ENSEMBLE MODELING OF NONLINEAR/LINEAR STOCHASTIC HYDROLOGIC PROCESSES FOR UPSCALING

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As physically-based hydrologic modeling practice expanded its domain from hillslopes to

large watersheds, geographical regions, continents and the whole globe,

a fundamental problem has emerged:

How to upscale the existing point-scale hydrologic conservation equations

for

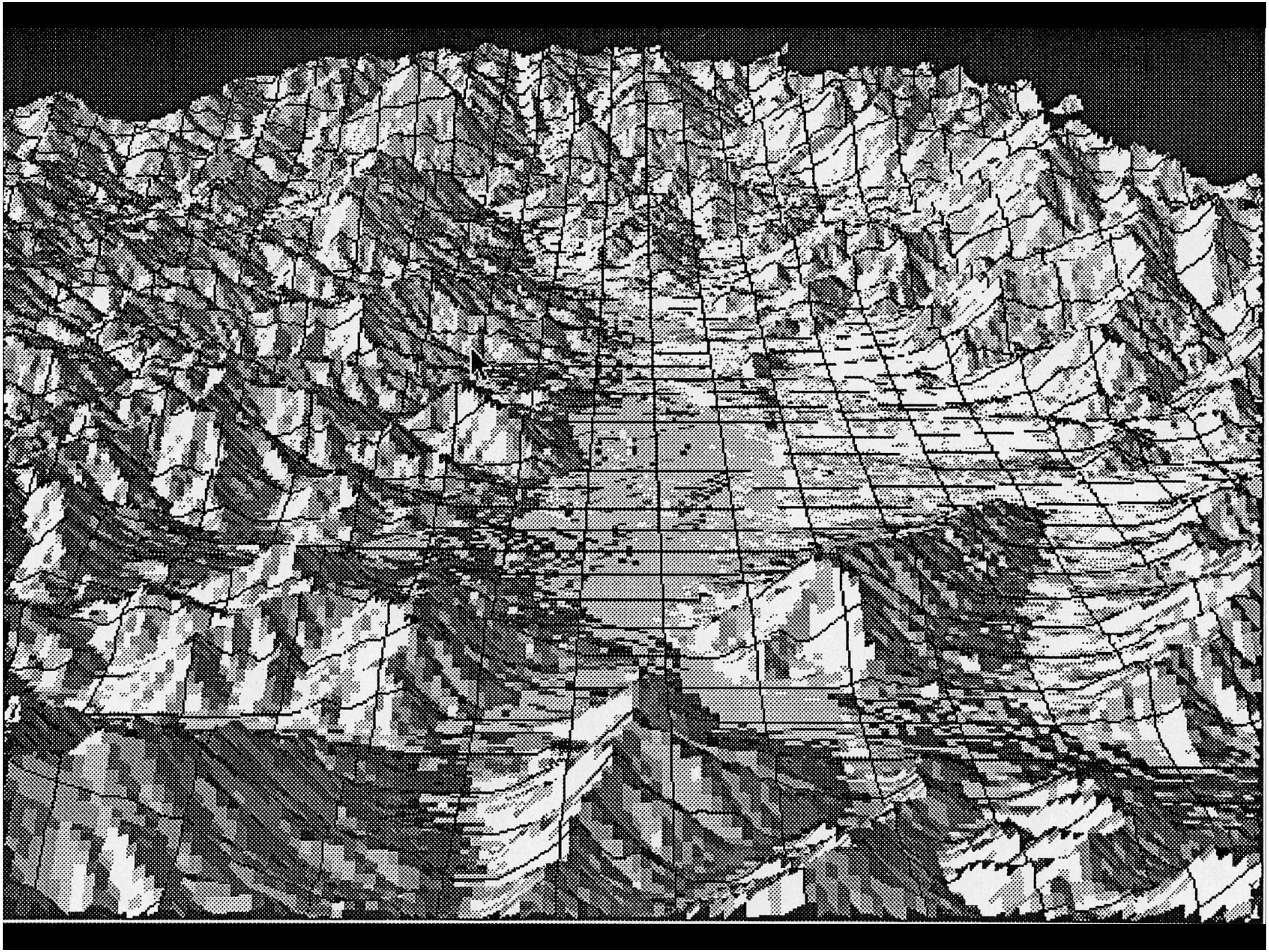
mass, momentum (and/or energy)
to the increasingly larger spatial scales,

in order to have

the conservation equations to be consistent with

the scale of the grid areas over which they will describe

the hydrologic processes.



Various Approaches to Upscaling of Hydrologic Conservation Equations:

I. **Averaging approaches**:

A. Volume/areal averaging approaches

B. Ensemble averaging approaches

1. Numerical probabilistic averaging approaches

2. Analytical ensemble averaging approaches

    a. Averaging based on analytical solutions to realizations
    b. Averaging based on the regular perturbations approach
    c. Averaging based upon decomposition theory of Adomian
    d. Averaging based on projector-operator approach
    e. Averaging based on cumulant expansion approach
        i. Cumulant expansion combined with spectral theory
        ii. Cumulant expansion combined with Lie group theory

II. **Approaches based upon similarity**

Lie Group Symmetries

Classical dimensional analysis

A. <u>Volume/Areal Averaging Approach</u>

The point-scale hydrologic conservation equation is integrated over a volumetric or areal domain, and then the resulting integrals are divided by the size of the domain.

With this approach it was possible to derive Darcy's Equation from the microscopic Navier-Stokes equations under many simplifying assumptions in order to obtain closure (Whittaker, 1999).

It was used in hydrology to reduce the hydrologic conservation equations from their original PDE forms at point scale to ODE forms at larger spatial scales:

a) Duffy (1996) reduced the unsaturated-saturated subsurface flow conservation equation from its original PDE form to a set of ODEs by means of volume averaging;

b) Tayfur and Kavvas (1994, 1998) reduced the rill and interrill overland flow equations from a 2-D PDE at point scale to an ODE at hillslope scale by volume averaging.

This approach has closure problems.

B. **<u>Ensemble averaging approaches</u>**:

Recognize that the point-scale hydrologic conservation equations become uncertain (stochastic PDEs) due to the uncertain values of their point-scale parameters and boundary conditions at the grid-area scale.

Accordingly, **the aim is**

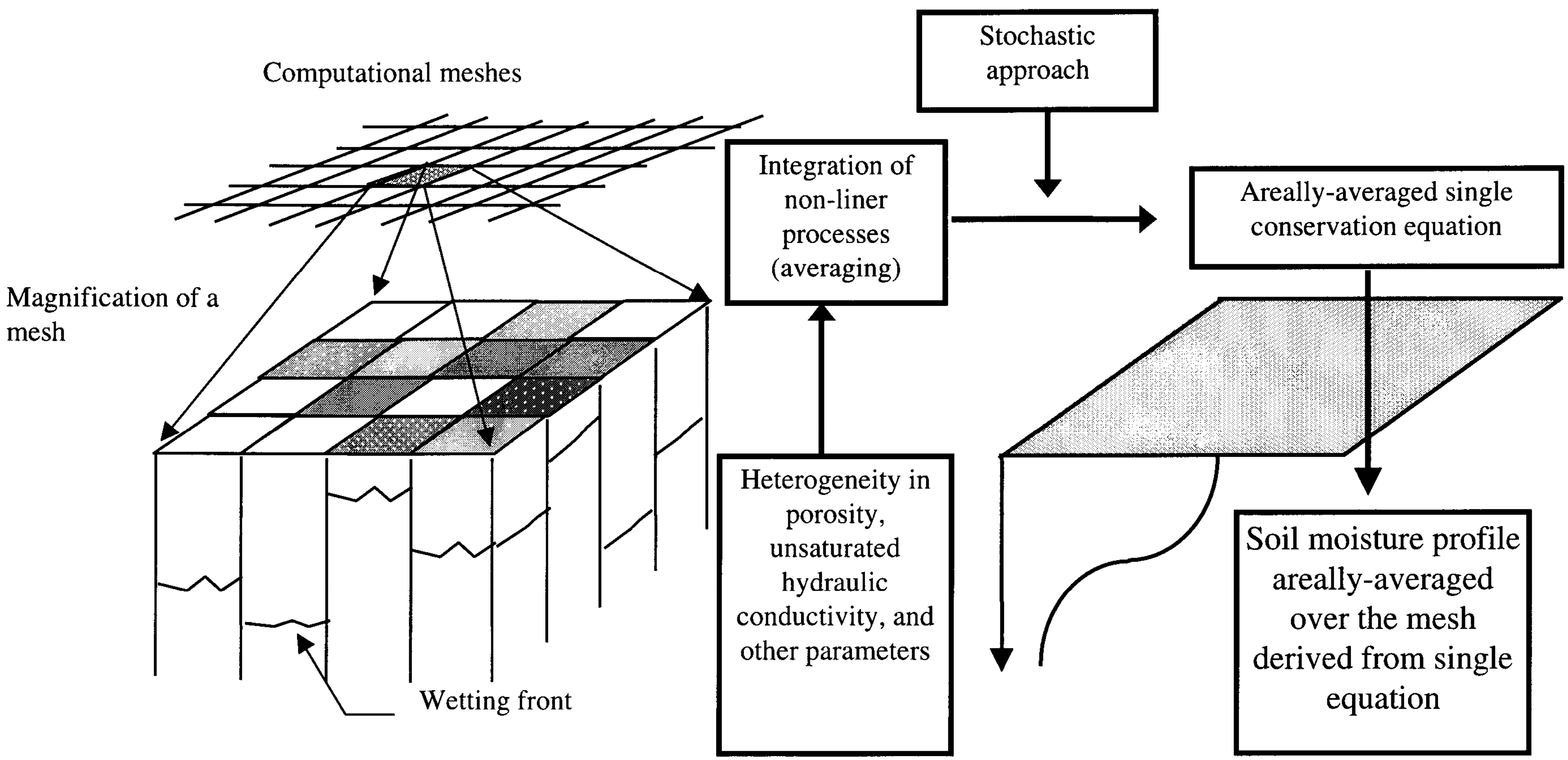
to obtain an ensemble average form of the

original point-scale conservation equation (as a stochastic PDE)

which will represent its

upscaled form at the scale of the modeling grid area.

Schematic description of averaging soil water profiles in a computational mesh



1. **<u>Numerical probabilistic averaging approaches</u>**:

(Avissar and Pielke, 1989; Entekhabi and Eagleson,1989; Avissar, 1991; Famiglietti and Wood, 1991; Kuchment, 2001, etc.)

Assign probability distributions for the parameters of the

point-scale conservation equation

in order to describe the parameters'

statistical variability within a grid area (subgrid variability).

Then using these probability distributions, numerically average

the point-scale conservation equations over the grid area in order to obtain

the grid-area-scale behavior of the corresponding hydrologic process.

2. **Analytical ensemble averaging approaches**:

a. **Averaging based on analytical solutions to realizations**:
(Serrano, 1992,1993,1995; Chen et al. 1994a,b; etc.)

Approach: Obtain the pathwise analytical solution to the conservation equation, and then take its ensemble average

Advantage: Possible to obtain exact analytical closures even in nonlinear problems

Successfully applied to the ensemble averaging of nonlinear unsaturated soil water flow and nonlinear Boussinesq equations

Drawback: Solutions are cumbersome and difficult to understand/use by third parties.

b. **<u>Averaging based on the regular perturbations approach</u>**:

The most often used approach in hydrology (Gelhar and Axness,1983; Dagan,1982,1984; Rubin, 1990, 1991; Graham and McLaughlin, 1989; Mantoglou and Gelhar, 1987a,b; Mantoglou, 1992; Tayfur and Kavvas, 1994; Horne and Kavvas, 1997; etc.)

<u>Approach</u>: Express each stochastic parameter and each state variable in the conservation equations by a sum of their corresponding mean and a small perturbation term. Then substitute this perturbation expression in place of the original parameter/state variable within the conservation equation. Then take the expectation of the resulting conservation equation to obtain an ensemble average equation for the considered hydrologic process.

<u>Advantage</u>: Straightforward to apply even in nonlinear cases.

<u>Drawbacks</u>:

Immediately results in a closure problem where the equation for the mean requires information about the behavior of higher moments. When one attempts to write an equation for the required higher moment, then that equation for the specific higher moment requires information about the behavior of even higher moments. Hence, one can close the system of equations only by means of some adhoc assumptions.

Its fundamental assumption of small magnitude fluctuations in the modeled process is often invalid in highly heterogeneous media, or when dealing with nonlinear dynamics.

c. <u>**Averaging based upon decomposition theory of Adomian**</u>:
(Adomian,1983; Serrano,1993; 1995a,b)

<u>Approach</u>: the state variable in the original conservation equation is decomposed into a series of component functions. Then, starting with the deterministic analytical solution to the original conservation equation, the other terms in the decomposition are determined recursively, where each successive component in the series decomposition representation is determined in terms of the preceding component.

<u>Advantages</u>: can accommodate any size of fluctuation; can be applied both to linear and nonlinear problems; avoids closure problems by adding successively smaller magnitudes to the solution.

<u>Drawbacks</u>: requires a pathwise analytical solution to the conservation equation in order to develop the corresponding ensemble average equation; however, such analytical solutions are unattainable for many nonlinear hydrologic processes.

d. **Averaging based on projector-operator approach** :

(Nakajima, 1958; Zwanzig,1960; Cushman,1991; Cushman and Ginn,1993)

Approach: Considers an operator which projects quantities onto their averages (Pu = <u>). Then applying this operator together with an operator that represents the difference between the actual variable and its mean (Du=u-<u>), derives an exact integro-differential equation for the ensemble average. This integro-differential equation is nonlocal.

Advantages: Avoids the closure problem.

Drawbacks: Applicable only to linear problems.
The obtained integro-differential equation is implicit in the state variable. Therefore, it requires further approximations for its explicit solution.

e. **Averaging based on cumulant expansion approach** :

(Kubo, 1959, 1962; van Kampen 1974,1976; Kabala and Sposito,1991,1994; Kavvas and Karakas,1996; Karakas and Kavvas, 2000; Kavvas,2002, 2003; Cayar and Kavvas, 2009)

Approach: Express the original conservation equation in terms of an operator equation with an average component and a fluctuating dynamic component. Solve the resulting initial value problem in order to obtain the ensemble average equation, expressed in terms of a series of cumulants (correlation functions) of increasing order. Truncation at any order cumulant yields an **exact closure at that order** (does not require any information on higher cumulants unlike the regular perturbation approach).

However, the resulting ensemble average equation is in terms of operators which need to be expressed explicitly for practical applications.

Two approaches for explicit expressions:

i. Cumulant expansion combined with spectral theory:
(Kabala and Sposito, 1991,1994)

Takes the Fourier transform of the cumulant expansion expression in order to develop an equation for the ensemble average in the Fourier space. Still needs to be inverted to the real time-space for practical applications.

ii. Cumulant expansion combined with Lie operator theory:

(Kavvas and Karakas, 1996; Wood and Kavvas, 1999; Karakas and Kavvas, 2000; Kavvas, 2002; Yoon and Kavvas, 2002; Ohara and Kavvas, 2007)

Reconizes that the operators in the cumulant expansion representation of the ensemble average conservation equation are Lie operators. Then it employs the Lie operator properties (Serre, 1965; Olver,1993) in order to obtain an explicit expression for the ensemble average conservation equation in real time-space.

<u>A general formula for the upscaling of linear hydrologic conservation equations from point-scale to next larger spatial scale:</u>

Any linear hydrologic conservation equation may be written in the operator form:

$$\frac{\partial h(x,t)}{\partial t} = A(\mathbf{x},t)\ h(\mathbf{x},t) \qquad (1)$$

where h is the state variable and A is the operator coefficient function.

"Master Key" differential equation

for the **upscaling of any linear hydrologic conservation equation (1)**

from point-scale to next larger scale (Kavvas, ASCE JHE,2002):

$$\frac{\partial \langle h(x_t,t)\rangle}{\partial t} = \langle A(x_t,t)\rangle \langle h(x_t,t)\rangle + \int_0^t ds\, Cov_o[A(x_t,t)\, ;\, A(x_{t-s},t - s)]\, \langle h(x_t,t)\rangle \quad (2)$$

to the order of the covariance time of the operator A. (Exact second order.) In equation (2), the Lagrangian location x_{t-s} is obtained from the known location x_t by

$$\mathbf{x}_{t-s} = \overleftarrow{\exp}\left(\int_{t-s}^{t} dt\, A_L(\mathbf{x}_t,t)\right)\mathbf{x}_t \quad (3)$$

$\mathbf{A}_L$ is that portion of $\langle \mathbf{A} \rangle$ which is made up of the linear combination of the first spatial derivatives.

As such, the time-ordered exponential operator $\overleftarrow{\exp}\,(\cdot)$ on the right-hand-side (rhs) of Eqn.(3) is a **Lie operator**.

Since this Lie operator is fundamentally a displacement operator, it displaces the spatial location $\mathbf{x}_t$ at time t to a location $\mathbf{x}_{t-s}$ at time t - s, Example: Groundwater solute transport by unsteady, spatially nonstationary stochastic flow (velocity v is a time-space stochastic process) is expressed by the following Darcy-scale conservation equation:

$$\frac{\partial c(x,t)}{\partial t} = - v_i(x,t)\frac{\partial c(x,t)}{\partial x_i} + D_{ji}\frac{\partial^2 c(x,t)}{\partial x_j \partial x_i} \qquad (4)$$

Eqn.(4) may be expressed as the operator equation

$$\frac{\partial c(x,t)}{\partial t} = A(\mathbf{x},t)\ c(\mathbf{x},t) \qquad (5)$$

where the operator A(x,t) is defined by

$$A(x,t) = - v_i(x,t)\ \frac{\partial}{\partial x_i} + D_{ji}\ \frac{\partial^2}{\partial x_j x_i}$$

Then substituting this definition of A(x,t) into the "Master Key" upscaling equation (2) one obtains

Upscaled conservation equation for solute transport at a spatial scale one step larger than the Darcy scale:

(Kavvas and Karakas, J. of Hydrol.,1996)

$$\frac{\partial\langle c(x_t,t)\rangle}{\partial t} = \{- \langle v_i(x_t,t)\rangle + \int_0^t ds\mathrm{Cov}_o\left[v_j(x_t,t)\ ;\ \frac{\partial v_i(x_{t-s};t-s)}{\partial x_j}\right]\}\frac{\partial\langle c(x_t,t)\rangle}{\partial x_i}$$

$$+ \{ D_{ji} + \int_0^t ds\ \mathrm{Cov}_o[v_j(x_t,t)\ ;\ v_i(x_{t-s},t-s)]\}\frac{\partial^2\langle c(x_t,t)\rangle}{\partial x_j \partial x_i}$$

to the order of the covariance time of A. In this case

$$A_L(x_t,t) = - \langle v_i(x_t,t)\rangle \frac{\partial}{\partial x_i}$$

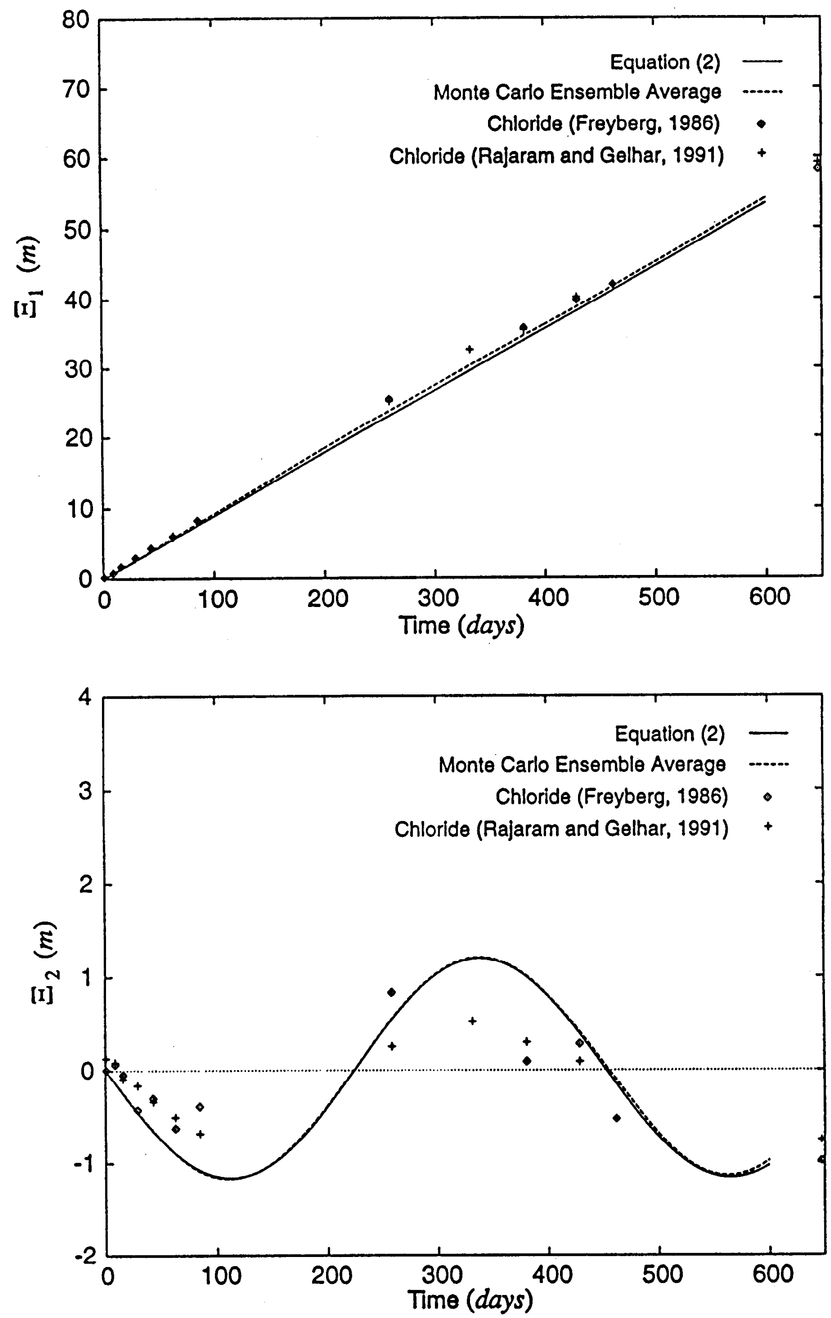
and

$$\mathbf{x}_{t-s} = \overline{\exp}\left[-\int_{t-s}^t dt\ \langle v_l(x,t)\rangle \frac{\partial}{\partial x_l}\right]\mathbf{x}_t$$

APPLICATION TO SOLUTE TRANSPORT OBSERVATIONS

AT BORDEN AQUIFER

(Wood,B.D. and M.L.Kavvas, WRR, 35(7),1999)



Comparison of the first spatial moments of the ensemble-averaged concentration field with data from the Borden site.

A fundamental complication:

The hydrologic conservation equations

are mostly nonlinear!

due to their nonlinear functional forms or

due to the parameters in these equations being dependent on the values of the equations' state variables.

Therefore, most point-scale hydrologic conservation equations are nonlinear stochastic partial differential equations (SPDEs)

or

nonlinear stochastic ordinary differential equations (SODs)

Any hydrologic process with a state variable h,
a parameter vector **a**, and a forcing function g,
may be described by a conservation equation
in the operator ordinary differential equation form

$$\frac{\partial h(\mathbf{x},t)}{\partial t} = \eta(h, \mathbf{a}, g ; \mathbf{x}, t) \qquad (6)$$

Richards equation as the conservation equation for one-dimensional vertical unsaturated flow in the vadose zone (Bear and Verruijt,1987; Chen et al.1994a):

$$\frac{\partial \theta_w}{\partial t} = \frac{\partial}{\partial z} D_z(\theta_w)\frac{\partial \theta_w}{\partial z} - \frac{\partial K_z(\theta_w)}{\partial z}$$

$$\eta_3(\theta_w, D_z, K_z; z, t) = \frac{\partial}{\partial z} D_z(\theta_w)\frac{\partial \theta_w}{\partial z} - \frac{\partial K_z(\theta_w)}{\partial z}$$

Then, the Richards equation takes the operator ODE form

$$\frac{\partial \theta_w}{\partial t} = \eta_3(\theta_w, D_z, K_z; z, t)$$

Differential Equation for the Upscaling of any original Point-scale Hydrologic Conservation Equation in terms of the Upscaled (ensemble average) State Variable <h> (Kavvas, JHE, 8(2), 2003)

$$\frac{d\langle h(\mathbf{x}_t,t)\rangle}{dt} = \langle \eta\big(h(\mathbf{x}_t,t), \mathbf{a}(\mathbf{x}_t,t), g(\mathbf{x}_t,t)\big)\rangle$$

$$+ \int_0^t ds \mathrm{Cov}_0\left[\frac{\partial \eta\big(h(\mathbf{x}_t,t),\mathbf{a}(\mathbf{x}_t,t),g(\mathbf{x}_t,t)\big)}{\partial h}; \eta\big(h(\mathbf{x}_{t-s},t-s),\mathbf{a}(\mathbf{x}_{t-s},t-s),g(\mathbf{x}_{t-s},t-s)\big)\right]$$

For predicting the ensemble average behavior of the hydrologic process

Eulerian-Lagrangian Fokker-Planck-Kolmogorov equation that corresponds to any stochastic nonlinear hydrologic conservation equation in the form of Eqn (6): (Kavvas, J. Hydrol. Engg., 8(2), 2003)

$$\frac{\partial P\big(h(\mathbf{x}_t,t),t\big)}{\partial t} = -\frac{\partial}{\partial h}\Big\{P\big(h(\mathbf{x}_t,t),t\big)\Big[<\eta\big(h(\mathbf{x}_t,t),\mathbf{a}(\mathbf{x}_t,t),g(\mathbf{x}_t,t)\big)>$$

$$+\int_0^t ds\,Cov_o\Big[\frac{\partial\eta\big(h(\mathbf{x}_t,t),\mathbf{a}(\mathbf{x}_t,t),g(\mathbf{x}_t,t)\big)}{\partial h};\eta\big(h(\mathbf{x}_{t-s},t-s),\mathbf{a}(\mathbf{x}_{t-s},t-s),g(\mathbf{x}_{t-s},t-s)\big)\Big]\Big]\Big\}$$

$$+\frac{1}{2}\frac{\partial^2}{\partial h^2}\Big\{2P\big(h(\mathbf{x}_t,t),t\big)\cdot$$

$$\cdot\int_0^t ds\,Cov_o\Big[\eta\big(h(\mathbf{x}_t,t),\mathbf{a}(\mathbf{x}_t,t),g(\mathbf{x}_t,t)\big);\eta\big(h(\mathbf{x}_{t-s},t-s),\mathbf{a}(\mathbf{x}_{t-s},t-s),g(\mathbf{x}_{t-s},t-s)\big)\Big]\Big\}$$

For determining the probabilistic (ENSEMBLE) behavior of the upscaled hydrologic process. (Valid only for finite correlation lengths)

The one-to-one correspondence between

any hydrologic conservation equation

as a linear or nonlinear stochastic PDE/ODE

and

a Fokker-Planck-Kolmogorov equation

in a mixed, nonlocal Eulerian-Lagrangian form,

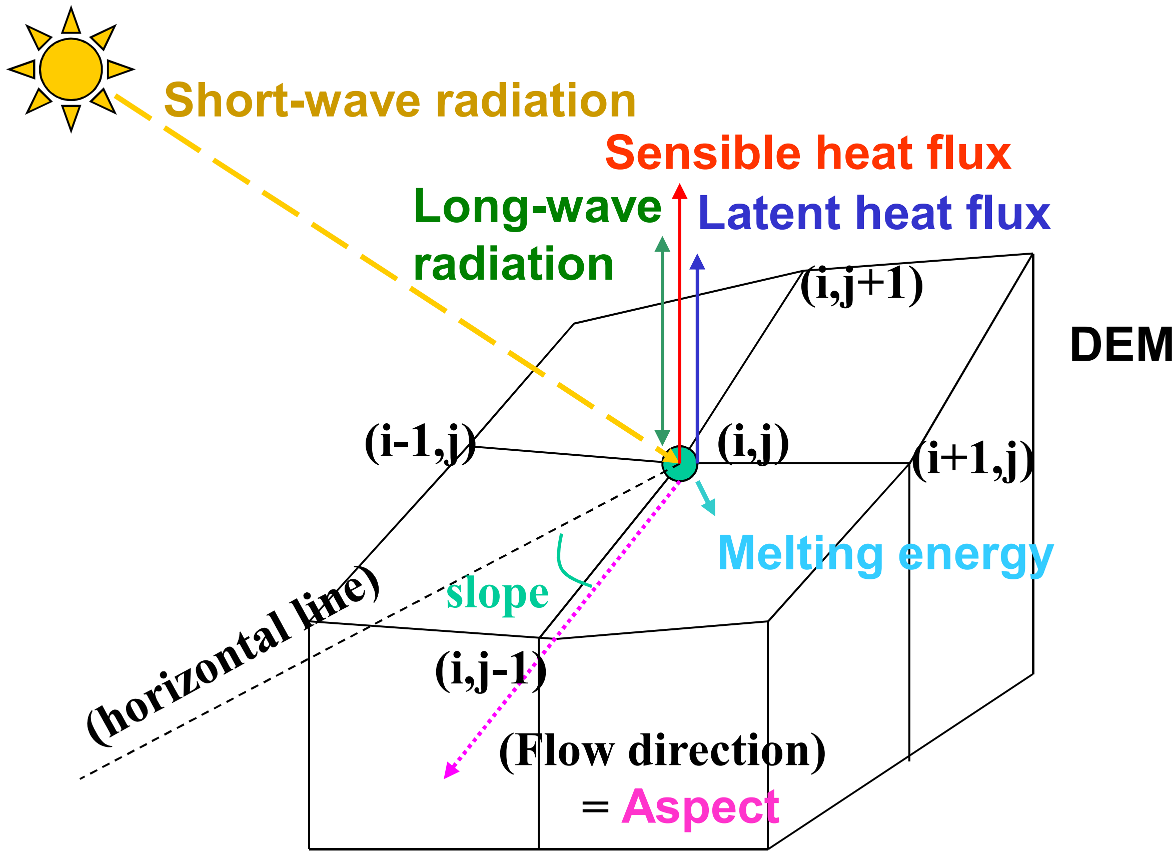
as shown above,

could be a valuable tool in modeling

the ensemble behavior of hydrologic/hydraulic processes

at evolving scales under uncertainty.

Snow Modeling



Top 2 factors affecting snow distribution

Snow accumulation process

1) Precipitation distribution
2) Wind effect

Snowmelt process

1) Shortwave radiation
2) Air temperature

Point-scale snow melt-accumulation model [Non-linear system of ODEs]

$$\begin{cases} \dfrac{dT_s}{dt} = \dfrac{En - M + sw}{D\rho_s C_s} \\ \dfrac{dD}{dt} = \underbrace{\dfrac{\rho_w}{\rho_s}(-E - M_r + sn)}_{\text{Mass exchange}} - \underbrace{\dfrac{D}{\rho_s}\dfrac{d\rho_s}{dt}}_{\text{Depth adjustment of density variation}} \end{cases}$$

Energy conservation eqn.
(For Snow temperature, T_s)

Mass conservation eqn.
(For Snow depth, D)

where

Energy exchange (kW):
Except short-wave radiation

$$En(T_s, D) = L_{in} - L_{out}(T_s, D) + H(T_s) - lE(T_s) + Q_p(T_s)$$

Evaporation (m): $E = \dfrac{lE}{L_v \rho_w}$

$M = sw + En(T_s, D)$

Snowmelt (m): $M_r = \dfrac{M}{L_f \rho_w (1 - W_0)}$

Evaluation of random variables and sources of stochasticity

(Ohara , Kavvas and Chen, 2008 Journal of Hydrol Engr)

State variables

1. Snow depth → Random variables
2. Snow temperature → Random variables

Sources of stochasticity

1. Short-wave radiation
2. Snow fall (Precipitation)

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3. Air temperature → Assumed deterministic
4. Vegetation effect → Assumed constant
5. Wind effect → Assumed constant

In terms of heterogeneity in a single cell

Point-scale snowmelt equations

$$\begin{cases} \dfrac{dT_s}{dt} = F_1\left(T_s, D, sw; \mathbf{x}, \mathbf{t}\right) \\ \dfrac{dD}{dt} = F_2\left(T_s, D, sn; \mathbf{x}, \mathbf{t}\right) \end{cases}$$

where

$$F_1\left(T_s, D, sw\right) = \frac{En - M + sw}{D\rho_s C_s}$$

$$F_2\left(T_s, D, sn\right) = \frac{\rho_w}{\rho_s}\left(-E - Mr\right) + \frac{\rho_w}{\rho_{ns}} sn - \frac{2}{3\eta}\rho_s D^2 e^{\left(-0.04 T_s - \mu\rho_s\right)}$$

Simplified FPE

$$\frac{\partial P(T_s, D, t)}{\partial t} = -\frac{\partial}{\partial T_s}\left\{\langle F_{1,t}\rangle P(T_s, D, t)\right\} - \frac{\partial}{\partial D}\left\{\langle F_{2,t}\rangle P(T_s, D, t)\right\}$$

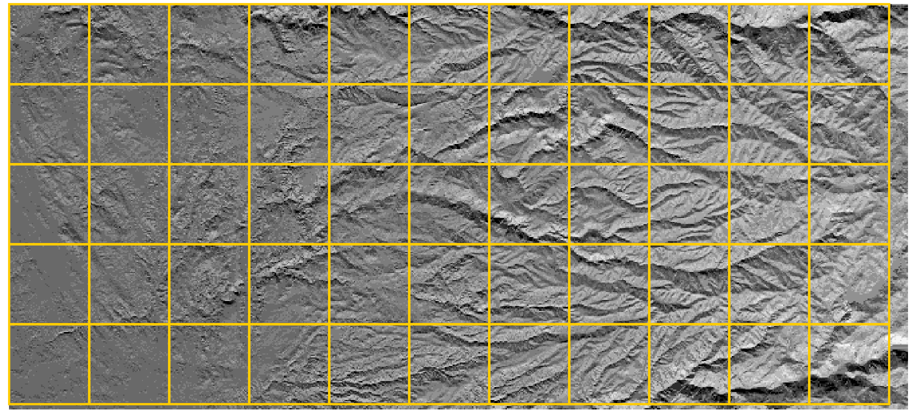
$$+ \frac{\partial^2}{\partial {T_s}^2}\left\{P(T_s, D, t)\int_0^t \left(\frac{1}{D\rho_s C_s}\right)^2 Cov_o\left[sw(\mathbf{x}_t, t); sw(\mathbf{x}_{t-s}, t-s)\right]ds\right\}$$

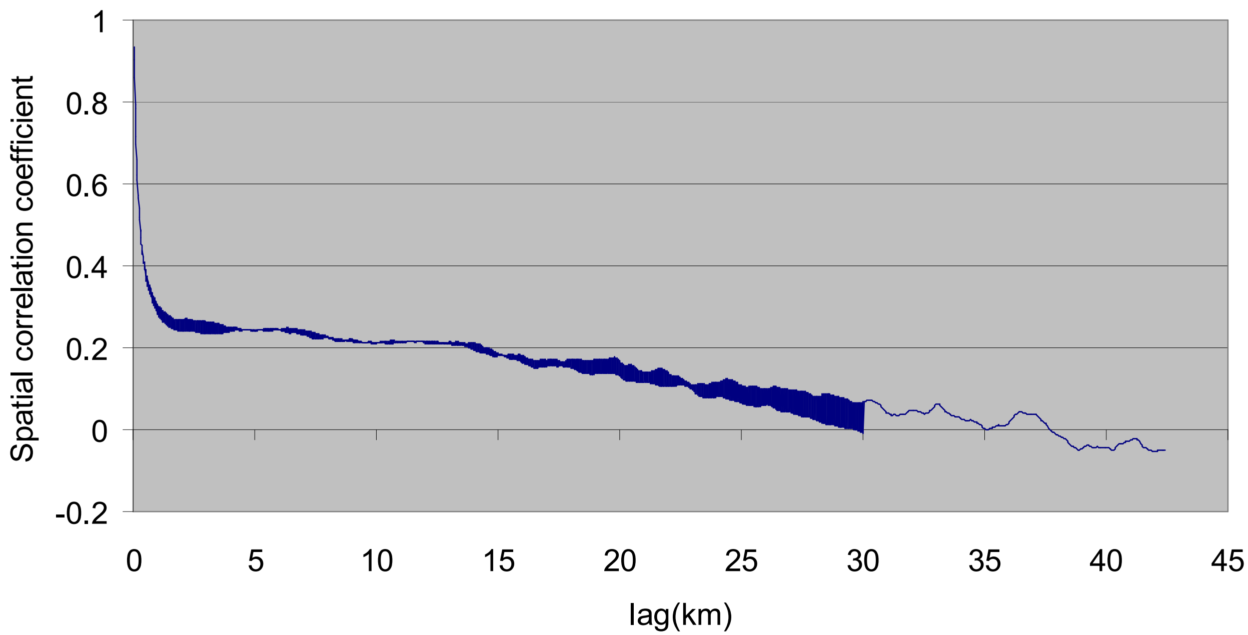
$$+ \frac{\partial^2}{\partial D^2}\left\{P(T_s, D, t)\int_0^t \left(\frac{\rho_w}{\rho_{ns}}\right)^2 Cov_o\left[sn(\mathbf{x}_t, t); sn(\mathbf{x}_{t-s}, t-s)\right]ds\right\}$$

How to evaluate these time-space dependent covariances?

$$Cov_o\left[sw(\mathbf{x}_t, t); sw(\mathbf{x}_{t-s}, t-s)\right] \approx Var\left[sw(x_t)\right]\delta(x_t - x_{t-s})$$

Spatial correlation function of estimated daily short-wave radiation

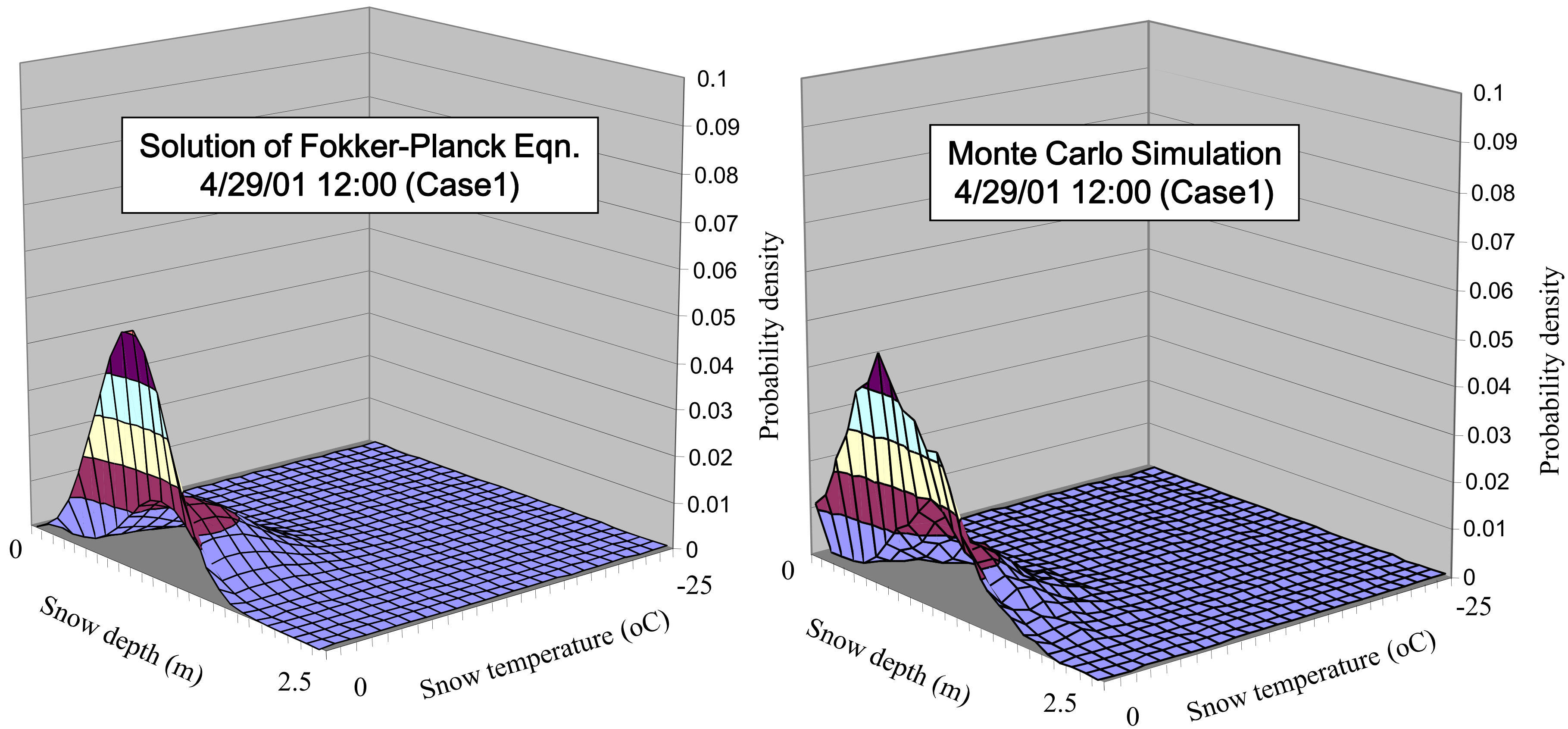




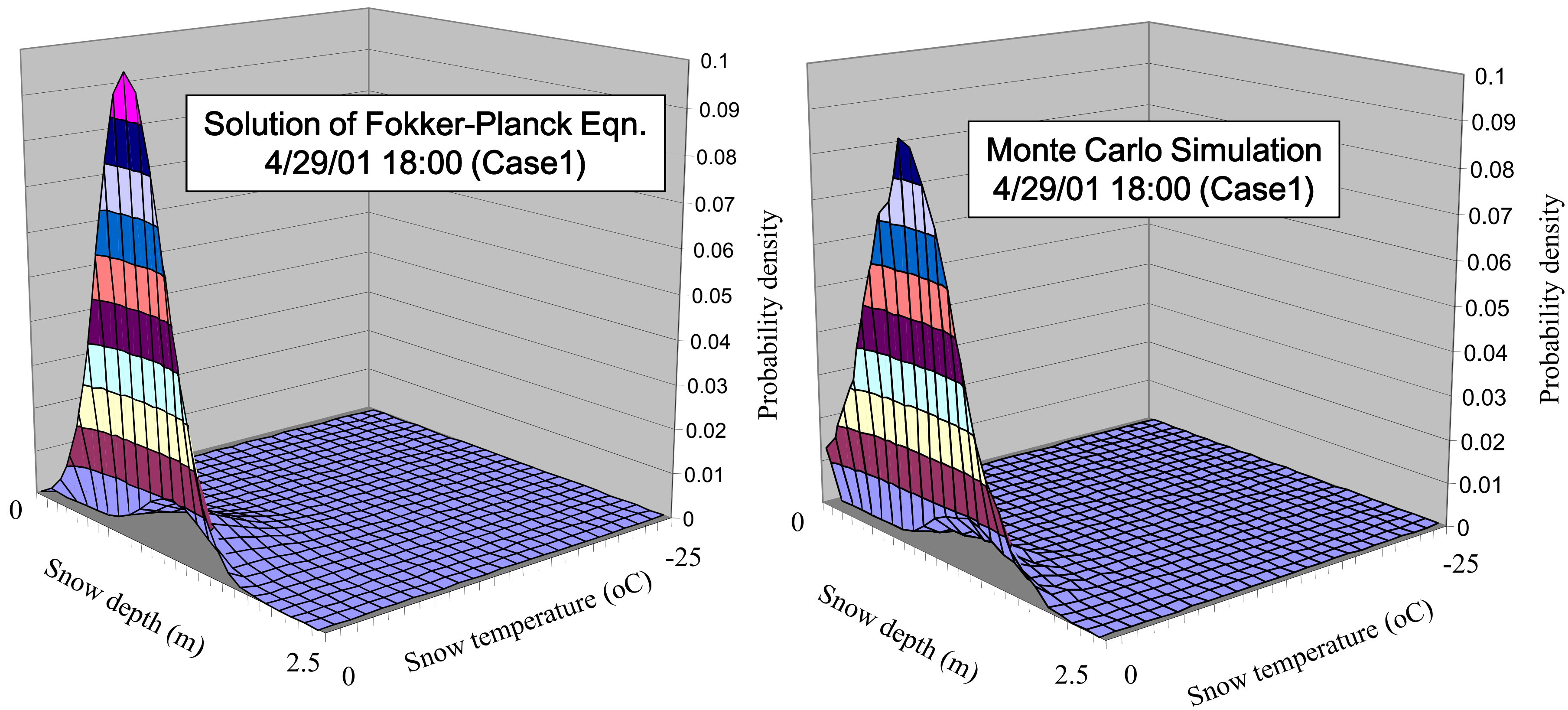
Simulation period

Case 1 : **Spring** [2001, April 29 – May 1 (3days)] for **melt** process

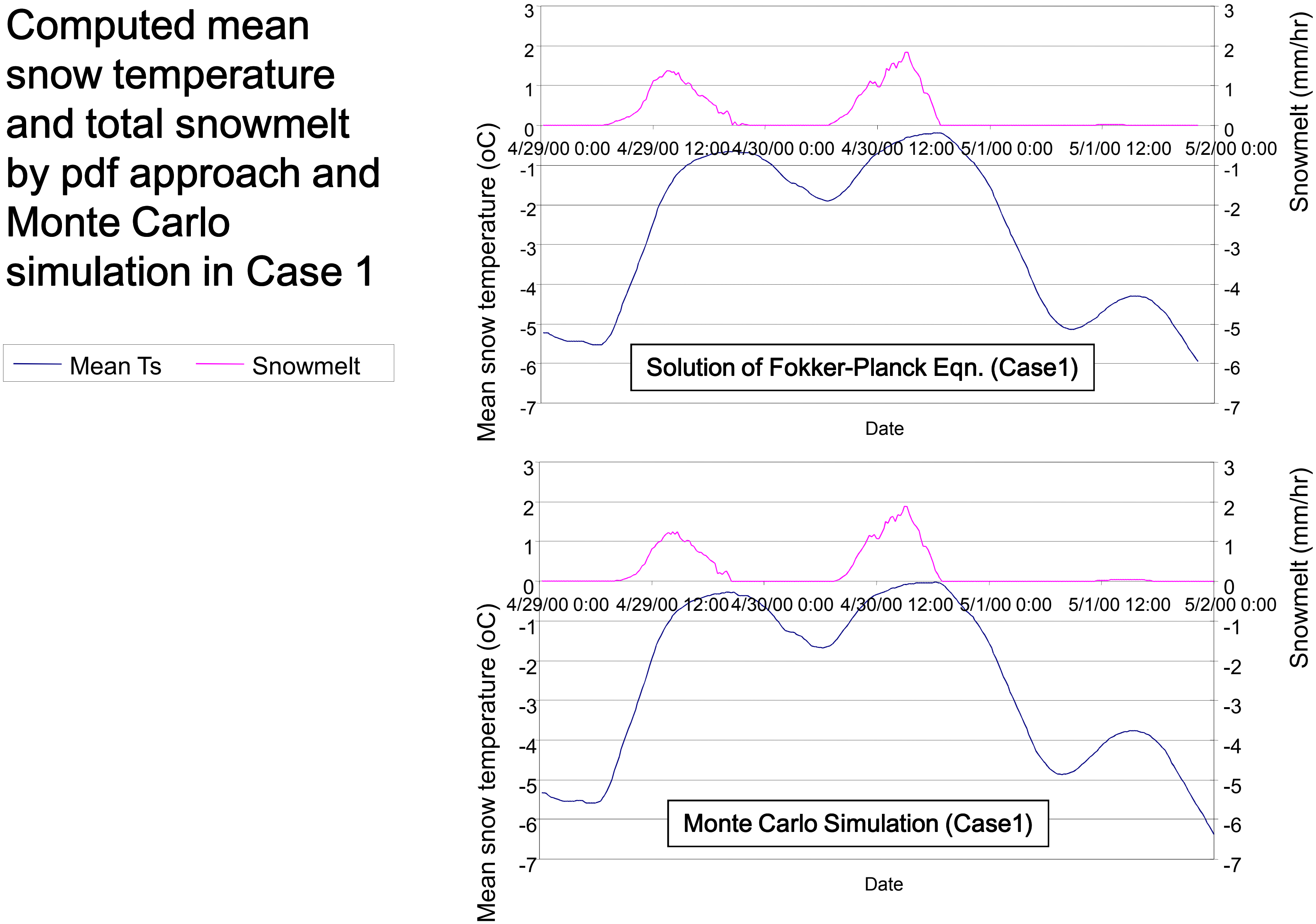
Time evolution of pdf obtained by Fokker-Planck equation and Monte Carlo simulation in Case 1 (b)



Time evolution of pdf obtained by Fokker-Planck equation and Monte Carlo simulation in Case 1 (c)



Computed mean snow temperature and total snowmelt by pdf approach and Monte Carlo simulation in Case 1



LIE GROUP SYMMETRIES FOR STOCHASTIC NONLINEAR DYNAMICS

The Modeling Approach (Cayar and Kavvas, JHE 2009):
For a multi-dimensional nonlinear stochastic partial differential equation system with its initial and boundary conditions that models a stochastic nonlinear dynamical process:

1) First, identify the Lie group of symmetry transformations that translate the original multi-dimensional stochastic space-time problem to a new space where the original problem is transformed into a (usually nonlinear) stochastic initial-value problem whose pathwise solution is the same as that of the original problem;
2) Second, after making the Lie group of symmetry transformations and ending up with a stochastic nonlinear initial value problem, determine **the equivalent** mixed Lagrangian-Eulerian Fokker-Planck-Kolmogorov equation (Kavvas, 2003) for the ensemble solution of the problem in terms of its evolutionary probability density function (PDF).
3) Then back-transform to the original space-time to obtain the ensemble solution to the original problem in the original space-time in terms of the space-time evolving PDF of the process state variable/variables.

4. From the space-time evolutionary PDF of the nonlinear stochastic process obtain the statistical functions of the process that are of practical interest (mean function, variance function, covariance function, etc.).

Stochastic 2-D Boussinesq Equation for Groundwater Flow in Heterogeneous Unconfined Aquifers:

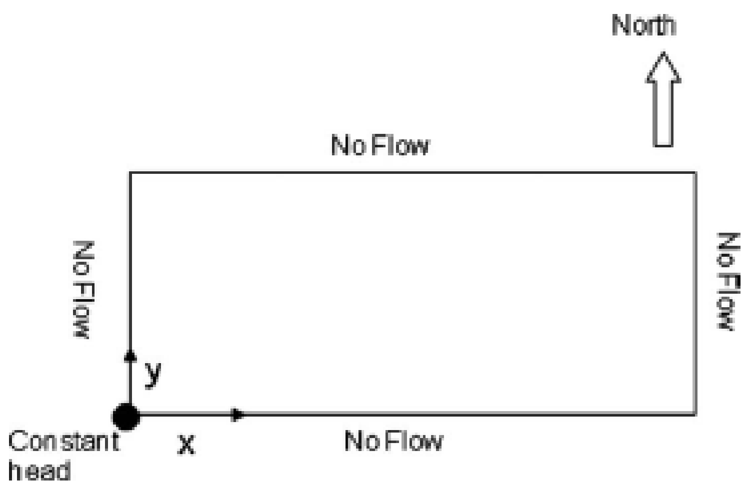
$$\frac{\partial}{\partial x}\left(\kappa h(x,y)\frac{\partial h}{\partial x}\right)+\frac{\partial}{\partial y}\left(\kappa h(x,y)\frac{\partial h}{\partial y}\right)=\frac{\partial h}{\partial t}$$

The following initial and boundary conditions are assigned for this problem:

$$h(x,y,0)=h_0 \quad \text{for} \quad x\rangle 0 \quad y\rangle 0 \quad t=0$$

$$h(0,0,t)=h_1 \quad \text{for} \quad x=0 \quad y=0 \quad t\geq 0$$

$$h(\infty,\infty,t)=h_0 \quad \text{for} \quad x=\infty \quad y=\infty \quad t\geq 0$$



After the appropriate Lie group of symmetry transformations (Cayar and Kavvas, JHE 2009):

$$\varsigma = x^2 + y^2 \qquad \alpha = t \qquad H(\varsigma,\alpha) = h(x,y,t) \qquad \upsilon = \frac{\varsigma}{\alpha} \qquad \Psi(\upsilon) = H(\varsigma,\alpha)$$

the original PDE may be reduced to the following nonlinear ODE (Cayar and Kavvas, JHE 2009)

$$\frac{\partial^2 \Psi}{\partial \upsilon^2} = -\frac{1}{4\kappa\Psi}\frac{\partial \Psi}{\partial \upsilon} - \frac{1}{\upsilon}\frac{\partial \Psi}{\partial \upsilon} - \frac{1}{\Psi}\left(\frac{\partial \Psi}{\partial \upsilon}\right)^2$$

with the boundary conditions:

$\Psi(\upsilon) = h_0$ *for* $\upsilon = \infty$

$\Psi(\upsilon) = h_1$ *for* $\upsilon = 0$

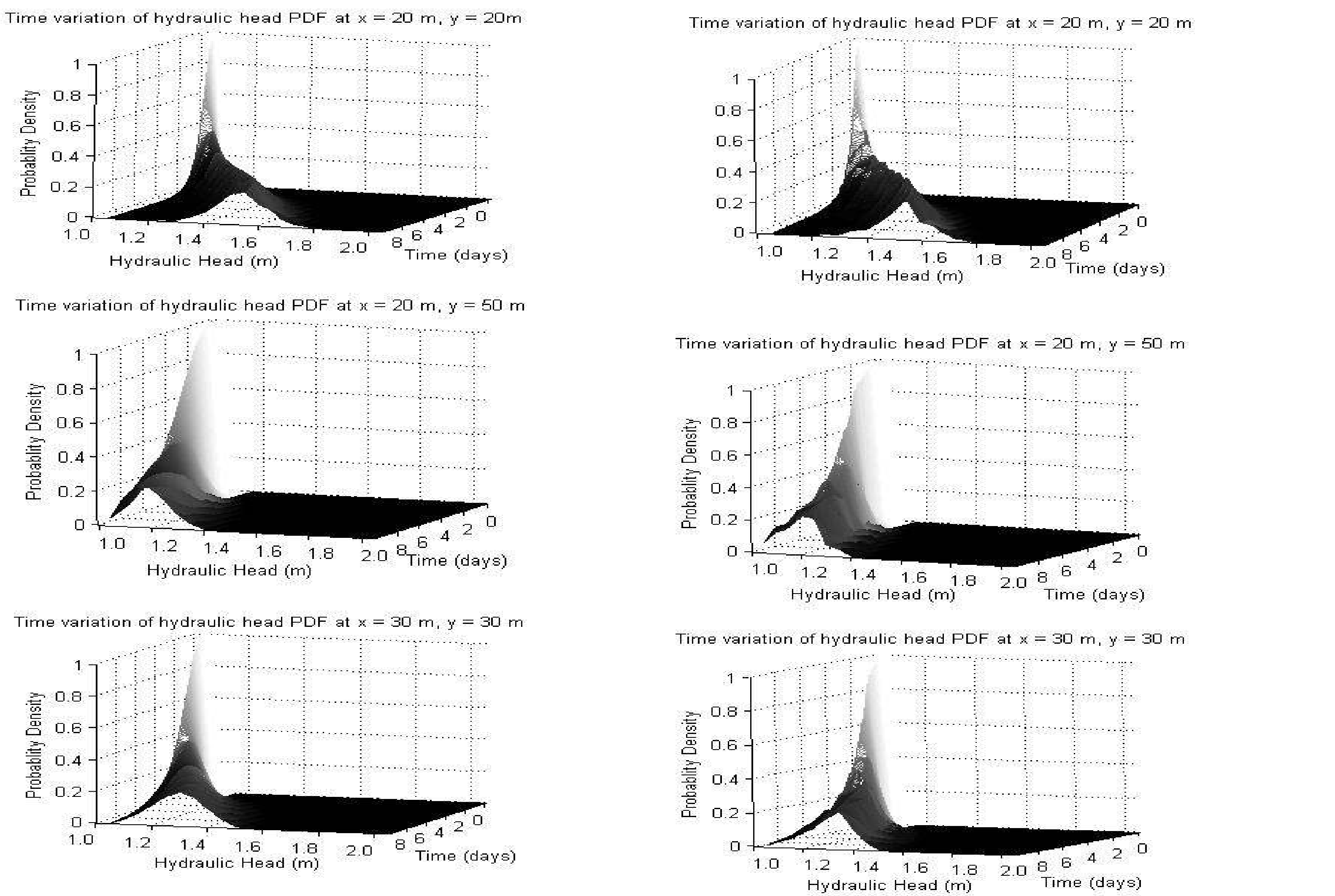
The above second-order nonlinear ODE can be transformed into a set of two first-order ODEs (Cayar and Kavvas, JHE 2009):

$$\frac{\partial \Psi}{\partial \upsilon} = \Omega(\upsilon)$$

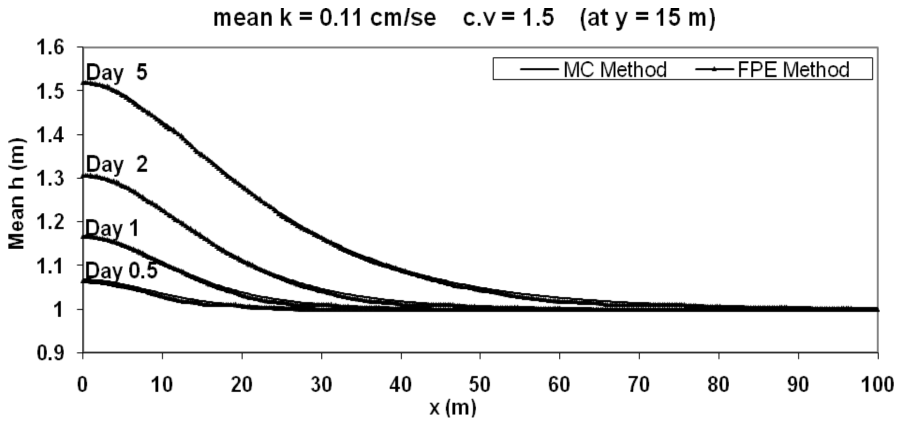
$$\frac{\partial \Omega}{\partial \upsilon} = -\Omega\left(\frac{1}{4\kappa\Psi} + \frac{1}{\upsilon}\right) - \frac{\Omega^2}{\Psi}$$

whose equivalent mixed Lagrangian-Eulerian Fokker-Planck-Kolmogorov Eqn. may be expressed after some simplifications, as (Cayar and Kavvas, JHE 2009):

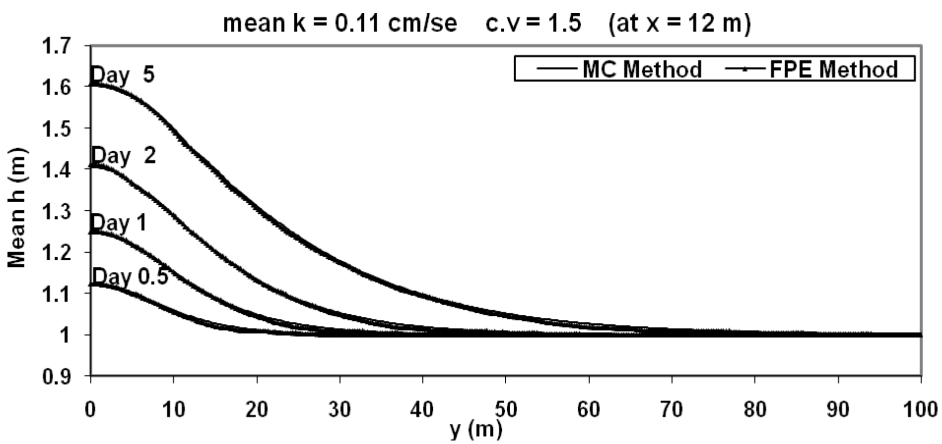
$$\frac{\partial P(\Psi,\Omega,\upsilon)}{\partial \upsilon} = -\frac{\partial}{\partial \Psi}\left(\langle\Omega\rangle P(\Psi,\Omega,\upsilon)\right) - \frac{\partial}{\partial \Omega}\left(-\langle\Omega\rangle\left(\frac{1}{4\langle\kappa\rangle\langle\Psi\rangle} + \frac{1}{\upsilon} + \frac{\langle\Omega\rangle}{\langle\Psi\rangle}\right)P(\Psi,\Omega,\upsilon)\right)$$
$$+\frac{\partial^2}{\partial \Psi^2}\left(Var(\Omega_\upsilon)P(\Psi,\Omega,\upsilon)\right) + \frac{\partial^2}{\partial \Omega^2}\left(\frac{\Omega^2}{16\Psi^2}Var\left(\frac{1}{\kappa}\right)P(\Psi,\Omega,\upsilon)\right)$$



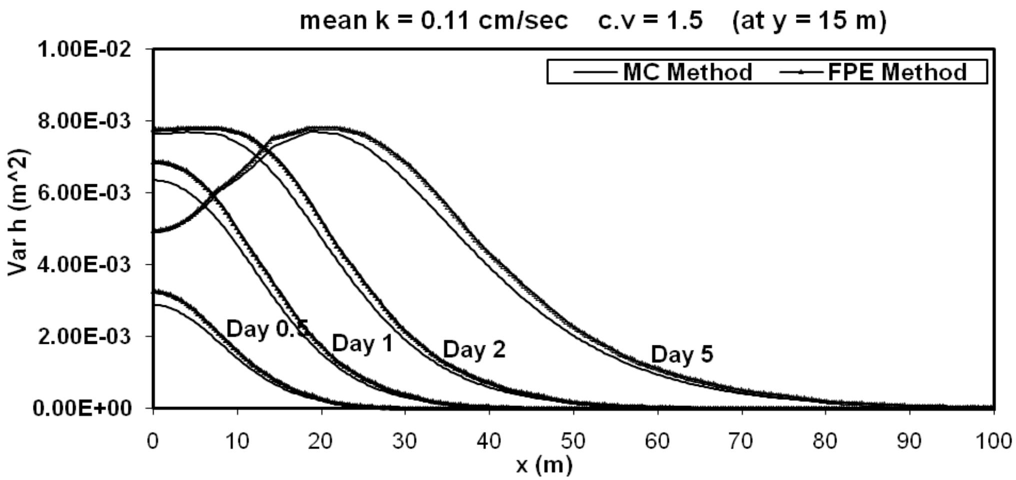
Probability density functions of Boussinesq unconfined heterogeneous aquifer flow in time and space (from Cayar and Kavvas, JHE 2009)



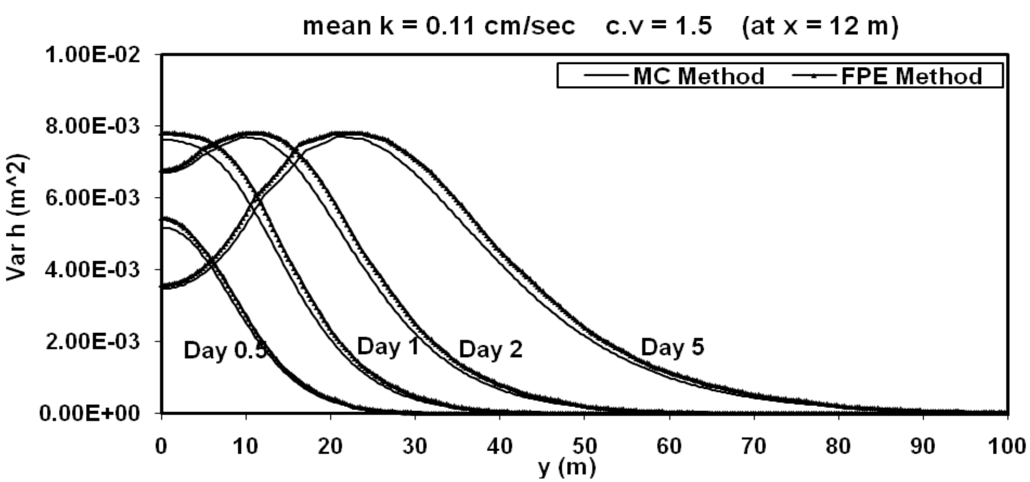
Mean hydraulic head comparison in x-direction (at y = 15) for C_v =1.5 (From Cayar and Kavvas, JHE 2009)



Mean hydraulic head comparison in y-direction (at x = 12) for C_v =1.5 (From Cayar and Kavvas, JHE 2009)



Comparison of variance of hydraulic head in x-direction (at y = 15) for C_v =1.5 (From Cayar and Kavvas, JHE 2009)



Comparison of variance of hydraulic head in y-direction (at x = 0) for C_v =1.5 (From Cayar and Kavvas, JHE 2009)

Welcome to the new world of ensemble (upscaled) hydrologic/hydraulics conservation equations !

1) While the original point-scale conservation equations are Eulerian;
the ensemble (upscaled) conservation equations are mixed Eulerian-Lagrangian;
hence: their solutions will require new computational approaches;

2) While the parameters of the existing point-scale conservation equations are at point-scale,
the parameters of the ensemble (upscaled) conservation equations are at the scale of the grid areas being modeled (eg. areal median saturated hydraulic conductivity, areal variance of log hydraulic conductivity, areal covariance of flow velocity, etc.)
hence: new parameter estimation methodologies will be required;

3) The spatial heterogeneities due to topography, soils, vegetation, land use/land cover, geology are incorporated explicitly into ensemble (upscaled) conservation equations by means of the newly emerging parameters on the areal variance/covariance of the point-scale parameters;

Especially; the areal dispersion of the point scale hydrologic dynamics (due to heterogeneity in land conditions and atmospheric boundary conditions) is explicitly modeled in the ensemble (upscaled) equations.

4) The hydrologic/hydraulic models which are based upon point-scale conservation equations with effective parameters may yield significantly incorrect predictions over highly heterogeneous flow domains.

In such flow domains it may be necessary to utilize ensemble (upscaled) conservation (governing) equations with their upscaled parameters.

M.L.Kavvas,
“Nonlinear hydrologic processes: conservation equations for their ensemble averages and probability distributions”,
ASCE Journal of Hydrologic Engineering, 2003

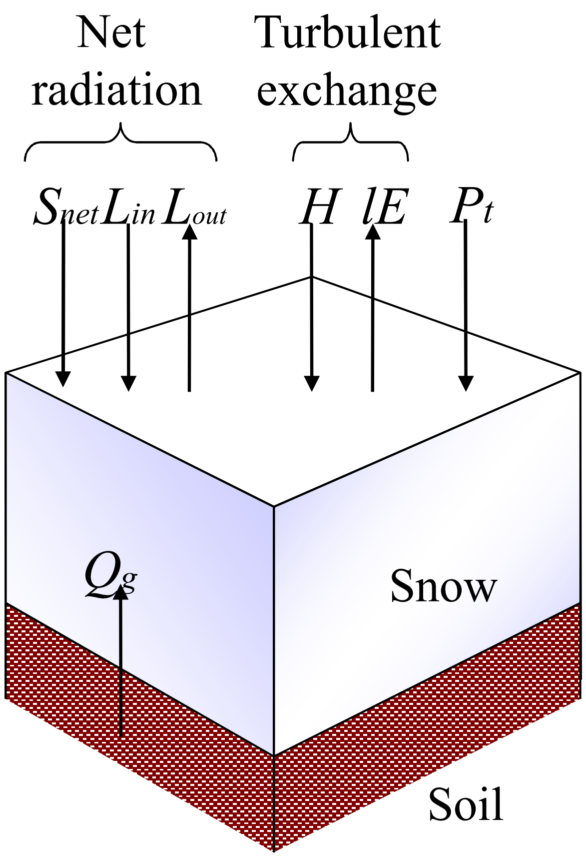
M.L.Kavvas and A. Karakas
"On the Stochastic Theory of Solute Transport by Unsteady and Steady Groundwater Flow in Heterogeneous Aquifers"
Journal of Hydrology, Vol. 179, pp.321-351, 1996

M.L.Kavvas
“General conservation equation for solute transport in heterogeneous porous media”
Journal of Hydrologic Engineering, Vol.6, No.4, 341-350, 2001

M.Cayar and M.L.Kavvas
“Symmetry in Nonlinear Hydrologic Dynamics under Uncertainty: Ensemble Modeling of 2D Boussinesq Equation for Unsteady Flow in Heterogeneous Aquifers”
Journal of Hydrologic Engineering, Vol. 14, No.10, 1173-1184, 2009

N.Ohara, M.L.Kavvas and Z.Q.Chen
“Stochastic upscaling for snow accumulation and melt processes with PDF approach”
Journal of Hydrologic Engineering, Vol. 13, No. 12, 1103-1118, 2008

Energy exchange over seasonal snow cover



S_{in} : **Short-wave radiation**
(Radiation from the sun)

L_{in} : **Incoming long-wave radiation**
(Emission of atmosphere)

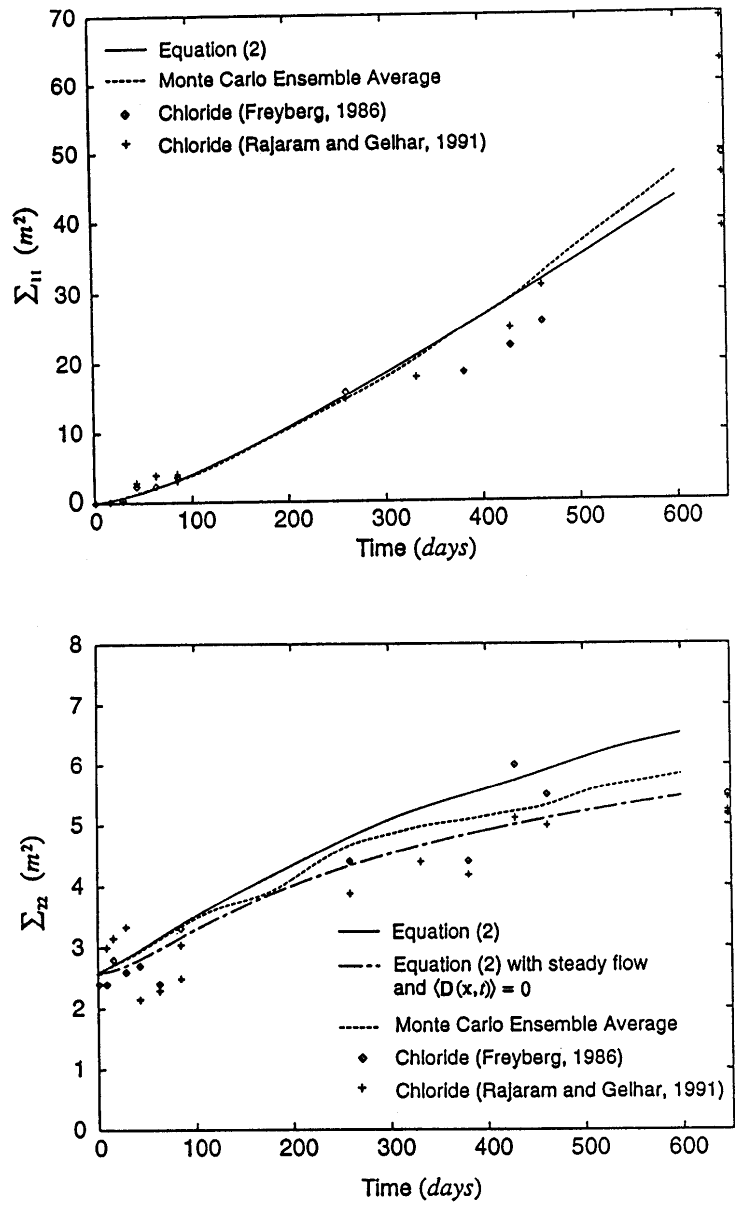
L_{out} : **Outgoing long-wave radiation**
(Emission of snow surface)

H : **Sensible heat flux**
(Heat flux from air)

lE_v : **Latent heat flux**
(Vaporization and condensation)

P_t : **Precipitation heat flux**
(Heat flux from rain drops)

Q_g : **Ground heat flux**
(Heat flux from soil under the snow)



Second spatial moments of the ensemble-averaged concentration field as determined from the upscaled transport equation and from the Monte Carlo simulation. The second moments calculated directly from the Borden aquifer data appear as points.

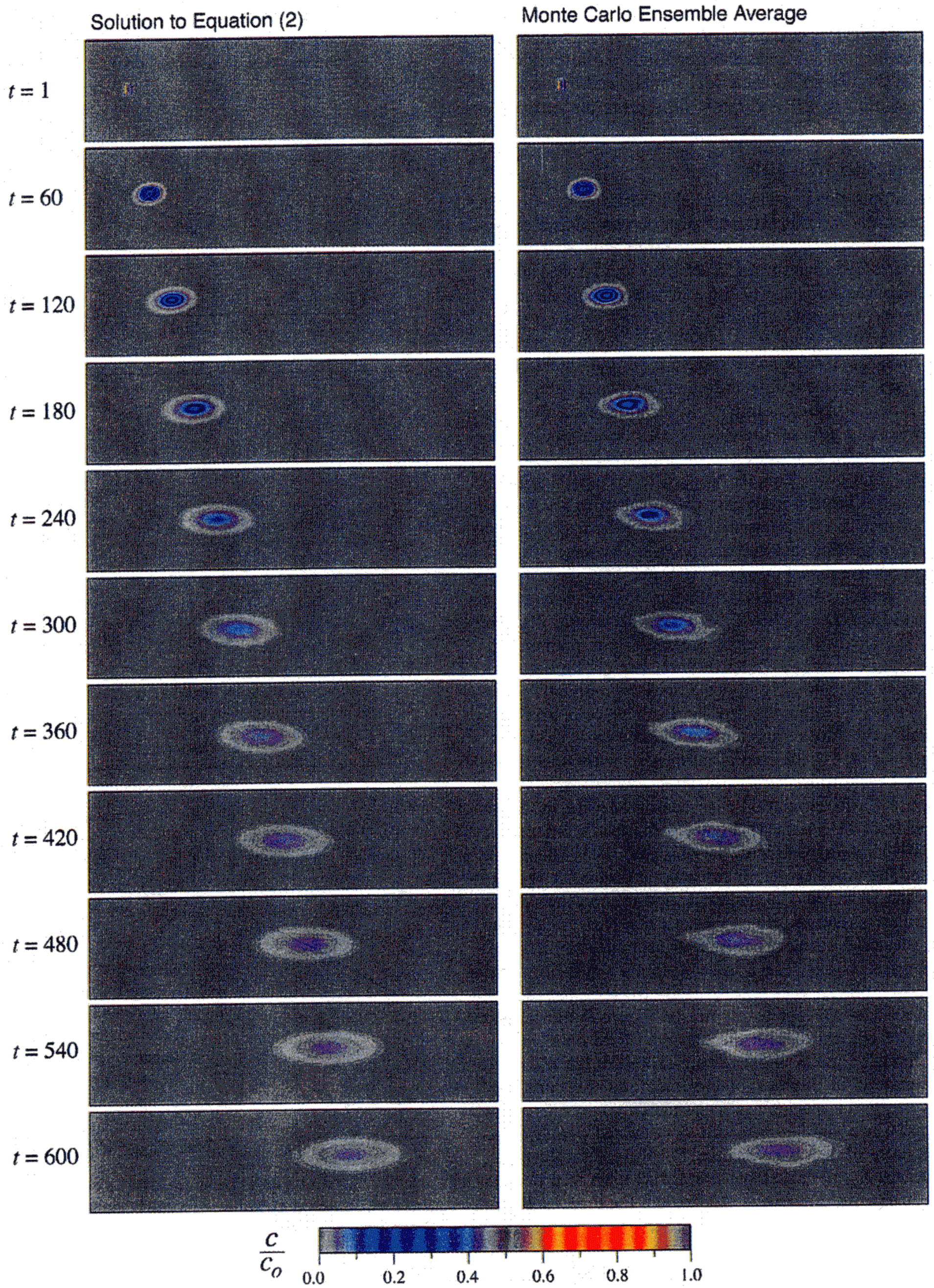
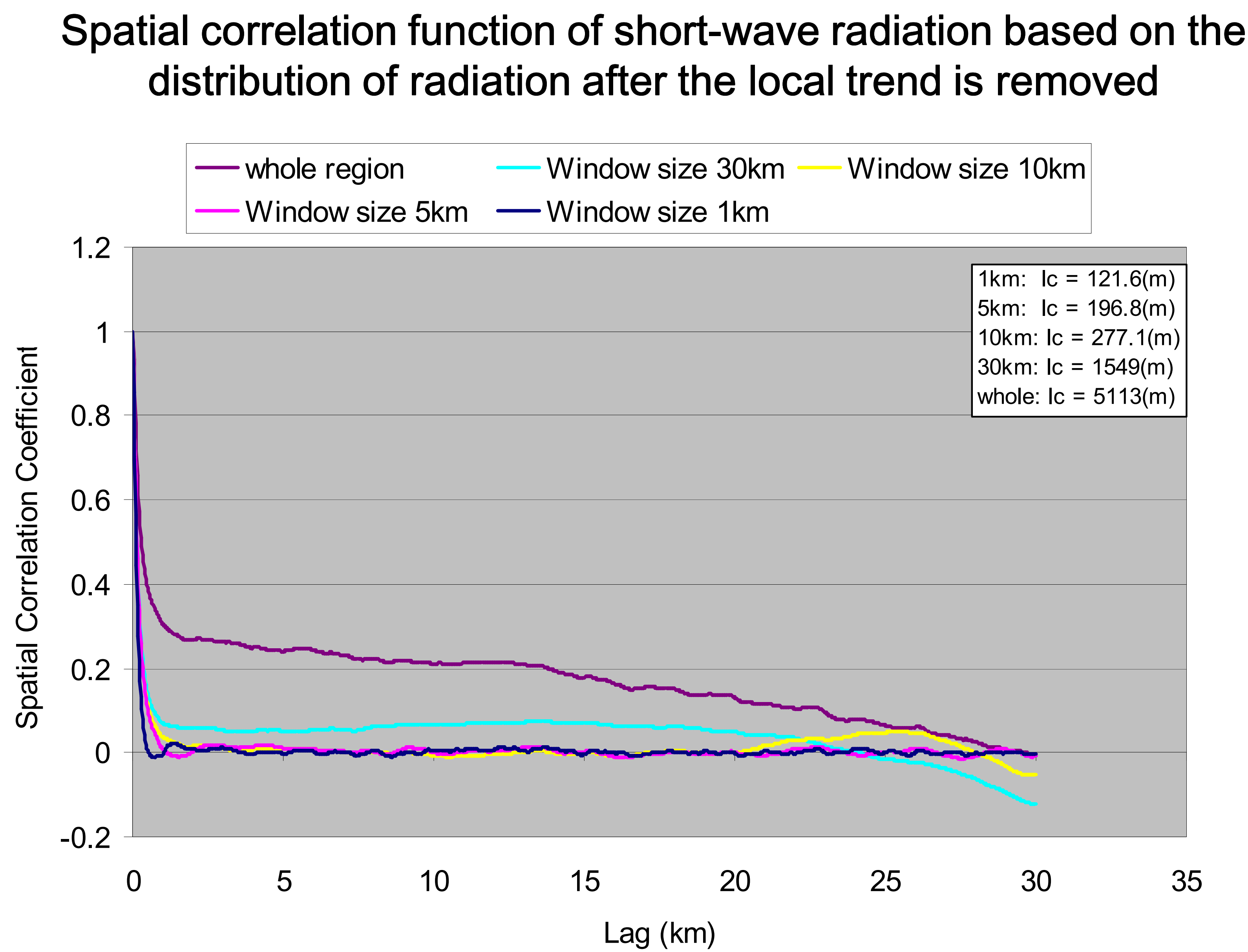


Plate 1. Ensemble-averaged concentration fields plotted for 11 output times (times are in days). The fields represent the ensemble-averaged concentration determined by the numerical solution of (2) and by the ensemble average of the Monte Carlo simulations. The effect of the transient field can be seen as a rotation of the principal axis of the plume.


Spatial correlation function of short-wave radiation based on the distribution of radiation after the local trend is removed
whole region
Window size 30km
Window size 10km
Window size 5km
Window size 1km
1km: lc = 121.6(m)
5km: lc = 196.8(m)
10km: lc = 277.1(m)
30km: lc = 1549(m)
whole: lc = 5113(m)
Spatial Correlation Coefficient
1.2
1
0.8
0.6
0.4
0.2
0
-0.2
0
5
10
15
20
25
30
35
Lag (km)

Fokker-Planck equation (FPE) for snow process

$$\frac{\partial P(T_s,D,t)}{\partial t} = -\frac{\partial}{\partial T_s}\left\{P(T_s,D,t)\left[\langle F_{1,t}\rangle - \int_0^t Cov_o\left[\frac{\partial F_{1,t}}{\partial T_s};F_{1,t-s}\right]ds - \int_0^t Cov_o\left[\frac{\partial F_{1,t}}{\partial D};F_{2,t-s}\right]ds\right]\right\}$$

$$-\frac{\partial}{\partial D}\left\{P(T_s,D,t)\left[\langle F_{2,t}\rangle - \int_0^t Cov_o\left[\frac{\partial F_{2,t}}{\partial T_s};F_{1,t-s}\right]ds - \int_0^t Cov_o\left[\frac{\partial F_{2,t}}{\partial D};F_{2,t-s}\right]ds\right]\right\}$$

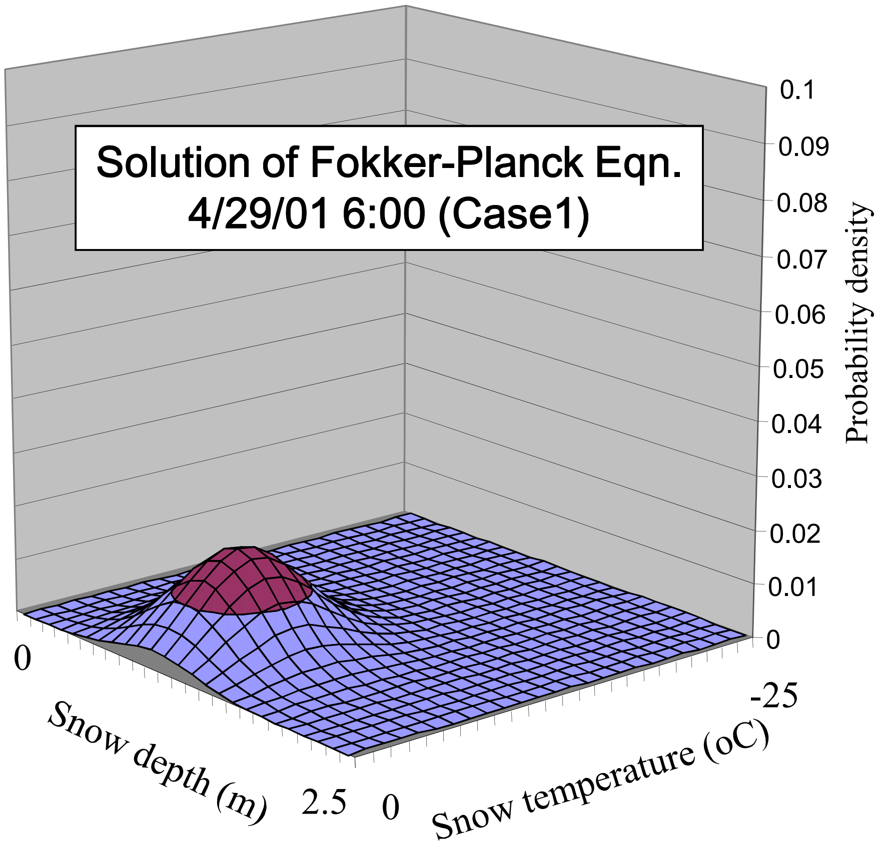
$$+\frac{1}{2}\frac{\partial^2}{\partial T_s^{\,2}}\left\{2P(T_s,D,t)\int_0^t Cov_o\left[F_{1,t};F_{1,t-s}\right]ds\right\} + \frac{1}{2}\frac{\partial^2}{\partial T_s\partial D}\left\{2P(T_s,D,t)\int_0^t Cov_o\left[F_{1,t};F_{2,t-s}\right]ds\right\}$$

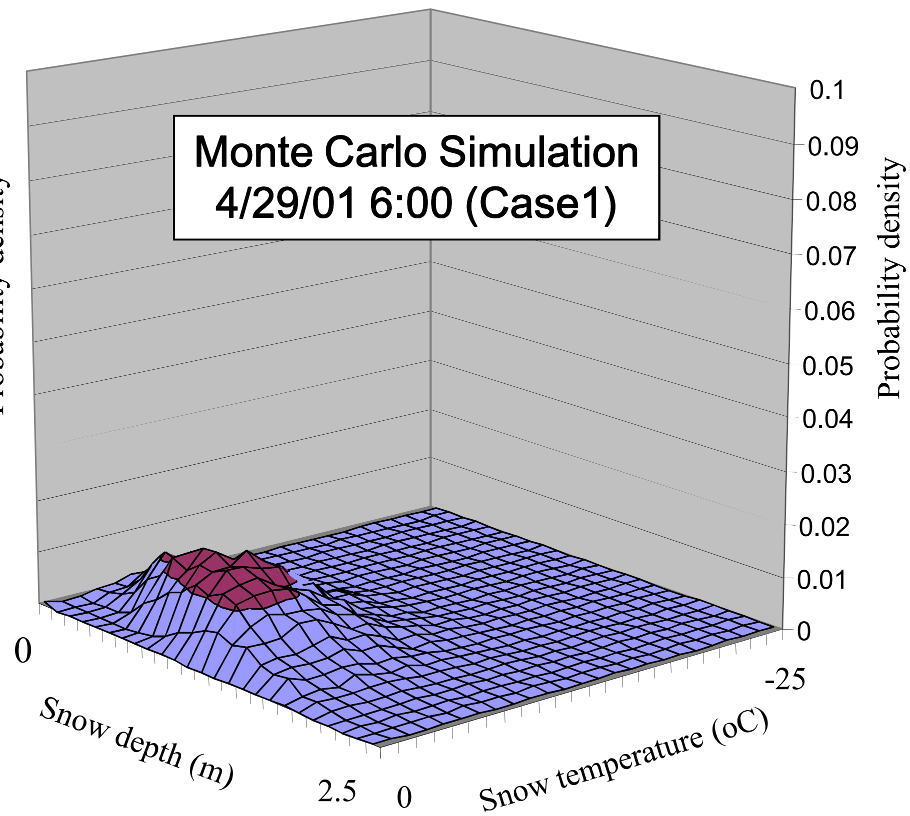
$$+\frac{1}{2}\frac{\partial^2}{\partial D\partial T_s}\left\{2P(T_s,D,t)\int_0^t Cov_o\left[F_{2,t};F_{1,t-s}\right]ds\right\} + \frac{1}{2}\frac{\partial^2}{\partial D^2}\left\{2P(T_s,D,t)\int_0^t Cov_o\left[F_{2,t};F_{2,t-s}\right]ds\right\}$$

$$F_1(T_s,D,sw) = \frac{En - M + sw}{D\rho_s C_s}$$

$$F_2(T_s,D,sn) = \frac{\rho_w}{\rho_s}(-E - Mr) + \frac{\rho_w}{\rho_{ns}}sn - \frac{2}{3\eta}\rho_s D^2 e^{(-0.04T_s - \mu\rho_s)}$$

Initial condition of pdf obtained by Fokker-Planck equation and Monte Carlo simulation in Case 1





Time evolution of pdf obtained by Fokker-Planck equation and Monte Carlo simulation in Case 1 (f)