k)/T_0}\left(f+\frac{k}{T_0}\right)\mathrm{e}^{\mathrm{j}2\pi f\tau}\,\mathrm{d}f \quad (3.117)$$

which is independent of t. This result reflects the fact that the Fourier transforms $\widetilde{X}_k(f)$ of the signals $\widetilde{x}_k(t)$ have non-overlapping support of width $1/T_0$

$$\widetilde{X}_k\left(f-\frac{k}{T_0}\right) = \begin{cases} X(f), & |f-k/T_0| \leqslant 1/(2T_0), \\ 0 & \text{otherwise}; \end{cases} \quad (3.118)$$

therefore, since only spectral components of $x(t)$ with frequencies separated by an integer multiple of $1/T_0$ can be correlated, then the only pairs of spectral components in $\widetilde{x}_k(t)$ and $\widetilde{x}_h(t)$ that can be correlated are those with the same frequency f. Thus, there is no spectral cross-correlation at distinct frequencies in $\widetilde{x}_k(t)$ and $\widetilde{x}_h(t)$.

It follows that a continuous-time wide-sense cyclostationary scalar signal $x(t)$ is equivalent to the infinite-dimensional vector-valued wide-sense stationary signal

$$[\ldots,\widetilde{x}_{-k}(t),\ldots,\widetilde{x}_{-1}(t),\widetilde{x}_0(t),\widetilde{x}_1(t),\ldots,\widetilde{x}_k(t),\ldots].$$

Another representation of a cyclostationary signal by stationary components is the *translation series representation* in terms of any complete orthonormal set of basis functions [3.23]. One example uses the Karhunen–Loève expansion of the cyclostationary signal $x(t)$ on the intervals $t \in [nT_0,(n+1)T_0)$ for all integers n, where T_0 is the period of cyclostationarity.

A harmonic series representation can also be obtained for an ACS process $x(t)$, provided that it belongs to the sub-class of the almost-periodically unitary processes [3.65]. In this case,

$$x(t) = \sum_{k=-\infty}^{+\infty} \widetilde{x}_k(t)\mathrm{e}^{\mathrm{j}2\pi\lambda_k t}, \quad (3.119)$$

where the frequencies λ_k are possibly incommensurate and the processes $\widetilde{x}_k(t)$ are singularly and jointly wide-sense stationary but not necessarily band limited. No translation series representation with stationary components has been demonstrated to exist for ACS processes.

3.10.2. Discrete-time processes and time series

A discrete-time wide-sense cyclostationary signal (process or time-series) $x(n)$ can be expressed in terms of a finite number of singularly and jointly wide-sense stationary signals $\widetilde{x}_k(n)$ with non overlapping bands [2.1,2.5,2.17,4.41,12.3,12.30]. That is, if N_0 is the period of cyclostationarity, and we define

$$\widetilde{x}_k(n) \triangleq [x(n)\mathrm{e}^{-\mathrm{j}2\pi(k/N_0)n}] \otimes h_0(n), \quad (3.120)$$

where $h_0(n) \triangleq (1/N_0)\,\mathrm{sinc}(n/N_0)$ is the ideal lowpass filter with monolateral bandwidth $1/(2N_0)$, then $x(n)$ can be expressed by the *harmonic series representation*:

$$x(n) = \sum_{k=0}^{N_0-1} \widetilde{x}_k(n)\mathrm{e}^{\mathrm{j}2\pi(k/N_0)n}. \quad (3.121)$$

Therefore, a discrete-time wide-sense cyclostationary scalar signal $x(n)$ is equivalent to the N_0-dimensional vector-valued wide-sense stationary signal

$$[\widetilde{x}_0(n),\widetilde{x}_1(n),\ldots,\widetilde{x}_{N_0-1}(n)].$$

A further decomposition of a discrete-time cyclostationary signal can be obtained in terms of sub-sampled components. Let $x(n)$ be a discrete-time real-valued wide-sense cyclostationary time-series with period N_0. The sub-sampled (or decimated) time-series:

$$x_i(n) \triangleq x(nN_0+i), \quad i=0,\ldots,N_0-1 \quad (3.122)$$

constitute what is called the *polyphase decomposition* of $x(n)$. Given the set of time series $x_i(n)$, $i=0,\ldots,N_0-1$, the original signal $x(n)$ can be reconstructed by using the synthesis formula:

$$x(n) = \sum_{i=0}^{N_0-1}\sum_{\ell\in\mathbb{Z}} x_i(\ell)\delta_{n-i-\ell N_0}, \quad (3.123)$$

where δ_γ is the Kronecker delta ($\delta_\gamma=1$ for $\gamma=0$ and $\delta_\gamma=0$ for $\gamma\neq 0$). The signal $x(n)$ is wide-sense cyclostationary with period N_0 if and only if the set of sub-sampled $x_i(n)$ are jointly wide-sense stationary [3.3]. In fact, due to the cyclostationarity of $x(n)$:

$$\begin{aligned}&\mathrm{E}^{\{\alpha\}}\{x_i(n+m)x_k(n)\}\\ &\quad=\mathrm{E}^{\{\alpha\}}\{x((n+m)N_0+i)x(nN_0+k)\}\\ &\quad=\mathrm{E}^{\{\alpha\}}\{x(mN_0+i)x(k)\},\end{aligned}\tag{3.124}$$

which is independent of n.

4. Ergodic properties and measurement of characteristics

4.1. Estimation of the cyclic autocorrelation function and the cyclic spectrum

Ergodic properties and measurements of characteristics are treated in [4.1–4.61]. Consistent estimates of second-order statistical functions of an ACS stochastic process can be obtained provided that the stochastic process has finite or "effectively finite" memory. Such a property is generally expressed in terms of mixing conditions or summability of second- and fourth-order cumulants. Under such mixing conditions, the *cyclic correlogram*

$$R_x^\alpha(\tau;t_0,T)\triangleq\frac{1}{T}\int_{t_0-T/2}^{t_0+T/2}x(t+\tau)x(t)\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\,\mathrm{d}t\tag{4.1}$$

is a consistent estimator of the cyclic autocorrelation function $\mathscr{R}_x^\alpha(\tau)$ (see (3.11)). Moreover,

$$\sqrt{T}[R_x^\alpha(\tau;t_0,T)-\mathscr{R}_x^\alpha(\tau)]$$

is an asymptotically ($T\to\infty$) zero-mean complex normal random variable for each τ and t_0. Consistency for estimators of the cyclic autocorrelation function for cyclostationary and/or ACS processes has been addressed in [2.1,3.1,4.3,4.13, 4.15,4.23,4.24,4.36,4.42,13.18]. The first treatment for ACS processes is in [4.13].

In the frequency domain, the *cyclic periodogram*

$$I_x^\alpha(t,f)\triangleq\frac{1}{T}X_T(t,f)X_T^*(t,f-\alpha),\tag{4.2}$$

where $X_T(t,\lambda)$ is defined according to (3.21), is an asymptotically unbiased but not consistent estimator of the cyclic spectrum $\mathscr{S}_x^\alpha(f)$ (see (3.19) and (3.20)). However, under the above-mentioned mixing conditions, the *frequency-smoothed cyclic periodogram*

$$\begin{aligned}&S_{x_T}^\alpha(t_0,f)_{\Delta f}\\ &\quad\triangleq\frac{1}{\Delta f}\int_{f-\Delta f/2}^{f+\Delta f/2}\frac{1}{T}X_T(t_0,\lambda)X_T^*(t_0,\lambda-\alpha)\,\mathrm{d}\lambda\end{aligned}\tag{4.3}$$

is a consistent estimator of the cyclic spectrum $\mathscr{S}_x^\alpha(f)$. Moreover,

$$\sqrt{T\,\Delta f}[S_{x_T}^\alpha(t,f)_{\Delta f}-\mathscr{S}_x^\alpha(f)]$$

is an asymptotically ($T\to\infty$, $\Delta f\to 0$, with $T\,\Delta f\to\infty$) zero-mean complex normal random variable for each f and t_0. The first detailed study of the variance of estimators of the cyclic spectrum is given in [2.8] and is based on the FOT framework. Consistency for estimators of the cyclic spectrum has been addressed in [2.1,4.14, 4.20,4.30,4.35,4.43,4.44,4.55,4.59,13.12].

In [2.5,2.8,4.17], it is shown that the *time-smoothed cyclic periodogram*:

$$\begin{aligned}S_{x_{1/\Delta f}}^\alpha(t,f)_T&\triangleq\frac{1}{T}\int_{t-T/2}^{t+T/2}\Delta f\\ &\quad\times X_{1/\Delta f}(s,f)X_{1/\Delta f}^*(s,f-\alpha)\,\mathrm{d}s\end{aligned}\tag{4.4}$$

is asymptotically equivalent to the frequency smoothed cyclic periodogram in the sense that

$$\begin{aligned}&\lim_{\Delta f\to 0}\lim_{T\to\infty}S_{x_T}^\alpha(t,f)_{\Delta f}\\ &\quad=\lim_{\Delta f\to 0}\lim_{T\to\infty}S_{x_{1/\Delta f}}^\alpha(t,f)_T,\end{aligned}\tag{4.5}$$

where the order of the two limits on each side cannot be reversed. Note that both the time-smoothed and frequency-smoothed cyclic periodograms exhibit spectral frequency resolution on the order of Δf and a cycle frequency resolution on the order of $1/T$ [4.17]. The problem of cyclic leakage in the estimate of a cyclic statistic at cycle frequency α arising from cyclic statistics at cycle frequencies different from α, first observed in [2.8], is addressed in [2.8,2.9,4.17]. In particular, it is shown that a strong stationary spectrally overlapping noise component added to a cyclostationary signal degrades the performance (bias and variance) of the estimators of cyclic statistics at nonzero cycle frequencies because of the leakage from the zero cycle frequency.

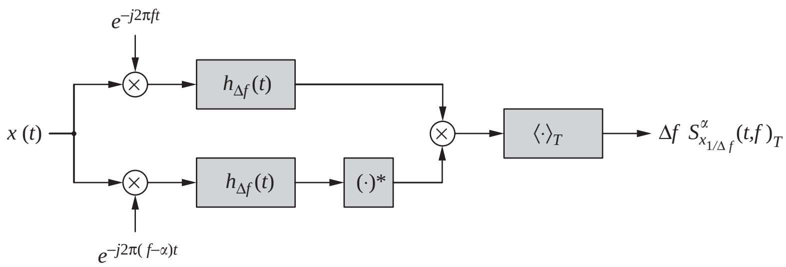

Fig. 4. Spectral correlation analyzer.

A spectral correlation analyzer can be realized by frequency shifting the signal $x(t)$ by two amounts differing by α, passing such frequency-shifted versions through two low-pass filters $h_{\Delta f}(t)$ with bandwidth Δf and unity pass-band height, and then correlating the output signals (see Fig. 4). The spectral correlation density function $S_x^{\alpha}(f)$ is obtained by normalizing the output by Δf, and taking the limit as the correlation time $T \to \infty$ and the bandwidth $\Delta f \to 0$, in this order [4.17].

Reliable spectral estimates with reduced computational requirements can be obtained by using nonlinear transformations of the data [4.39,4.51]. Strict-sense ergodic properties of ACS processes, referred to as *cycloergodicity* in the strict sense, were first treated in depth in [4.13]; see also [2.15]. A survey of estimation problems is given in [4.41]. Cyclostationary feature measurements in the nonstochastic approach are treated in considerable depth in [4.17] and also in [4.31]. Problems arising from the presence of jitter in measurements are addressed in [4.53,4.61]. Computationally efficient digital implementations of cyclic spectrum analyzers are developed and analyzed in [4.25,4.32,4.38,4.45,4.48].

For measurements of cyclic higher-order statistics see [4.50,4.56,4.57,13.7,13.9,13.12,13.14,13.18,13.29].

For further references, see the general treatments [2.1,2.2,2.5,2.8,2.9,2.11,2.15,2.18] and also see [3.22,3.35,3.39,3.59,3.72,11.10,11.13,11.20,12.49,12.51]. Measurements on cyclostationary random fields are treated in [20.3,20.5]. The problem of measurement of statistical functions for more general classes of nonstationary signals is considered in [21.1,21.5,21.12,21.14,21.15].

4.2. Two alternative approaches to the analysis of measurements on time series

In the FOT probability approach, probabilistic parameters are defined through infinite-time averages of functions of a single time series (such as products of time- and frequency-shifted versions of the time series) rather than through expected values or ensemble averages of a stochastic process. Estimators of the FOT probabilistic parameters are obtained by considering finite-time averages of the same quantities involved in the infinite-time averages. Therefore, assuming the above-mentioned limits exist (that is, the infinite-time averages exist), their asymptotic estimators converge by definition to the true values, which are exactly the infinite-time averages, without the necessity of requiring ergodicity properties as in the stochastic process framework. Thus, in the FOT probability framework, the kind of convergence of the estimators to be considered as the data-record length approaches infinity is the convergence of the function sequence of the finite-time averages (indexed by the data-record length). Therefore, unlike the stochastic process framework where convergence must be defined, for example, in the "stochastic mean-square sense" [3.59,4.41,13.18,13.26] or "almost sure sense" [4.35,4.56] or "in distribution", the convergence in the FOT probability framework must be considered "pointwise", in the "temporal mean-square sense" [2.8,2.9,2.31], or in the "sense of generalized functions (distributions)" [23.9].

By following the guidelines in [2.8,2.9,23.1], let us consider the convergence of time series in the temporal mean-square sense (t.m.s.s.). Given a time series $z(t)$ (such as a lag product of another time series), we define

$$z_\beta(t)_T \triangleq \frac{1}{T}\int_{t-T/2}^{t+T/2} z(u)\mathrm{e}^{-\mathrm{j}2\pi\beta u}\,\mathrm{d}u, \tag{4.6}$$

$$z_\beta \triangleq \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{+T/2} z(u)\mathrm{e}^{-\mathrm{j}2\pi\beta u}\,\mathrm{d}u \tag{4.7}$$

and we assume that

$$\lim_{T\to\infty} z_\beta(t)_T = z_\beta \quad \text{(t.m.s.s.)} \quad \forall\beta\in\mathbb{R}, \tag{4.8}$$

that is,

$$\lim_{T\to\infty}\langle|z_\beta(t)_T - z_\beta|^2\rangle_t = 0 \quad \forall\beta\in\mathbb{R}. \tag{4.9}$$

It can be shown that, if the time series $z(t)$ has finite-average-power (i.e., $\langle|z(t)|^2\rangle_t<\infty$), then the set $B\triangleq\{\beta\in\mathbb{R}: z_\beta\neq 0\}$ is countable, the series $\sum_{\beta\in B}|z_\beta|^2$ is summable [2.9] and, accounting for (4.8), it follows that

$$\lim_{T\to\infty}\sum_{\beta\in B} z_\beta(t)_T\mathrm{e}^{\mathrm{j}2\pi\beta t} = \sum_{\beta\in B} z_\beta\mathrm{e}^{\mathrm{j}2\pi\beta t} \quad \text{(t.m.s.s.)}. \tag{4.10}$$

The magnitude and phase of z_β are the amplitude and phase of the finite-strength additive complex sinewave with frequency β contained in the time series $z(t)$. Moreover, the right-hand side in (4.10) is just the almost-periodic component contained in the time series $z(t)$.

The function $z_\beta(t)_T$ is an estimator of z_β based on the observation $\{z(u), u\in[t-T/2, t+T/2]\}$. It is worthwhile to emphasize that, in the FOT probability framework, probabilistic functions are defined in terms of the almost-periodic component extraction operation, which plays the same role as that played by the statistical expectation operation in the stochastic process framework [2.8,2.15]. Therefore,

$$\begin{aligned}\mathrm{bias}\{z_\beta(t)_T\} &\triangleq \mathrm{E}^{\{\alpha\}}\{z_\beta(t)_T\} - z_\beta \\ &\simeq \langle z_\beta(t)_T\rangle_t - z_\beta,\end{aligned} \tag{4.11}$$

$$\begin{aligned}\mathrm{var}\{z_\beta(t)_T\} &\triangleq \mathrm{E}^{\{\alpha\}}\{|z_\beta(t)_T - \mathrm{E}^{\{\alpha\}}\{z_\beta(t)_T\}|^2\} \\ &\simeq \langle|z_\beta(t)_T - \langle z_\beta(t)_T\rangle_t|^2\rangle_t,\end{aligned} \tag{4.12}$$

where the approximation becomes exact equality in the limit as $T\to\infty$. Thus, unlike the stochastic process framework where the variance accounts for fluctuations of the estimates over the ensemble of sample paths, in the FOT probability framework the variance accounts for the fluctuations of the estimates in the time parameter t, viz., the central point of the finite-length time series segment adopted for the estimation. Therefore, the assumption that the estimator asymptotically approaches the true value (the infinite-time average) in the mean-square sense is equivalent to the statement that the estimator is mean-square consistent in the FOT probability sense. In fact, from (4.9), (4.11), and (4.12) it follows that

$$\begin{aligned}\langle|z_\beta(t)_T - z_\beta|^2\rangle_t &\simeq \mathrm{var}\{z_\beta(t)_T\} \\ &\quad + |\mathrm{bias}\{z_\beta(t)_T\}|^2\end{aligned} \tag{4.13}$$

and this approximation become exact as $T\to\infty$. In such a case, estimates obtained by using different time segments asymptotically do not depend on the central point of the segment.

5. Manufactured signals: modelling and analysis

5.1. General aspects

Cyclostationarity in manmade communications signals is due to signal processing operations used in the construction and/or subsequent processing of the signal, such as modulation, sampling, scanning, multiplexing and coding operations [5.1–5.23].

The analytical cyclic spectral analysis of mathematical models of analog and digitally modulated signals was first carried out in [2.5] for stochastic processes and in [5.9,5.10] for nonstochastic time series. The effects of multiplexing are considered in [5.13]. Continuous-phase frequency-modulated signals are treated in [5.12,5.18,5.19,5.23]. The effects of timing jitter on the cyclostationarity properties of communications signals are addressed in [2.8,5.17,5.21].

On this general subject, see the general treatments [2.5,2.8,2.9,2.11,2.13], and also see [2.15, 6.11,7.22].

5.2. Examples of communication signals

In this section, two fundamental examples of cyclostationary communication signals are considered and their wide-sense cyclic statistics are described. The derivations of the cyclic statistics can be accomplished by using the results of Sections 3.6 and 3.7. Then a third example of a more sophisticated communication signal is considered.

5.2.1. Double side-band amplitude-modulated signal

Let $x(t)$ be the (real-valued) double side-band amplitude-modulated (DSB-AM) signal:

$$x(t) \triangleq s(t)\cos(2\pi f_0 t + \phi_0). \tag{5.1}$$

The cyclic autocorrelation function and cyclic spectrum of $x(t)$ are [5.9]:

$$\begin{aligned} R_x^{\alpha}(\tau) = {} & \tfrac{1}{2} R_s^{\alpha}(\tau)\cos(2\pi f_0 \tau) \\ & + \tfrac{1}{4}\{R_s^{\alpha+2f_0}(\tau)e^{-j2\pi f_0\tau}e^{-j2\phi_0} \\ & + R_s^{\alpha-2f_0}(\tau)e^{j2\pi f_0\tau}e^{j2\phi_0}\}, \end{aligned} \tag{5.2}$$

$$\begin{aligned} S_x^{\alpha}(f) = {} & \tfrac{1}{4}\{S_s^{\alpha}(f-f_0) + S_s^{\alpha}(f+f_0) \\ & + S_s^{\alpha+2f_0}(f+f_0)e^{-j2\phi_0} \\ & + S_s^{\alpha-2f_0}(f-f_0)e^{j2\phi_0}\}, \end{aligned} \tag{5.3}$$

respectively. If $s(t)$ is a wide-sense stationary signal then $R_s^{\alpha}(\tau) = R_s^0(\tau)\delta_{\alpha}$ and

$$R_x^{\alpha}(\tau) = \begin{cases} \frac{1}{2} R_s^0(\tau)\cos(2\pi f_0\tau), & \alpha = 0, \\ \frac{1}{4} R_s^0(\tau)e^{\pm j2\pi f_0\tau}e^{\pm j2\phi_0}, & \alpha = \pm 2f_0, \\ 0 & \text{otherwise}, \end{cases} \tag{5.4}$$

$$S_x^{\alpha}(f) = \begin{cases} \frac{1}{4}\{S_s^0(f-f_0) + S_s^0(f+f_0)\}, & \alpha = 0, \\ \frac{1}{4} S_s^0(f \mp f_0)e^{\pm j2\phi_0}, & \alpha = \pm 2f_0, \\ 0 & \text{otherwise}. \end{cases} \tag{5.5}$$

Thus, $x(t)$ is cyclostationary with period $1/(2f_0)$.

In Fig. 5(a) the magnitude of the cyclic autocorrelation function $R_x^{\alpha}(\tau)$, as a function of α and τ, and in Fig. 5(b) the magnitude of the cyclic spectrum $S_x^{\alpha}(f)$, as a function of α and f, are reported for the DSB-AM signal (5.1) with stationary modulating signal $s(t)$ having triangular autocorrelation function.

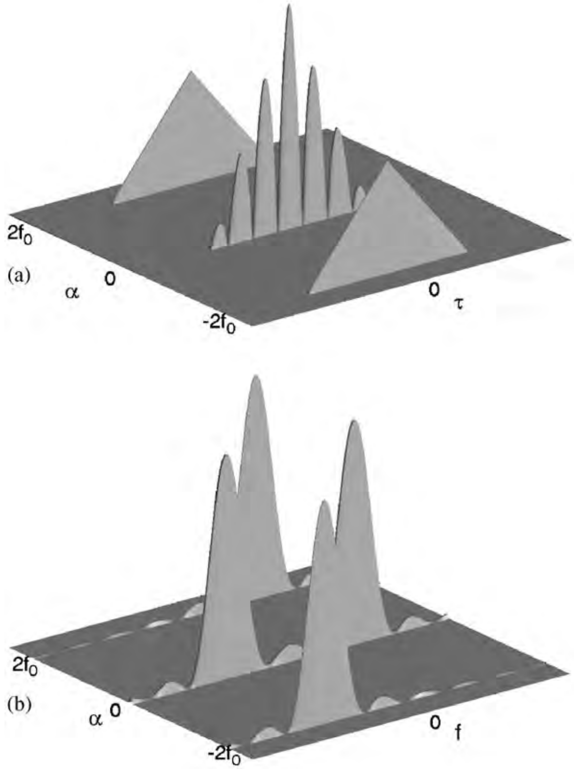

Fig. 5. (a) Magnitude of the cyclic autocorrelation function $R_x^{\alpha}(\tau)$, as a function of α and τ, and (b) magnitude of the cyclic spectrum $S_x^{\alpha}(f)$, as a function of α and f, for the DSB-AM signal (5.1).

5.2.2. Pulse-amplitude-modulated signal

Let $x(t)$ be the complex-valued pulse-amplitude modulated (PAM) signal:

$$x(t) \triangleq \sum_{k\in\mathbb{Z}} a_k q(t - kT_0), \tag{5.6}$$

where $q(t)$ is a complex-valued square integrable pulse and $\{a_k\}_{k\in\mathbb{Z}}$, $a_k \in \mathbb{C}$, is an ACS sequence whose cyclostationarity is possibly induced by framing, multiplexing, or coding [13.21].

The (conjugate) cyclic autocorrelation function and (conjugate) cyclic spectrum of $x(t)$

are [5.9,5.10]:

$$R^{\alpha}_{xx^{(*)}}(\tau) = \frac{1}{T_0}\sum_{m\in\mathbb{Z}} [\widetilde{R}^{\widetilde{\alpha}}_{aa^{(*)}}(m)]_{\widetilde{\alpha}=\alpha T_0} \times r^{\alpha}_{qq^{(*)}}(\tau - mT_0), \tag{5.7}$$

$$S^{\alpha}_{xx^{(*)}}(f) = \frac{1}{T_0} [\widetilde{S}^{\widetilde{\alpha}}_{aa^{(*)}}(\nu)]_{\nu=fT_0,\widetilde{\alpha}=\alpha T_0} \times Q(f)Q^{(*)}((-)(\alpha - f)), \tag{5.8}$$

respectively, where

$$\widetilde{R}^{\widetilde{\alpha}}_{aa^{(*)}}(m) \triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{k=-N}^{N} a_{k+m}a_k^{(*)}\mathrm{e}^{-\mathrm{j}2\pi\widetilde{\alpha}k}, \tag{5.9}$$

$$\widetilde{S}^{\widetilde{\alpha}}_{aa^{(*)}}(\nu) \triangleq \sum_{m\in\mathbb{Z}} \widetilde{R}^{\widetilde{\alpha}}_{aa^{(*)}}(m)\mathrm{e}^{-\mathrm{j}2\pi\nu m} \tag{5.10}$$

are the (conjugate) cyclic autocorrelation function and the (conjugate) cyclic spectrum, respectively, of the sequence $\{a_k\}_{k\in\mathbb{Z}}$,

$$Q(f) \triangleq \int_{\mathbb{R}} q(t)\mathrm{e}^{-\mathrm{j}2\pi ft}\,\mathrm{d}t \tag{5.11}$$

and

$$r^{\alpha}_{qq^{(*)}}(\tau) \triangleq q(\tau) \otimes [q^{(*)}(-\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha\tau}] = \int_{\mathbb{R}} q(t+\tau)q^{(*)}(t)\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\,\mathrm{d}t. \tag{5.12}$$

If the sequence $\{a_k\}_{k\in\mathbb{Z}}$ is wide-sense stationary and white, then

$$\widetilde{R}^{\widetilde{\alpha}}_{aa^*}(m) = \widetilde{R}^{0}_{aa^*}(0)\delta_{(\widetilde{\alpha} \bmod 1)}\delta_m, \tag{5.13}$$

where *mod* denotes the modulo operation. In this case, the cyclic autocorrelation function and the cyclic spectrum of $x(t)$ become

$$R^{\alpha}_{xx^*}(\tau) = \frac{\widetilde{R}^{0}_{aa^*}(0)}{T_0}\,\delta_{(\alpha T_0 \bmod 1)} r^{\alpha}_{qq^*}(\tau), \tag{5.14}$$

$$S^{\alpha}_{xx^*}(f) = \frac{\widetilde{R}^{0}_{aa^*}(0)}{T_0}\,\delta_{(\alpha T_0 \bmod 1)} Q(f)Q^*(f-\alpha), \tag{5.15}$$

respectively. Thus, $x(t)$ exhibits cyclostationarity with cycle frequencies $\alpha = k/T_0, k \in \mathbb{Z}$; that is, $x(t)$ is cyclostationary with period T_0.

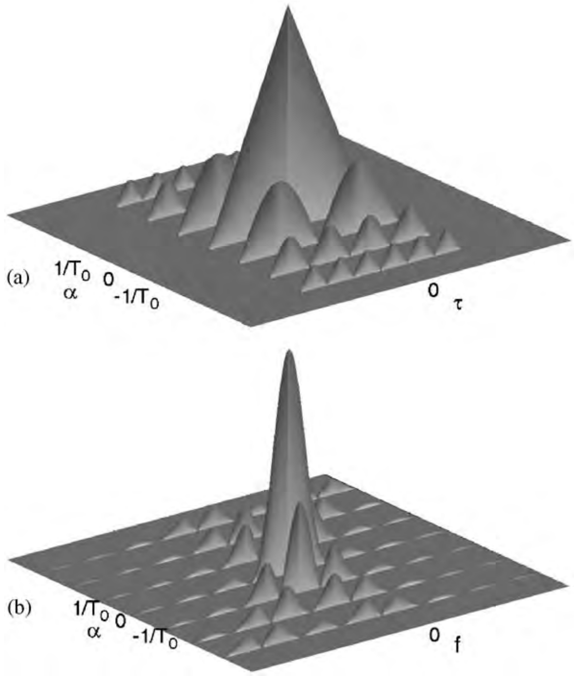

Fig. 6. (a) Magnitude of the cyclic autocorrelation function $R^{\alpha}_{x}(\tau)$, as a function of α and τ, and (b) magnitude of the cyclic spectrum $S^{\alpha}_{x}(f)$, as a function of α and f, for the PAM signal (5.6).

In Fig. 6(a) the magnitude of the cyclic autocorrelation function $R^{\alpha}_{x}(\tau)$, as a function of α and τ, and in Fig. 6(b) the magnitude of the cyclic spectrum $S^{\alpha}_{x}(f)$, as a function of α and f, are reported for the PAM signal (5.6) with stationary modulating sequence $\{a_k\}_{k\in\mathbb{Z}}$, and rectangular pulse $q(t) \triangleq \mathrm{rect}((t - T_0/2)/T_0)$. In this case, (5.12) reduces to

$$r^{\alpha}_{qq}(\tau) = \mathrm{e}^{-\mathrm{j}\pi\alpha(T_0-\tau)}\,\mathrm{rect}\left(\frac{\tau}{2T_0}\right) \times \left(1 - \frac{|\tau|}{T_0}\right) T_0\,\mathrm{sinc}(\alpha(T_0 - |\tau|)). \tag{5.16}$$

5.2.3. Direct-sequence spread-spectrum signal

Let $x(t)$ be the direct-sequence spread-spectrum (DS-SS) baseband PAM signal

$$x(t) \triangleq \sum_{k\in\mathbb{Z}} a_k q(t - kT_0), \tag{5.17}$$

where $\{a_k\}_{k\in\mathbb{Z}}$, $a_k \in \mathbb{C}$, is an ACS sequence, T_0 is the symbol period, and

$$q(t) \triangleq \sum_{n=0}^{N_c-1} c_n p(t - nT_c) \tag{5.18}$$

is the spreading waveform. In (5.18), $\{c_0, \ldots, c_{N_c-1}\}$ is the N_c-length spreading sequence (code) with $c_n \in \mathbb{C}$, and T_c is the chip period such that $T_0 = N_c T_c$.

The (conjugate) cyclic autocorrelation function and (conjugate) cyclic spectrum of $x(t)$ are [2.8,8.24]:

$$R^{\alpha}_{xx^{(*)}}(\tau) = \frac{1}{T_0} \sum_{m\in\mathbb{Z}} [\widetilde{R}^{\widetilde{\alpha}}_{aa^{(*)}}(m)]_{\widetilde{\alpha}=\alpha T_0} \times r^{\alpha}_{pp^{(*)}}(\tau - mT_0) \otimes \gamma^{\alpha}_{cc^{(*)}}(\tau), \tag{5.19}$$

$$S^{\alpha}_{xx^{(*)}}(f) = \frac{1}{T_0} [\widetilde{S}^{\widetilde{\alpha}}_{aa^{(*)}}(v)]_{v=fT_0,\widetilde{\alpha}=\alpha T_0} \times P(f) P^{(*)}((-)(\alpha - f)) \times \widetilde{\Gamma}^{\widetilde{\alpha}}_{cc^{(*)}}(v)|_{v=fT_c,\widetilde{\alpha}=\alpha T_c}, \tag{5.20}$$

where

$$\gamma^{\alpha}_{cc^{(*)}}(\tau) \triangleq \sum_{n_1=0}^{N_c-1} \sum_{n_2=0}^{N_c-1} c_{n_1} c^{(*)}_{n_2} \mathrm{e}^{-\mathrm{j}2\pi\alpha n_2 T_c} \times \delta(\tau - (n_1 - n_2)T_c) \tag{5.21}$$

and

$$\widetilde{\Gamma}^{\widetilde{\alpha}}_{cc^{(*)}}(v) \triangleq \mathscr{C}(v)\mathscr{C}^{(*)}((-)(\widetilde{\alpha} - v)) \tag{5.22}$$

with

$$\mathscr{C}(v) \triangleq \sum_{n=0}^{N_c-1} c_n \mathrm{e}^{-\mathrm{j}2\pi v n}. \tag{5.23}$$

6. Natural signals: modelling and analysis

Cyclostationarity occurs in data arising from a variety of natural (not man-made) phenomena due to the presence of periodic mechanisms in the phenomena [6.1–6.27]. In climatology and atmospheric science, cyclostationarity is due to rotation and revolution of the earth [6.1,6.2,6.9,6.12,6.13, 6.21–6.24]. Applications in hydrology are considered in [6.3,6.11,6.14,6.16,6.17,6.19,6.25,6.26]; for periodic ARMA modelling and the prediction problem in hydrology, see [12.16,12.20,12.21, 12.23]; for the modelling of ocean waves as a two-dimensional cyclostationary random field see [20.1].

A patent on a speech recognition technique exploiting cyclostationarity is [6.27].

7. Communications systems: analysis and design

7.1. General aspects

Cyclostationarity properties of modulated signals can be suitably exploited in the analysis and design of communications systems (see [7.1–7.101]) since the signals involved are typically ACS (see Section 5).

The cyclostationary nature of interference in communications systems is characterized in [7.5,7.6,7.8,7.9,7.18,7.25,7.47,7.98]. The problem of optimum filtering (cyclic Wiener filtering) of ACS signals is addressed in Section 7.2.

The problems of synchronization, signal parameter and waveform estimation, channel identification and equalization, and signal detection and classification are treated in Sections 8–11.

On the analysis and design of communications systems, see the general treatments [2.5,2.8, 2.11,2.13], and also see [3.23,3.47,5.9,5.10,5.14, 10.7,22.8].

7.2. Cyclic Wiener filtering

The problem of optimum linear filtering consists of designing the linear transformation of the data $x(t)$ that minimizes the mean-squared error of the filter output relative to a desired signal, say $d(t)$. In the case of complex data, the optimum filter is obtained by processing both $x(t)$ and $x^*(t)$, leading to the linear-conjugate-linear (LCL) structure [2.8,2.9]. If $d(t)$ and $x(t)$ are jointly ACS signals, the optimum filtering is referred to (as first suggested in [7.31]) as *cyclic Wiener filtering* (and also *frequency-shift* (*FRESH*) *filtering*). FRESH filtering consists of periodically or almost-periodically time-variant filtering of $x(t)$ and $x^*(t)$ and adding the results, where the frequency shifts are chosen in accordance with the cycle frequencies of $x(t)$ and $d(t)$ (e.g., cycle frequencies of the signal of interest and interference, $x(t)=d(t)+n(t)$ where $n(t)$ is the interference) [7.19,7.21,7.31]. The problem of optimum LPTV and LAPTV filtering was first addressed in [2.2,3.23] and, in terms of cyclic autocorrelations and cyclic spectra in [2.5,2.8].

Let

$$\widehat{d}(t) \triangleq y(t)+y_c(t) = \int_{\mathbb{R}} h(t,u)x(u)\,\mathrm{d}u + \int_{\mathbb{R}} h^c(t,u)x^*(u)\,\mathrm{d}u \tag{7.1}$$

be the LCL estimate of the desired signal $d(t)$ obtained from the data $x(t)$. To minimize the mean-squared error

$$\mathrm{E}^{\{\alpha\}}\{|\widehat{d}(t)-d(t)|^2\} \tag{7.2}$$

a necessary and sufficient condition is that the error signal be orthogonal to the data (orthogonality condition [2.5]); that is,

$$\mathrm{E}^{\{\alpha\}}\{[\widehat{d}(t+\tau)-d(t+\tau)]x^*(t)\}=0 \quad \forall t\in\mathbb{R}\ \ \forall\tau\in\mathbb{R}, \tag{7.3a}$$

$$\mathrm{E}^{\{\alpha\}}\{[\widehat{d}(t+\tau)-d(t+\tau)]x(t)\}=0 \quad \forall t\in\mathbb{R}\ \ \forall\tau\in\mathbb{R}. \tag{7.3b}$$

If $x(t)$ and $d(t)$ are singularly and jointly ACS,

$$\mathrm{E}^{\{\alpha\}}\{x(t+\tau)x^{(*)}(t)\} = \sum_{\alpha\in A_{xx^{(*)}}} R^{\alpha}_{xx^{(*)}}(\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha t}, \tag{7.4}$$

$$\mathrm{E}^{\{\alpha\}}\{d(t+\tau)x^{(*)}(t)\} = \sum_{\gamma\in F_{dx^{(*)}}} R^{\gamma}_{dx^{(*)}}(\tau)\mathrm{e}^{\mathrm{j}2\pi\gamma t}, \tag{7.5}$$

then it follows that the optimum filters are LAPTV:

$$h(t,u)=\sum_{\sigma\in G} h_\sigma(t-u)\mathrm{e}^{\mathrm{j}2\pi\sigma u}, \tag{7.6}$$

$$h^c(t,u)=\sum_{\eta\in G_c} h^c_\eta(t-u)\mathrm{e}^{\mathrm{j}2\pi\eta u}. \tag{7.7}$$

By substituting (7.1), (7.6), and (7.7) into (7.3a) and (7.3b), we obtain the system of simultaneous filter design [7.31,9.51]:

$$\sum_{\sigma\in G} \mathrm{e}^{\mathrm{j}2\pi\sigma\tau}[R^{\gamma-\sigma}_{xx^*}(\tau)\otimes h_\sigma(\tau)] + \sum_{\eta\in G_c} \mathrm{e}^{\mathrm{j}2\pi\eta\tau}[R^{\eta-\gamma}_{xx}(\tau)^*\otimes h^c_\eta(\tau)] = R^{\gamma}_{dx^*}(\tau) \quad \forall\gamma\in F_{dx^*}, \tag{7.8a}$$

$$\sum_{\sigma\in G} \mathrm{e}^{\mathrm{j}2\pi\sigma\tau}[R^{\gamma-\sigma}_{xx}(\tau)\otimes h_\sigma(\tau)] + \sum_{\eta\in G_c} \mathrm{e}^{\mathrm{j}2\pi\eta\tau}[R^{\eta-\gamma}_{xx^*}(\tau)^*\otimes h^c_\eta(\tau)] = R^{\gamma}_{dx}(\tau) \quad \forall\gamma\in F_{dx}, \tag{7.8b}$$

where the fact that $R^{\beta}_{x^*x^*}(\tau)=R^{-\beta}_{xx}(\tau)^*$ and $R^{\alpha}_{x^*x}(\tau)=R^{-\alpha}_{xx^*}(\tau)^*$ is used and, in (7.8a), for each $\gamma\in F_{dx^*}$, the sums are extended to all frequency shifts $\sigma\in G$ and $\eta\in G_c$ such that $\gamma-\sigma\in A_{xx^*}$ and $\eta-\gamma\in A_{xx}$; and, in (7.8b), for each $\gamma\in F_{dx}$, the sums are extended to all frequency shifts $\sigma\in G$ and $\eta\in G_c$ such that $\gamma-\sigma\in A_{xx}$ and $\eta-\gamma\in A_{xx^*}$.

Eqs. (7.8a) and (7.8b) can be re-expressed in the frequency domain:

$$\sum_{\sigma\in G} S^{\gamma-\sigma}_{xx^*}(f-\sigma)H_\sigma(f-\sigma) + \sum_{\eta\in G_c} S^{\eta-\gamma}_{xx}(\eta-f)^*H^c_\eta(f-\eta) = S^{\gamma}_{dx^*}(f) \quad \forall\gamma\in F_{dx^*}, \tag{7.9a}$$

$$\sum_{\sigma\in G} S^{\gamma-\sigma}_{xx}(f-\sigma)H_\sigma(f-\sigma) + \sum_{\eta\in G_c} S^{\eta-\gamma}_{xx^*}(\eta-f)^*H^c_\eta(f-\eta) = S^{\gamma}_{dx}(f) \quad \forall\gamma\in F_{dx}, \tag{7.9b}$$

where $H_\sigma(f)$ and $H^c_\eta(f)$ are the Fourier transforms of $h_\sigma(t)$ and $h^c_\eta(t)$, respectively.

Applications of FRESH filtering to interference suppression have been considered in [7.56,7.58, 7.68,7.73,7.76,7.77,7.79–7.81,7.85,7.94,7.98]. Adaptive FRESH filtering is addressed in [7.11, 7.14,7.19,7.20,7.21,7.23,7.24,7.27,7.30,7.46,7.57,7.69, 7.74,7.75,7.81].

Cyclic Wiener filtering or FRESH filtering can be recognized to be linked to FSE's, RAKE filters, adaptive demodulators, and LMMSE despreaders for recovery of baseband symbol data from PAM, GMSK/GFSK, and DSSS signals, and for separation of baseband symbol streams in overlapped signal environments, as well as multicarrier or direct frequency diversity spread spectrum systems for adaptive transmission and combining. In particular, on blind and nonblind adaptive demodulation of PAM signals and LMMSE blind despreading of short-code DSSS/CDMA signals using FSE's and RAKE filtering structures, see [7.12,7.15–7.17,7.28,7.29,7.36,7.38,7.39,7.44,7.48, 7.50,7.51,7.53,7.59,7.61,7.65–7.67]. On direct frequency-diverse multicarrier transceivers, see [7.32–7.34,7.52].

Patents of inventions for the analysis and design of communications systems exploiting cyclostationarity are [7.35,7.37,7.42,7.45,7.70,7.83,7.86,7.87, 7.92,7.95,7.96,7.99–7.101]. Patents of inventions exploiting the FRESH filtering are [7.55,7.93].

8. Synchronization

8.1. Spectral line generation

Let $x(t)$ be a real-valued second-order wide-sense ACS time series. According to the results of Section 3.3, the second-order lag product $x(t+\tau)x(t)$ can be decomposed into the sum of its almost-periodic component and a residual term $\ell_x(t,\tau)$ not containing any finite-strength additive sinewave component (see (3.44) and (3.46)):

$$\begin{aligned} x(t+\tau)x(t) &= \mathrm{E}^{\{\alpha\}}\{x(t+\tau)x(t)\} + \ell_x(t,\tau) \\ &= \sum_{\alpha \in A} R_x^{\alpha}(\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha t} + \ell_x(t,\tau), \end{aligned} \quad (8.1)$$

where

$$\langle \ell_x(t,\tau)\mathrm{e}^{-\mathrm{j}2\pi\alpha t}\rangle_t \equiv 0 \quad \forall \alpha \in \mathbb{R}. \quad (8.2)$$

For communications signals, the cycle frequencies $\alpha \in A$ are related to parameters such as sinewave carrier frequency, pulse rate, symbol rate, frame rate, sampling frequency, etc. (see Section 5). Therefore, the extraction of the almost-periodic component in (8.1) leads to a signal suitable for synchronization purposes. For example, if $x(t)$ is the binary PAM signal defined in (5.6) with $q(t)$ real and duration limited to an interval strictly less than T_0, and $a_k \in \{-1,1\}$, then it follows that

$$x^2(t) = \sum_{k \in \mathbb{Z}} q^2(t-kT_0) \quad (8.3)$$

and $\ell_x(t,0)=0$. Therefore, the synchronization signal $x^2(t)$ is periodic with period T_0.

More generally, by definition, ACS signals enable spectral lines to be generated by passage through a stable nonlinear time-invariant transformation (see Sections 10.4 and 13). That is, quadratic or higher-order nonlinear time-invariant transformations of an ACS signal give rise to time series containing finite-strength additive sinewave components whose frequencies are the second or higher-order cycle frequencies of the original signal. All synchronization schemes can be recognized to exploit the second- or higher-order cyclostationarity features of signals [8.13]. Cyclostationarity properties are exploited for synchronization in [8.1–8.36].

The spectral analysis of timing waveforms with re-generated spectral lines is treated in [8.1,8.8,8.11,8.13–8.15]. Phase-lock loops are analyzed in [8.2,8.5–8.7]. Blind or non-data-aided synchronization algorithms are described in [8.20–8.26,8.28–8.33,8.35,8.36]. See also [2.8,5.1].

Patents on synchronization techniques exploiting cyclostationarity are [8.17,8.27,8.34].

9. Signal parameter and waveform estimation

Cyclostationarity properties can be exploited to design signal selective algorithms for signal parameter and waveform estimation [9.1–9.100]. In fact, if the desired and interfering signals have different cyclic parameters such as carrier frequency or baud rate, then they exhibit cyclostationarity at different cycle frequencies and, consequently, parameters of the desired signal can be extracted by estimating cyclic statistics of the received data, consisting of desired signal plus interfering signal, at a cycle frequency exhibited by the desired signal but not by the interference. This signal selectivity by exploitation of cyclostationarity was first suggested in [4.8]. Parameters that can be estimated in the presence of interference and/or high noise include carrier frequency and phase, pulse rate and phase, signal power level, modulation indices, bandwidths, time- and frequency-difference of arrival, direction of arrival, and so on.

Signal selective time-difference-of-arrival estimation algorithms are considered in [9.11,9.12, 9.19,9.31,9.33,9.36–9.38,9.54,9.56,9.72,9.75,9.83]. In [9.84,9.90], both time-difference-of-arrival and frequency-difference-of arrival are estimated. Array processing problems, including spatial filtering and direction finding, are treated in [7.15,9.7,9.9, 9.10,9.15,9.17,9.18,9.20–9.22,9.26–9.30,9.32,9.34, 9.39–9.44,9.47–9.50,9.53,9.57–9.65,9.68,9.73,9.74, 9.79,9.80,9.82,9.86–9.88,9.91,9.94,9.95,9.97,9.99,9.100].

The polynomial-phase signal parameter estimation is addressed in [9.67,9.71,9.78,9.93]. Harmonics in additive and multiplicative noise are considered in [9.55,9.66,9.69,9.70,9.76,9.77]. But, there are difficulties in application of some of this work—difficulties associated with lack of ergodic properties of stochastic-process models adopted for polynomial-phase signals and signals with coupled harmonics.

On this subject see the general treatments [2.2,2.5,2.8,2.9,2.11], and also see [3.23,3.60,5.12, 7.1,7.2,7.7,7.19,7.21,7.24,8.24,11.1,21.6].

Patents of inventions on signal parameter and waveform estimation exploiting cyclostationarity are [9.16,9.45,9.46,9.52,9.81,9.96].

10. Channel identification and equalization

10.1. General aspects

Cyclostationarity-based techniques have been exploited for channel identification and equalization [10.1–10.57]. Linear and nonlinear systems, and time-invariant, periodically and almost-periodically time-variant systems, have been considered. Also, techniques for noisy input/output measurement, and blind adaptation algorithms, have been developed for LTI systems.

On this subject see the general treatments [2.2,2.5,2.8,2.9,2.11], and also see [7.25,10.36,8.24, 9.59,13.20,13.28,14.11].

Patents of inventions on blind system identification exploiting cyclostationarity are [10.3,10.6, 10.10,10.11,10.39,10.56].

10.2. LTI-system identification with noisy-measurements

Cyclostationarity-based techniques can be exploited in channel identification and equalization problems in order to separate the desired and disturbance contributions in noisy input/output measurements, provided that there is at least one cycle frequency of the desired signal that is not shared by the disturbance.

Let us consider the problem of estimating the impulse-response function $h(t)$ or, equivalently, the harmonic-response function:

$$H(f) \triangleq \int_{\mathbb{R}} h(t) \mathrm{e}^{-\mathrm{j}2\pi f t} \, \mathrm{d}t \tag{10.1}$$

of an LTI system with input/output relation

$$y(t) = h(t) \otimes x(t) \tag{10.2}$$

on the basis of the observed noisy signals $v(t)$ and $z(t)$

$$v(t) = x(t) + n(t), \tag{10.3}$$

$$z(t) = y(t) + m(t), \tag{10.4}$$

where $x(t)$, $n(t)$, and $m(t)$ are zero-mean time series.

By assuming $x_1 = x_2 = x$, $y_1 = y$, $y_2 = x$, $h_1 = h$ (LTI), and $h_2 = \delta$ in (3.77), and choosing $(*)$ to be conjugation, (3.81) specializes to

$$S^{\alpha}_{yx^*}(f) = S^{\alpha}_{xx^*}(f) H(f). \tag{10.5}$$

Therefore, from the model (10.3) and (10.4), we obtain

$$S^{\alpha}_{vv^*}(f) = S^{\alpha}_{xx^*}(f) + S^{\alpha}_{nn^*}(f), \tag{10.6}$$

$$\begin{aligned} S^{\alpha}_{zv^*}(f) &= S^{\alpha}_{yx^*}(f) + S^{\alpha}_{mn^*}(f) \\ &= S^{\alpha}_{xx^*}(f) H(f) + S^{\alpha}_{mn^*}(f) \end{aligned} \tag{10.7}$$

provided that $x(t)$ is uncorrelated with both $n(t)$ and $m(t)$.

Eqs. (10.6) and (10.7) reveal the ability of cyclostationarity-based algorithms to be signal selective. In fact, under the assumption that $n(t)$

does not exhibit cyclostationarity at cycle frequency α (i.e., $S^{\alpha}_{nn^*}(f) \equiv 0$) and $n(t)$ and $m(t)$ do not exhibit joint cyclostationarity with cycle frequency α (i.e., $S^{\alpha}_{mn^*}(f) \equiv 0$), the harmonic-response function for the system is given by

$$H(f) = \frac{S^{\alpha}_{yx^*}(f)}{S^{\alpha}_{xx^*}(f)} = \frac{S^{\alpha}_{zv^*}(f)}{S^{\alpha}_{vv^*}(f)}. \tag{10.8}$$

That is, $H(f)$ can be expressed in terms of the cyclic spectra of the noisy input and output signals. Therefore, (10.8) provides a system identification formula that is intrinsically immune to the effects of noise and interference as first observed in [2.8,10.5]. Thus, this identification method is highly tolerant to disturbances in practice, provided that a sufficiently long integration time is used for the cyclic spectral estimates.

The identification of LTI systems based on noisy input/output measurements is considered in [2.5, 2.8,9.11,9.12,9.36,9.37,10.5,10.12,10.14,10.18,10.20, 10.24,10.55].

10.3. Blind LTI-system identification and equalization

By specializing (3.84) to the case of the LTI system (10.2), we get

$$S^{\alpha}_{yy^*}(f) = S^{\alpha}_{xx^*}(f)H(f)H^*(f-\alpha) \tag{10.9}$$

which, for $\alpha = 0$, reduces to the input/output relationship in terms of power spectra:

$$S^{0}_{yy^*}(f) = S^{0}_{xx^*}(f)|H(f)|^2. \tag{10.10}$$

From (10.9) and (10.10) it follows that input/output relationships for LTI systems in terms of cyclic statistics, unlike those in terms of autocorrelation functions and power spectra, preserve phase information of the harmonic-response function $H(f)$. Thus, cyclostationarity properties of the output signal are suitable to be exploited for recovering both phase and magnitude of the system harmonic-response function as first observed in [10.7].

Cyclostationarity properties are exploited for blind identification (without measurements of the system input) of linear systems and for blind equalization techniques in [9.59,10.1,10.7,10.15, 10.16,10.19,10.21,10.22,10.25,10.26,10.28–10.33, 10.36–10.38,10.42–10.45,10.47,10.48,10.50–10.53].

10.4. Nonlinear-system identification

Let $y(t)$ the output signal of a Volterra system excited by the input signal $x(t)$:

$$y(t) = \sum_{n=1}^{+\infty} \int_{\mathbb{R}^n} k_n(\tau_1, \ldots, \tau_n) x(t+\tau_1) \cdots x(t+\tau_n)\, \mathrm{d}\tau_1 \cdots \mathrm{d}\tau_n. \tag{10.11}$$

Accounting for the results of Sections 3.3 and 13, the input lag-product waveform can be decomposed into the sum of an almost-periodic component, referred to as the temporal moment function, and a residual term not containing any finite-strength additive sinewave component (see (13.3)):

$$x(t+\tau_1) \cdots x(t+\tau_n) = \sum_{\alpha \in A_{\boldsymbol{x}}} \mathscr{R}^{\alpha}_{\boldsymbol{x}}(\boldsymbol{\tau}) \mathrm{e}^{\mathrm{j}2\pi\alpha t} + \ell_{\boldsymbol{x}}(t, \boldsymbol{\tau}). \tag{10.12}$$

Thus, identification and equalization techniques for Volterra systems excited by ACS signals make use of higher-order cyclostationarity properties. In fact, there are potentially substantial advantages to using cyclostationary input signals, relative to stationary input signals, for purposes of Volterra system modelling and identification as first observed in [10.13].

Both time-invariant and almost-periodically time-variant nonlinear systems are treated in [10.4,10.8,10.13,10.23,10.35,10.41,10.46,13.16].

11. Signal detection and classification, and source separation

Signal detection techniques designed for cyclostationary signals take account of the periodicity or almost periodicity of the signal autocorrelation function [11.1–11.39]. Single-cycle and multicycle detectors exploit one or multiple cycle frequencies, respectively. The detection problem for additive Gaussian noise is addressed in [11.2,11.5,11.7, 11.9–11.11,11.15,11.19,11.21,11.26,11.27,11.32], and for non-Gaussian noise, in [11.16,11.18, 11.24,11.25]. The problem of signal detection in cyclostationary noise is treated in [11.2,11.8, 11.12,11.17]. Tests for the presence of cyclostationarity are proposed in [11.14,11.20,11.31,12.31] by exploiting the asymptotic properties of the cyclic correlogram and in [11.6,11.13] by exploiting the properties of the support of the spectral correlation function. Modulation classification techniques are proposed in [11.23,11.28–11.30]; see also [13.22]. The problem of cyclostationary source separation is considered in [11.34,11.35,11.37,11.39].

On this subject see the general treatments [2.5,2.11,2.13], and also see [3.60,7.72,7.74,9.13,9.14,9.19,11.11,13.30,13.31].

Patents on detection and signal recognition exploiting cyclostationarity are [11.22,11.36,11.38].

See also the URL http://www.sspi-tech.com for information on general purpose, automatic, communication-signal classification software systems.

12. Periodic AR and ARMA modelling and prediction

Periodic autoregressive (AR) and autoregressive moving average (ARMA) (discrete-time) systems are characterized by input/output relationships described by difference equations with periodically time-varying coefficients and system orders [12.36,12.37]:

$$\sum_{k=0}^{P(n)} a_k(n)y(n-k) = \sum_{m=0}^{Q(n)} b_m(n)x(n-m), \qquad (12.1)$$

where $x(n)$ and $y(n)$ are the input and output signals, respectively, and the coefficients $a_k(n)$ and $b_m(n)$ and the orders $P(n)$ and $Q(n)$ are periodic functions with the same period N_0. Thus, periodic ARMA systems are a special case of discrete-time periodically time-varying systems. When they are excited by a stationary or cyclostationary input signal $x(n)$, they give rise to a cyclostationary output signal $y(n)$. When they are excited by an almost-cyclostationary input signal, they give rise to an almost-cyclostationary output signal. The problem of fitting an AR model to data $y(n)$ is equivalent to the problem of solving for a linear predictor for $y(n)$, where $x(n)$ is the model-fitting-error time series.

Periodic AR and ARMA systems are treated in [12.1–12.53]. The modelling and prediction problem is treated in [12.1–12.8,12.12–12.16,12.19,12.21,12.23–12.26,12.28–12.30,12.32,12.34–12.37,12.41,12.43,12.45,12.46,12.50,12.52]. The parameter estimation problem is addressed in [12.9,12.11,12.17,12.22,12.27,12.31,12.36,12.38–12.40,12.47–12.49,12.51].

On the subject of periodic AR and ARMA modelling and prediction, see the general treatments [2.5,2.8,2.11,2.13], and also see [3.30,3.69,14.23,15.3] for the prediction problem; [4.3,4.5,4.12,4.27,4.29] for the parameter estimation problem; [6.2,6.9,6.13,6.16] for modelling of atmospheric and hydrologic signals; [16.1,16.2,16.5,16.7] for applications to econometrics; and [21.4] for application to modelling helicopter noise.

13. Higher-order statistics

13.1. Introduction

As first defined in [13.4], a signal $x(t)$ is said to exhibit higher-order cyclostationarity (HOCS) if there exists a homogeneous non-linear transformation of $x(t)$ of order greater than two such that the output of this transformation contains finite-strength additive sinewave components. Motivations to study HOCS properties of signals include the following:

(1) Signals not exhibiting second-order cyclostationarity can exhibit HOCS [13.13].
(2) Narrow-band filtering can destroy second-order (wide-sense) cyclostationarity. In fact, let us consider the input/output relationship for LTI systems in terms of cyclic spectra (10.9). If the bandwidth of $H(f)$ is smaller than the smallest nonzero second-order cycle frequency of the input signal $x(t)$, then the output signal $y(t)$ does not exhibit second-order wide-sense cyclostationarity. The signal $y(t)$, however, can exhibit HOCS (see also [3.13]).
(3) The exploitation of HOCS can be useful for signal classification [13.22]. Specifically, different communication signals, even if they exhibit the same second-order cyclostationarity

properties, can exhibit different cyclic features of higher order. Moreover, different behaviors can be obtained for different conjugation configurations [13.9,13.14,13.17].

(4) Exploitation of HOCS can be useful for many estimation problems, as outlined below.

On the subject of higher-order cyclostationarity and its applications see [13.1–13.31]; in addition, see [3.81] for the wavelet decomposition; [4.56] for estimation issues; [5.23] for the cyclic higher-order properties of CPM signals; [7.74] for blind adaptive detection; [8.8] for the spectral analysis of PAM signals; [9.55,9.61,9.67,9.69–9.71,9.80, 9.85,9.93,9.99] for application to signal parameter and waveform estimation; [10.13,10.23,10.24, 10.34,10.40,10.41,10.46] for applications to nonlinear system identification and equalization; [11.20] for applications to signal detection; [11.28–11.30] for applications to signal classification; [11.34] for applications to source separation; [15.18,15.21,15.23] for applications in acoustics and mechanics; [21.9–21.11,21.13] for the higher-order characterization of generalized almost-cyclostationary signals.

13.2. Higher-order cyclic statistics

In this section, cyclic higher-order (joint) statistics for both continuous-time and discrete-time time series are presented in the FOT probability framework as first introduced for continuous time in [13.4,13.5] and developed in [13.9,13.13,13.14, 13.17], and, for discrete and continuous time, in [13.20]. For treatments within the stochastic framework, see [13.10,13.12,13.15,13.18,13.26] or extend to higher-order statistics the link between the two frameworks discussed in Section 3.4.

13.2.1. Continuous-time time series

Let us consider the column vector $\boldsymbol{x}(t) \triangleq [x_1^{(*)_1}(t), \ldots, x_N^{(*)_N}(t)]^{\mathsf{T}}$ whose components are N not necessarily distinct complex-valued continuous-time time-series and $(*)_k$ represents optional complex conjugation of the kth signal $x_k(t)$. The N time-series exhibit joint Nth-order wide-sense cyclostationarity with cycle frequency $\alpha \neq 0$ if at least one of the Nth-order *cyclic temporal cross-moment functions* (CTCMFs)

$$\mathscr{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau}) \triangleq \left\langle \prod_{k=1}^{N} x_k^{(*)_k}(t+\tau_k) \mathrm{e}^{-\mathrm{j}2\pi\alpha t} \right\rangle_t, \tag{13.1}$$

where $\boldsymbol{\tau} \triangleq [\tau_1, \ldots, \tau_N]^{\mathsf{T}}$ is not identically zero. Thus, N time series exhibit wide-sense joint Nth-order cyclostationarity with cycle frequency $\alpha \neq 0$ if, for some $\boldsymbol{\tau}$, the lag product waveform

$$L_{\boldsymbol{x}}(t, \boldsymbol{\tau}) \triangleq \prod_{k=1}^{N} x_k^{(*)_k}(t+\tau_k) \tag{13.2}$$

contains a finite-strength additive sinewave component with frequency α, whose amplitude and phase are the magnitude and phase of $\mathscr{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})$, respectively.

The Nth-oder *temporal cross-moment function* (TCMF) is defined by

$$\begin{aligned} \mathscr{R}_{\boldsymbol{x}}(t, \boldsymbol{\tau}) &\triangleq \mathrm{E}^{\{\alpha\}}\{L_{\boldsymbol{x}}(t, \boldsymbol{\tau})\} \\ &= \sum_{\alpha \in A_{\boldsymbol{x}}} \mathscr{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau}) \mathrm{e}^{\mathrm{j}2\pi\alpha t} \end{aligned} \tag{13.3}$$

where $A_{\boldsymbol{x}}$ is the countable set (not depending on $\boldsymbol{\tau}$) of the Nth-order cycle frequencies of the time series $x_1^{(*)_1}(t+\tau_1), \ldots, x_N^{(*)_N}(t+\tau_N)$ (for the given conjugation configuration).

The N-dimensional Fourier transform of the CTCMF

$$\mathscr{S}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}) \triangleq \int_{\mathbb{R}^N} \mathscr{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau}) \mathrm{e}^{-\mathrm{j}2\pi \boldsymbol{f}^{\mathsf{T}} \boldsymbol{\tau}} \, \mathrm{d}\boldsymbol{\tau}, \tag{13.4}$$

where $\boldsymbol{f} \triangleq [f_1, \ldots, f_N]^{\mathsf{T}}$, is called the Nth-order *cyclic spectral cross-moment function* (CSCMF) and can be written as

$$\mathscr{S}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}) = S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}') \delta(\boldsymbol{f}^{\mathsf{T}} \mathbf{1} - \alpha), \tag{13.5}$$

where $\mathbf{1}$ is the vector $[1, \ldots, 1]^{\mathsf{T}}$, and prime denotes the operator that transforms a vector $\boldsymbol{u} \triangleq [u_1, \ldots, u_K]^{\mathsf{T}}$ into the reduced-dimension version $\boldsymbol{u}' \triangleq [u_1, \ldots, u_{K-1}]^{\mathsf{T}}$. The function $S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}')$, referred to as the *reduced-dimension CSCMF* (RD-CSCMF), can be expressed as

$$S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}') = \int_{\mathbb{R}^{N-1}} R_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau}') \mathrm{e}^{-\mathrm{j}2\pi \boldsymbol{f}'^{\mathsf{T}} \boldsymbol{\tau}'} \, \mathrm{d}\boldsymbol{\tau}', \tag{13.6}$$

where

$$R_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau}') \triangleq \mathscr{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau})|_{\tau_N=0} \tag{13.7}$$

is the *reduced-dimension CTCMF* (RD-CTCMF).

The RD-CSCMF can also be expressed as

$$S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}') = \lim_{T\to\infty} \lim_{Z\to\infty} \frac{1}{Z} \int_{t-Z/2}^{t+Z/2} \frac{1}{T} \times X_{N,T}^{(*)_N}(u, (-)_N(\alpha - \boldsymbol{f}'^{\mathsf{T}}\mathbf{1})) \times \prod_{k=1}^{N-1} X_{k,T}^{(*)_k}(u, (-)_k f_k)\,\mathrm{d}u, \tag{13.8}$$

where

$$X_{k,T}(t, f_k) \triangleq \int_{t-T/2}^{t+T/2} x_k(s) \mathrm{e}^{-\mathrm{j}2\pi f_k s}\,\mathrm{d}s \tag{13.9}$$

and $(-)_k$ denotes an optional minus sign that is linked to the optional conjugation $(*)_k$. Eq. (13.8) reveals that N time-series exhibit joint Nth-order wide-sense cyclostationarity with cycle frequency α (i.e., $\mathscr{R}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau}) \not\equiv 0$ or, equivalently, $S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}') \not\equiv 0$) if and only if the Nth-order temporal cross-moment of their spectral components at frequencies f_k, whose sum is equal to α, is nonzero. In fact, by using

$$h_B(t) \triangleq B\,\mathrm{rect}(Bt) \tag{13.10}$$

with $B \triangleq 1/T$ and $\mathrm{rect}(t) = 1$ if $|t| \leqslant \frac{1}{2}$ and $\mathrm{rect}(t) = 0$ if $|t| > \frac{1}{2}$, Eq. (13.8) can be re-written as

$$S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}') = \lim_{B\to 0} \lim_{Z\to\infty} \frac{1}{Z} \int_{t-Z/2}^{t+Z/2} \frac{1}{B^{N-1}} [(x_N^{(*)_N}(u) \times \mathrm{e}^{-\mathrm{j}2\pi(\alpha - \boldsymbol{f}'^{\mathsf{T}}\mathbf{1})u}) \otimes h_B(u)] \times \prod_{k=1}^{N-1} [(x_k^{(*)_k}(u) \mathrm{e}^{-\mathrm{j}2\pi f_k u}) \otimes h_B(u)]\,\mathrm{d}u \tag{13.11}$$

which is the Nth-order temporal cross-moment of low-pass-filtered versions of the frequency-shifted signals $x_k^{(*)_k}(t)\mathrm{e}^{-\mathrm{j}2\pi f_k t}$ when the sum of the frequency shifts is equal to α and the bandwidth B approaches zero. Such a property is the generalization to the order $N > 2$ of the spectral correlation property of signals that exhibit second-order cyclostationarity. Moreover, relation (13.4) between the Nth-order cyclic spectral moment function (13.5), (13.8) and Nth-order cyclic temporal moment (13.1) is, at first pointed out in [13.4], the Nth-order generalization of the *Cyclic Wiener Relation*.

Let us note that in general the function $R_{\boldsymbol{x}}^{\alpha}(\boldsymbol{\tau}')$ is not absolutely integrable because it does not in general decay as $\|\boldsymbol{\tau}'\| \to \infty$, but rather it oscillates. Thus, $S_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f}')$ can contain impulses and, consequently, $\mathscr{S}_{\boldsymbol{x}}^{\alpha}(\boldsymbol{f})$ can contain products of impulses. In [13.13], it is shown that the RD-CSCMF $S_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}')$ can contain impulsive terms if the vector $\boldsymbol{f}$ with $f_N = \beta - \sum_{k=1}^{N-1} f_k$ lies on the β-submanifold; i.e., if there exists at least one partition $\{\mu_1, \ldots, \mu_p\}$ of $\{1, \ldots, N\}$ with $p > 1$ such that each sum $\alpha_{\mu_i} = \sum_{k\in\mu_i} f_k$ is a $|\mu_i|$th-order cycle frequency of $x(t)$, where $|\mu_i|$ is the number of elements in μ_i.

In the spectral-frequency domain, a well-behaved function can be introduced starting from the Nth-order *temporal cross-cumulant* function (TCCF):

$$\begin{aligned}\mathscr{C}_{\boldsymbol{x}}(t, \boldsymbol{\tau}) &\triangleq \mathrm{cum}\{x_k^{(*)_k}(t + \tau_k),\ k = 1, \ldots, N\} \\ &= (-j)^N \frac{\partial^N}{\partial\omega_1 \cdots \partial\omega_N} \log_e \mathrm{E}^{\{\alpha\}} \left\{ \exp\left[\mathrm{j} \sum_{k=1}^{N} \omega_k x_k^{(*)_k}(t + \tau_k) \right] \right\} \Bigg|_{\boldsymbol{\omega}=\mathbf{0}} \\ &= \sum_{\mathsf{P}} \left[(-1)^{p-1}(p-1)! \prod_{i=1}^{p} \mathscr{R}_{\boldsymbol{x}_{\mu_i}}(t, \boldsymbol{\tau}_{\mu_i}) \right],\end{aligned} \tag{13.12}$$

where $\boldsymbol{\omega} \triangleq [\omega_1, \ldots, \omega_N]^{\mathsf{T}}$, P is the set of distinct partitions of $\{1, \ldots, N\}$, each constituted by the subsets $\{\mu_i,\ i = 1, \ldots, p\}$, $\boldsymbol{x}_{\mu_i}$ is the $|\mu_i|$-dimensional vector whose components are those of $\boldsymbol{x}$ having indices in μ_i. In (13.12), the almost-periodic component extraction operator $\mathrm{E}^{\{\alpha\}}\{\cdot\}$ extracts the frequencies of the $2N$-variate fraction-of-time joint probability density function of the real and imaginary parts of the time-series $x_k(t)$ ($k = 1, \ldots, N$) according to

$$\begin{aligned}&\mathrm{E}^{\{\alpha\}}\{g(x_1^{(*)_1}(t + \tau_1), \ldots, x_N^{(*)_N}(t + \tau_N))\} \\ &\quad = \int_{\mathbb{R}^{2N}} g(\xi_1^{(*)_1}, \ldots, \xi_N^{(*)_N}) \times f^{\{\alpha\}}_{x_{1\mathrm{r}}(t+\tau_1)x_{1\mathrm{i}}(t+\tau_1)\cdots x_{N\mathrm{r}}(t+\tau_N)x_{N\mathrm{i}}(t+\tau_N)} (\xi_{1\mathrm{r}}, \xi_{1\mathrm{i}}, \ldots, \xi_{N\mathrm{r}}, \xi_{N\mathrm{i}}) \\ &\quad\quad \times \mathrm{d}\xi_{1\mathrm{r}}\,\mathrm{d}\xi_{1\mathrm{i}} \cdots \mathrm{d}\xi_{N\mathrm{r}}\,\mathrm{d}\xi_{N\mathrm{i}},\end{aligned} \tag{13.13}$$

where $x_{k\mathrm{r}}(t) \triangleq \mathrm{Re}[x_k(t)]$, $x_{k\mathrm{i}}(t) \triangleq \mathrm{Im}[x_k(t)]$, $\xi_{k\mathrm{r}} \triangleq \mathrm{Re}[\xi_k]$, $\xi_{k\mathrm{i}} \triangleq \mathrm{Im}[\xi_k]$, and

$$g(\xi_1^{(*)_1}, \ldots, \xi_N^{(*)_N}) \triangleq \exp\left[\mathrm{j}\sum_{k=1}^{N} \omega_k \xi_k^{(*)_k}\right]. \tag{13.14}$$

In fact, by taking the N-dimensional Fourier transform of the coefficient of the Fourier series expansion of the almost-periodic function (13.12)

$$\mathscr{C}_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau}) \triangleq \langle \mathscr{C}_{\boldsymbol{x}}(t, \boldsymbol{\tau}) \mathrm{e}^{-\mathrm{j}2\pi\beta t} \rangle_t \tag{13.15}$$

which is referred to as the Nth-order *cyclic temporal cross-cumulant function* (CTCCF), one obtains the Nth-order *cyclic spectral cross-cumulant function* (CSCCF) $\mathscr{P}_{\boldsymbol{x}}^{\beta}(\boldsymbol{f})$. It can be written as

$$\mathscr{P}_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}) = P_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}')\delta(\boldsymbol{f}^{\mathsf{T}}\mathbf{1} - \beta), \tag{13.16}$$

where

$$P_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}') \triangleq \int_{\mathbb{R}^{N-1}} C_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau}') \mathrm{e}^{-\mathrm{j}2\pi\boldsymbol{f}'^{\mathsf{T}}\boldsymbol{\tau}'}\, \mathrm{d}\boldsymbol{\tau}' \tag{13.17}$$

is the Nth-order *cyclic cross-polyspectrum* (CCP), and

$$C_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau}') \triangleq \mathscr{C}_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau})|_{\tau_N=0} \tag{13.18}$$

is called the *reduced-dimension CTCCF* (RD-CTCCF). The cyclic cross-polyspectrum is a well-behaved function under the mild conditions that the time-series $x_N(t)$ and $x_k(t+\tau_k)$ ($k = 1, \ldots, N-1$) are asymptotically ($|\tau_k| \to \infty$) independent (in the FOT probability sense) so that $C_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau}') \to 0$ as $\|\boldsymbol{\tau}'\| \to \infty$ and, moreover, there exists an $\varepsilon > 0$ such that $|C_{\boldsymbol{x}}^{\beta}(\boldsymbol{\tau}')| = o(\|\boldsymbol{\tau}'\|^{-N+1-\varepsilon})$ as $\|\boldsymbol{\tau}'\| \to \infty$. Furthermore, except on a β-submanifold, the CCP $P_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}')$ is coincident with the RD-CSCMF $S_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}')$.

The CCP can also be expressed as

$$\begin{aligned} P_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}') = \lim_{T\to\infty} \frac{1}{T} \mathrm{cum}\{& X_{N,T}^{(*)_N}(t, (-)_N(\alpha - \boldsymbol{f}'^{\mathsf{T}}\mathbf{1})), \\ & X_{k,T}^{(*)_k}(t, (-)_k f_k), \\ & \quad k = 1, \ldots, N-1\}, \end{aligned} \tag{13.19}$$

where, in the computation of the cumulant in (13.19), the stationary FOT expectation operation can be adopted as $T \to \infty$. Eq. (13.19) reveals that the CCP of N time series is the Nth-order cross-cumulant of their spectral components at frequencies f_k whose sum is equal to β. In fact, Eq. (13.19) can be re-written as

$$\begin{aligned} P_{\boldsymbol{x}}^{\beta}(\boldsymbol{f}') = \lim_{B\to 0} \frac{1}{B^{N-1}} \mathrm{cum}\{& [(x_N^{(*)_N}(t)\mathrm{e}^{-\mathrm{j}2\pi(\alpha - \boldsymbol{f}'^{\mathsf{T}}\mathbf{1})t}) \\ & \otimes h_B(t)], [(x_k^{(*)_k}(t)\mathrm{e}^{-\mathrm{j}2\pi f_k t}) \otimes h_B(t)], \\ & k = 1, \ldots, N-1\} \end{aligned} \tag{13.20}$$

which is the Nth-order temporal cross-cumulant of low-pass filtered versions of the frequency-shifted signals $x_k^{(*)_k}(t)\mathrm{e}^{-\mathrm{j}2\pi f_k t}$ when the sum of the frequency shifts is equal to β and the bandwidth B approaches zero.

As first shown in [13.5] and then, in more detail, in [13.9,13.13,13.15,13.17], the Nth-order temporal cumulant function of a time series provides a mathematical characterization of the notion of a pure Nth-order sinewave. It is that part of the sinewave present in the Nth-order lag product waveform that remains after removal of all parts that result from products of sinewaves in lower order lag products obtained by factoring the Nth-order product.

13.2.2. Discrete-time time series

In accordance with the definition for the continuous-time case, N not necessarily distinct discrete-time complex-valued time series $x_k(n)$, $n \in \mathbb{Z}$, exhibit joint Nth-order wide-sense cyclostationarity with cycle frequency $\widetilde{\alpha} \neq 0$ ($\widetilde{\alpha} \in [-\frac{1}{2}, \frac{1}{2}[$) if at least one of the Nth-order CTCMF's

$$\widetilde{\mathscr{R}}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{m}) \triangleq \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \prod_{k=1}^{N} x_k^{(*)_k}(n + m_k) \mathrm{e}^{-\mathrm{j}2\pi\widetilde{\alpha}n} \tag{13.21}$$

is not identically zero. In (13.21), $\boldsymbol{x}(n) \triangleq [x_1^{(*)_1}(n), \ldots, x_N^{(*)_N}(n)]^{\mathsf{T}}$ and $\boldsymbol{m} \triangleq [m_1, \ldots, m_N]^{\mathsf{T}}$.

The magnitude and phase of the CTCMF (13.21) are the amplitude and phase of the sinewave component with frequency $\widetilde{\alpha}$ contained in the discrete-time lag product whose temporal expected value is the discrete-time Nth-order TCMF.

The N-fold discrete Fourier transform of the CTCMF

$$\widetilde{\mathscr{S}}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{v}) \triangleq \sum_{\boldsymbol{m}\in\mathbb{Z}^N} \widetilde{\mathscr{R}}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{m}) \mathrm{e}^{-\mathrm{j}2\pi\boldsymbol{m}^{\mathsf{T}}\boldsymbol{v}}, \tag{13.22}$$

where $\boldsymbol{v} \triangleq [v_1, \ldots, v_N]^\mathsf{T}$, is called the Nth-order CSCMF and can be written as

$$\widetilde{\mathscr{S}}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{v}) = \widetilde{S}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{v}') \sum_{r=-\infty}^{+\infty} \delta(\widetilde{\alpha} - \boldsymbol{v}^\mathsf{T}\mathbf{1} - r), \tag{13.23}$$

where the function $\widetilde{S}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{v}')$ is the Nth-order RD-CSCMF, which can be expressed as the $(N-1)$-fold discrete Fourier transform

$$\widetilde{S}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{v}') = \sum_{\boldsymbol{m}' \in \mathbb{Z}^{N-1}} \widetilde{R}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{m}') \mathrm{e}^{-\mathrm{j}2\pi \boldsymbol{m}'^\mathsf{T} \boldsymbol{v}'} \tag{13.24}$$

of the Nth-order RD-CTCMF

$$\widetilde{R}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{m}') \triangleq \widetilde{\mathscr{R}}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{m})|_{m_N=0}. \tag{13.25}$$

Once the Nth-order discrete-time TCCF $\widetilde{\mathscr{C}}_{\boldsymbol{x}}(k, \boldsymbol{m})$ is defined analogously to the continuous-time case, the Nth-order CTCCF is given by

$$\widetilde{\mathscr{C}}_{\boldsymbol{x}}^{\widetilde{\beta}}(\boldsymbol{m}) \triangleq \lim_{N \to \infty} \frac{1}{2N+1} \sum_{n=-N}^{N} \widetilde{\mathscr{C}}_{\boldsymbol{x}}(n, \boldsymbol{m}) \mathrm{e}^{-\mathrm{j}2\pi \widetilde{\beta} n}. \tag{13.26}$$

Its discrete Fourier transform, referred to as the Nth-order CSCCF, is given by

$$\widetilde{\mathscr{P}}_{\boldsymbol{x}}^{\widetilde{\beta}}(\boldsymbol{v}) = \widetilde{P}_{\boldsymbol{x}}^{\widetilde{\beta}}(\boldsymbol{v}') \sum_{r=-\infty}^{+\infty} \delta(\widetilde{\beta} - \boldsymbol{v}^\mathsf{T}\mathbf{1} - r), \tag{13.27}$$

where the function $\widetilde{P}_{\boldsymbol{x}}^{\widetilde{\beta}}(\boldsymbol{v}')$ is referred to as the Nth-order CCP, which can be expressed as the $(N-1)$-fold discrete Fourier transform of the Nth-order RD-CTCCF

$$\widetilde{C}_{\boldsymbol{x}}^{\widetilde{\beta}}(\boldsymbol{m}') \triangleq \widetilde{\mathscr{C}}_{\boldsymbol{x}}^{\widetilde{\beta}}(\boldsymbol{m})|_{m_N=0}. \tag{13.28}$$

Finally, it can be easily shown that the following periodicity properties hold:

$$\widetilde{\mathscr{R}}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{m}) = \widetilde{\mathscr{R}}_{\boldsymbol{x}}^{\widetilde{\alpha}+p}(\boldsymbol{m}), \quad p \in \mathbb{Z}, \tag{13.29}$$

$$\widetilde{\mathscr{S}}_{\boldsymbol{x}}^{\widetilde{\alpha}}(\boldsymbol{v}) = \widetilde{\mathscr{S}}_{\boldsymbol{x}}^{\widetilde{\alpha}+p}(\boldsymbol{v}+\boldsymbol{q}), \quad p \in \mathbb{Z}, \ \boldsymbol{q} \in \mathbb{Z}^N, \tag{13.30}$$

$$\widetilde{\mathscr{C}}_{\boldsymbol{x}}^{\widetilde{\beta}}(\boldsymbol{m}) = \widetilde{\mathscr{C}}_{\boldsymbol{x}}^{\widetilde{\beta}+p}(\boldsymbol{m}), \quad p \in \mathbb{Z}, \tag{13.31}$$

$$\widetilde{\mathscr{P}}_{\boldsymbol{x}}^{\widetilde{\beta}}(\boldsymbol{v}) = \widetilde{\mathscr{P}}_{\boldsymbol{x}}^{\widetilde{\beta}+p}(\boldsymbol{v}+\boldsymbol{q}), \quad p \in \mathbb{Z}, \ \boldsymbol{q} \in \mathbb{Z}^N. \tag{13.32}$$

Analogous relations can be stated for the reduced-dimension statistics.

14. Applications to circuits, systems, and control

Applications of cyclostationarity to circuits, systems, and control are in [14.1–14.31]. In circuit theory, cyclostationarity has been exploited in modelling noise [14.1,14.2,14.12,14.17,14.19,14.20, 14.24,14.26,14.28]. In system theory, cyclostationary or almost cyclostationary signals arise in dealing with periodically or almost-periodically time variant systems (see Section 3.6) [14.4–14.8, 14.13,14.14,14.23,14.25]. In control theory, cyclostationarity has been exploited in [14.3,14.10,14.16, 14.18,14.21,14.22].

On this subject, also see [9.4,9.59,10.5,10.24, 13.15,13.28,15.1,16.3].

Patents on applications of cyclostationarity to system and circuit analysis and design are [14.27,14.29,14.31].

15. Applications to acoustics and mechanics

Applications of cyclostationarity to acoustics and mechanics are in [15.1–15.24]. Cyclostationarity has been exploited in acoustics and mechanics for modelling road traffic noise [15.2,15.3], for analyzing music signals [15.6], and for describing the vibration signals in mechanical systems. In mechanical systems with moving parts, such as engines, if some parameter such as speed, temperature, and load torque can be assumed to be constant, then the dynamic physical processes generate vibrations that can be modelled as originating from periodic mechanisms such as rotation and reciprocation of gears, belts, chains, shafts, propellers, pistons, and so on [2.5]. Consequently, vibration signals exhibit periodic behavior of one type or another with periods related to the engine cycle. Often, the observed vibration signals contain both an almost-periodic and an almost-cyclostationary component [15.1,15.5,15.8,15.10, 15.14,15.18,15.23]. Even if the almost-cyclostationary component has a power smaller than the power of the almost-periodic component, it can be useful; for example it can be successfully exploited in early diagnosis of gear faults [15.4,15.8,15.9, 15.11,15.13,15.14,15.16,15.19–15.21,15.24].

Additional applications in the field of acoustics and mechanics can be found in [4.33,4.43,4.49, 14.21,21.2].

Patents on applications of cyclostationarity to engine diagnosis are [15.12,15.17].

16. Applications to econometrics

In high-frequency financial time series, such as asset return, the repetitive patterns of openings and closures of markets, the number of active markets throughout the day, seasonally varying preferences, and so forth, are sources of periodic variations in financial-market volatility and other statistical parameters. Autoregressive models with periodically varying parameters provide appropriate descriptions of seasonally varying economic time series [16.1–16.13]. Furthermore, neglecting the periodic behavior gives rise to a loss in forecast efficiency [2.15,16.8,16.10].

On this subject see also [12.40].

17. Applications to biology

Applications of cyclostationarity to biology are in [17.1–17.18]. Applications of the concept of spectral redundancy (spectral correlation or cyclostationarity) have been proposed in medical image signal processing [17.6,17.8], and nondestructive evaluation [17.2]. Methods of averaging were developed for estimating the generalized spectrum that allow for the meaningful characterization of phase information when classic assumptions of stationarity do not hold. This has led to many significant performance improvements in ultrasonic tissue characterization, particularly for liver and breast tissues [17.5,17.10–17.18].

A patent on the application of cyclostationarity to cholesterol detection is [17.4].

18. Level crossings

Level crossings of cyclostationary signals have been characterized in [18.1–18.10].

19. Queueing

Exploitation of cyclostationarity in queueing theory in computer networks is treated in [19.1–19.4,19.6]. Queueing theory in car traffic is treated in [19.5].

On this subject, also see [14.8].

20. Cyclostationary random fields

A periodically correlated or cyclostationary random field is a second-order random field whose mean and correlation have periodic structure [20.6,20.7]. Specifically, a random field $x(n_1, n_2)$ indexed on $\mathbb{Z}^2$ is called strongly periodically correlated with period (N_{01}, N_{02}) if and only if there exists no smaller $N_{01} > 0$ and $N_{02} > 0$ for which the mean and correlation satisfy

$$\mathrm{E}\{x(n_1 + N_{01}, n_2 + N_{02})\} = \mathrm{E}\{x(n_1, n_2)\}, \tag{20.1}$$

$$\begin{aligned} &\mathrm{E}\{x(n_1 + N_{01}, n_2 + N_{02})x^*(n_1' + N_{01}, n_2' + N_{02})\} \\ &\quad = \mathrm{E}\{x(n_1, n_2)x^*(n_1', n_2')\} \end{aligned} \tag{20.2}$$

for all n_1, n_2, n_1', $n_2' \in \mathbb{Z}$. Cyclostationary random fields are treated in [20.1–20.7].

21. Generalizations of cyclostationarity

21.1. General aspects

Generalizations of cyclostationary processes and time series are treated in [21.1–21.17]. The problem of statistical function estimation for general nonstationary persistent signals is addressed in [21.1,21.5] and limitations of previously proposed approaches are exposed. In [21.2–21.4,21.8], the class of the correlation autoregressive processes is studied. Nonstationary signals that are not ACS can arise from linear time-variant, but not almost-periodically time-variant, transformations of ACS signals. Such transformations occur, for example, in mobile communications when the product of transmitted-signal bandwidth and observation interval is not much smaller than the ratio between the propagation speed in the medium and the relative radial speed between transmitter and receiver [8.35,21.11]. Two models for the output signals of such transformations are the generalized almost-cyclostationary signals [21.7,21.9–21.11, 21.13,21.15–21.17], and the spectrally correlated signals [21.6,21.12,21.14]. Moreover, communications signals with parameters, such as the carrier frequency and the baud rate, that vary slowly with time cannot be modelled as ACS but, rather, can be modelled as generalized almost-cyclostationary if the observation interval is large enough [21.9].

21.2. Generalized almost-cyclostationary signals

A continuous-time complex-valued time series $x(t)$ is said to be *wide-sense generalized almost-cyclostationary* (GACS) if the almost-periodic component of its second-order lag product admits a (generalized) Fourier series expansion with both coefficients and frequencies depending on the lag parameter τ [21.9,21.10]:

$$\mathrm{E}^{\{\alpha\}}\{x(t+\tau)x^*(t)\} = \sum_{n\in\mathbb{I}} R_{xx^*}^{(n)}(\tau)\mathrm{e}^{\mathrm{j}2\pi\alpha_n(\tau)t}. \tag{21.1}$$

In (21.1), $\mathbb{I}$ is a countable set, the frequencies $\alpha_n(\tau)$ are referred to as *lag-dependent cycle frequencies*, and the coefficients, referred to as *generalized cyclic autocorrelation functions*, are given by

$$R_{xx^*}^{(n)}(\tau) \triangleq \langle x(t+\tau)x^*(t)\mathrm{e}^{-\mathrm{j}2\pi\alpha_n(\tau)t}\rangle_t. \tag{21.2}$$

For GACS signals, the cyclic autocorrelation function can be expressed in terms of the generalized cyclic autocorrelation functions by the following relationship:

$$R_{xx^*}^{\alpha}(\tau) = \sum_{n\in\mathbb{I}} R_{xx^*}^{(n)}(\tau)\delta_{\alpha-\alpha_n(\tau)}. \tag{21.3}$$

Moreover, it results that

$$\begin{aligned} A_\tau &\triangleq \{\alpha \in \mathbb{R} : R_{xx^*}^{\alpha}(\tau) \neq 0\} \\ &= \bigcup_{n\in\mathbb{I}} \{\alpha \in \mathbb{R} : \alpha = \alpha_n(\tau)\}. \end{aligned} \tag{21.4}$$

That is, for GACS signals, the support in the (α,τ) plane of the cyclic autocorrelation function $R_{xx^*}^{\alpha}(\tau)$ consists of a countable set of curves described by the equations $\alpha = \alpha_n(\tau)$, $n \in \mathbb{I}$. For the GACS signals, the set

$$A \triangleq \bigcup_{\tau\in\mathbb{R}} A_\tau \tag{21.5}$$

is not necessarily countable.

The ACS signals are obtained as the special case of GACS signals for which the functions $\alpha_n(\tau)$ are constant with respect to τ and are equal to the cycle frequencies. In such a case the support of the cyclic autocorrelation function in the (α,τ) plane consists of lines parallel to the τ axis and the generalized cyclic autocorrelation functions are coincident with the cyclic autocorrelation functions. Moreover, the set A turns out to be countable in this case (see (3.13)).

The higher-order characterization of the GACS signals in the FOT probability framework is provided in [21.9]. Linear filtering is addressed in [21.10,21.11] where the concept of expectation of the impulse-response function in the FOT probability framework is also introduced. The problem of sampling a GACS signal is considered in [21.13] where it is shown that the discrete-time signal obtained by uniformly sampling a continuous-time GACS signal is an ACS signal. The estimation of the cyclic autocorrelation function for GACS processes is addressed in [21.15,21.17] in the stochastic process framework. In [21.16], a survey of GACS signals is provided.

21.3. Spectrally correlated signals

A continuous-time complex-valued second-order harmonizable stochastic process $x(t)$ is said to be *spectrally correlated* (SC) if its *Loève bifrequency spectrum* can be expressed as [21.14]

$$\begin{aligned} \mathscr{S}_{xx^*}(f_1,f_2) &\triangleq \mathrm{E}\{X(f_1)X^*(f_2)\} \\ &= \sum_{n\in\mathbb{I}} S_{xx^*}^{(n)}(f_1)\delta(f_2-\Psi_n(f_1)), \end{aligned} \tag{21.6}$$

where $\mathbb{I}$ is a countable set, the curves $f_2 = \Psi_n(f_1)$, $n \in \mathbb{I}$, describe the support of $\mathscr{S}_{xx^*}(f_1,f_2)$, and the functions $S_{xx^*}^{(n)}(f_1)$, referred to as the *spectral correlation density functions*, describe the density of the Loève bifrequency spectrum on its support curves. The case of linear support curves is considered in [21.6,21.12]. The ACS processes are obtained as the special case of

SC processes for which the support curves are lines with unity slope.

The *bifrequency spectral correlation density function*

$$\bar{\mathcal{S}}_{xx^*}(f_1,f_2) \triangleq \sum_{n\in\mathbb{I}} S_{xx^*}^{(n)}(f_1)\delta_{f_2-\Psi_n(f_1)} \tag{21.7}$$

is the density of the Loève bifrequency spectrum on its support curves $f_2 = \Psi_n(f_1)$. In [21.14] it is shown that, if the location of the support curves is unknown, the bifrequency spectral correlation density function $\bar{\mathcal{S}}_{xx^*}(f_1,f_2)$ can be reliably estimated by the time-smoothed cross-periodogram only if the slope of the support curves is not too far from unity.

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