Non-Bayesian Quickest Change Detection With Stochastic Sample Right Constraints

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Abstract—In this paper, we study the design and analysis of optimal detection scheme for sensors that are deployed to monitor the change in the environment and are powered by energy harvested from the environment. In this type of applications, detection delay is of paramount importance. We model this problem as quickest change detection problem with stochastic energy constraints. In particular, a wireless sensor powered by renewable energy takes observations from a random sequence, whose distribution will change at an unknown time. Such a change implies events of interest. The energy in the sensor is consumed by taking observations and is replenished randomly. The sensor cannot take observations if there is no energy left in the battery. Our goal is to design optimal power allocation and detection schemes to minimize the worst case detection delay, which is the difference between the time when the change occurs and the time when an alarm is raised. Two types of average run length (ARL) constraints, namely an algorithm level ARL constraint and a system level ARL constraint, are considered. We propose a low complexity scheme in which the energy allocation rule is to spend energy to take observations as long as the battery is not empty and the detection scheme is the Cumulative Sum test. We show that this scheme is optimal for the formulation with the algorithm level ARL constraint and is asymptotically optimal for the formulations with the system level ARL constraint.

Index Terms—Cumulative Sum test, energy harvesting sensor, non-Bayesian quickest change detection, sequential detection.

I. INTRODUCTION

RECENTLY, the study of sensor networks powered by renewable energy harvested from the environment has attracted considerable attention [1]–[8]. Compared with the sensor networks powered by batteries, the sensor networks powered by renewable energy have several unique features such as unlimited life span and high dependence on the environment, etc. Optimal power management schemes for each individual sensor and scheduling protocols for the whole network have been developed to optimize utility functions of communication related metrics such as channel capacity [8], transmission delay [4], [5], [7], transmission rate or network throughput [1]–[3], [6], [7]. However, besides these communication related metrics, there are other signal processing related performance metrics that are also important for sensor networks targeted for certain applications. For example, if a sensor network is deployed to monitor the health of a bridge, then the detection delay between the time when a structural problem occurs and the time when an alarm is raised is of interest. As another example, if a sensor network is deployed for intrusion detection, then the detection delay and the false alarm probability are of interest. Until now, these alternative but important performance metrics have not been investigated for sensors powered by renewable energy.

Power management schemes are of great importance for both renewable energy powered and battery powered wireless sensor networks [9]. In this paper, we focus on the design of optimal power management schemes for renewable energy powered wireless sensor networks when the detection delay is of interest. In such applications, wireless sensors are deployed to monitor the change in the environment. Such changes typically imply certain activities of interest. For example, in the bridge monitoring, a change may imply that a certain structural problem has occurred in the bridge. As the result, it is of paramount importance to minimize the detection delay after the presence of such a change, hence we formulate our problem as a quickest change detection problem, see a recent book [10] and recent surveys on this topic [11], [12]. Besides monitoring the health of a bridge, quickest change detection also has many other potential applications, such as the quality control [13], network intrusion detection [14], cognitive radio [15], etc. We note that the detection delay in the quickest change detection problem refers to the delay between the time when a change occurs and the time when an alarm is raised. It is not the delay from time zero to the time when an alarm is raised, since we are interested in the change.

Non-Bayesian quickest change detection is one of the most important formulations, which was first studied by G. Lordon [16] and M. Pollak [17]. In the standard non-Bayesian formulation, there is a sequence of observations \( \{X_k, k = 1, 2, \ldots\} \) with a fixed but unknown change point \( t \). Before the change point \( t \), the sequence \( X_1, \ldots, X_{t-1} \) are independent and identically distributed (i.i.d.) with probability density function (pdf) \( f_e \), and after \( t \), the sequence are i.i.d. with pdf \( f_o \). Under an average run length (ARL) to false alarm constraint, namely the expected duration to a false alarm is at least \( \gamma \). Lorden’s setup is to minimize the “worst-worst case” detection delay \( \sup_{1 \leq t \leq T \leq T+1} \mathbb{E}_t[|T - t + 1|^+ |X_1, \ldots, X_{t-1}] \), where \( T \) is the stopping time at which an alarm is raised, while Pollak’s setup is to minimize the “worst case” conditional average detection delay...
delay \sup_{t} E_t[(T-t) | T \geq t]. It has been shown that the Cumulative Sum (CUSUM) procedure is optimal for Lordon’s setup [18] and that the CUSUM procedure [19] and several variations of the Shiryaev-Roberts (SR) procedure are asymptotically optimal for Pollak’s setup [17], [20], [21]. Since no prior information about the change point is required, these non-Bayesian setups are very attractive for practical applications.

In the above mentioned classic setups, there is no energy constraint and the observations can be taken at every time slot. In this paper, we extend Lorden’s and Pollak’s setups to sensors that are powered by renewable energy. In this case, the energy stored in the sensor is replenished by a random process and consumed by taking observations. The sensor cannot take observations if there is no energy left. Hence, the sensor cannot take observations at every time instant anymore. The sensor needs to plan its use of power carefully. Moreover, the stochastic nature of the energy replenishing process will certainly affect the performance of change detection schemes. Since the energy collected by the harvester in each time instant is not a constant but a random variable, this brings new optimization challenges.

We first consider the scenario in which a unit of energy arrives with probability \( p \) at each time instant. For Lorden’s setup, two types of ARL constraints are considered in this paper. The first type is an algorithm level ARL constraint, which puts a lower bound on the expected number of observations taken by the sensor before it runs a false alarm. With zero initial energy, we show that the optimal detection procedure is the well known CUSUM procedure, and the optimal power allocation scheme is to allocate the energy as soon as it is harvested. The second type ARL constraint is on a system level, which puts a lower bound on the expected duration to a false alarm. In this case, for any initial energy level, we show that the CUSUM procedure and the immediate power allocation strategy is asymptotically optimal when the system ARL goes to infinity. For Pollak’s setup, we discuss the problem only with the system level ARL in detail. As we can see later, the immediate power allocation coupled with the CUSUM detection is actually asymptotically optimal for both the system level ARL and the algorithm level ARL.

We then consider a more general energy arriving model, in which more than one unit of energy can arrive at each time instant. In this scenario, we show that a simple energy allocation policy, in which the sensor takes samples as long as there is energy left in the battery, coupled with the CUSUM test is asymptotically optimal for both Lorden’s and Pollak’s setups when the system level ARL goes to infinity.

There have been some existing works on the quickest change detection problem that take the sample cost into consideration. The first main line of existing works considers the problem under a Bayesian setup. The main difference between the Bayesian setup and the non-Bayesian setup is that in the Bayesian setup, the change point is modeled as a random variable with a known distribution. No such assumption is made in the non-Bayesian setup. [14] considers the design of detection strategy that strikes a balance between the detection delay, false alarm probability and the number of sensors being active. In particular, [14] considers a wireless network with multiple sensors monitoring the Bayesian change in the environment. Based on the observations from sensors at each time slot, the fusion center decides how many sensors should be active in the next time slot to save energy. [22] discusses the Bayesian quickest change detection problem with a constraint that only a finite number of observations can be taken. [23] takes the average number of observations into consideration, and provides the optimal solution along with low-complexity but asymptotically optimal rules. In [24], the authors propose a DE-CUSUM scheme for the non-Bayesian setup and show that it is asymptotically optimal.

The remainder of this paper is organized as follows. The mathematical model is given in Section II. Section III presents the optimal solution for Lorden’s problem under the algorithm level ARL constraint and the performance analysis for the optimal solution. In Section IV, we present asymptotically optimal solutions for Lorden’s and Pollak’s problems under the system level ARL constraint. Section V presents our results for a more general energy arriving model. Numerical examples are given in Section VI to illustrate the results obtained in this work. Finally, Section VII offers concluding remarks.

## II. Problem Formulation

Let \( \{X_k, k = 1, 2, \ldots\} \) be a sequence of random variables whose distribution changes at a fixed but unknown time \( t \). Before \( t \), the \( \{X_k\} \)’s are i.i.d. with pdf \( f_0 \); after \( t \), they are i.i.d. with pdf \( f_1 \). The pre-change pdf \( f_0 \) and post-change pdf \( f_1 \) are perfectly known by the sensor. We use \( P_t \) and \( E_t \) to denote the probability measure and the expectation with the change happening at \( t \), respectively, and use \( P_{\infty} \) and \( E_{\infty} \) to denote the case \( t = \infty \).

We assume that the energy arrives randomly at each time slot. To facilitate the presentation and set up notation, we present the model for the case when the energy arriving process is a Bernoulli process with parameter \( p \) in this section. A more general model will be considered in Section V. Specifically, we use \( \nu = \{\nu_1, \nu_2, \ldots, \nu_k, \ldots\} \) to denote the energy arriving process with \( \nu_k \in \{0, 1\} \), in which \( \nu_k = 1 \) indicates that a unit of energy is collected by the energy harvester at time slot \( k \) and \( \nu_k = 0 \) means that no energy is harvested. \( \{\nu_k\} \) is i.i.d. over \( k \). Moreover, we use \( P^\nu \) to denote its probability measure (correspondingly, we use \( E^\nu \) to denote the expectation with respect to the measure \( P^\nu \)), and we have \( P^\nu(\nu_k = 1) = p \).

The sensor can decide how to allocate these collected energies. Let \( \mu = \{\mu_1, \mu_2, \ldots, \mu_k, \ldots\} \) be the power allocation strategy with \( \mu_k \in \{0, 1\} \), in which \( \mu_k = 1 \) means that the wireless sensor spends a unit of energy on taking an observation at time slot \( k \), while \( \mu_k = 0 \) means that no energy is spent at time \( k \) and hence no observation is taken.

The sensor’s battery has a finite capacity \( C \). The energy arriving process and the energy utilizing process will affect the amount of energy in the battery. We use \( E_k \) to denote the energy left in the battery at the end of time slot \( k \). \( E_k \) evolves according to:

\[
E_k = \min[C, E_{k-1} + \nu_k - \mu_k].
\]

Denote \( E_0 \) as the initial energy stored in the sensor, we have \( 0 \leq E_0 \leq C \). The energy allocation policy \( \mu \) must obey the
causality constraint, namely the energy cannot be used before it is harvested. The energy causality constraint can be written as
\begin{equation}
E_k \geq 0 \quad k = 1, 2, \ldots
\end{equation}

We use \( \mathcal{U} \) to denote the set of all \( \mu \)'s that satisfy (1).

The sensor spends energy to take observation. The observation sequence is denoted as \( \{Z_k, k = 1, 2, \ldots\} \), where
\begin{equation}
Z_k = \begin{cases} X_k & \text{if } \mu_k = 1 \\ \phi & \text{if } \mu_k = 0 \end{cases}
\end{equation}

We call an observation \( Z_k \) a non-trivial observation if \( \mu_k = 1 \), i.e., if the observation is taken from the environment. Denote \( \{\hat{X}_k, k = 1, 2, \ldots\} \) as the non-trivial observation sequence, which is the subsequence of \( \{Z_k\} \) with all its non-trivial elements.

\( \{Z_k\} \)'s are not necessarily conditionally (conditioned on the change point) i.i.d. due to the existence of \( \{\mu_k\} \). The distribution of \( Z_k \) is related to both \( \mu_k \) and \( X_k \). Therefore, we use \( P^\mu_{\text{tr}} \) and \( E^\mu_{\text{tr}} \) to denote the probability measure and the expectation of the observation sequence \( \{Z_k\} \) with the change happening at \( t \), respectively. However, we notice that \( \{\hat{X}_k\} \) is a conditionally i.i.d. sequence, since \( \hat{X}_k \) is generated by either \( f_0 \) or \( f_1 \), depending on whether this observation is taken before the change point \( t \) or after \( t \).

In this paper, we want to find a stopping time \( T \), at which the sensor will declare that a change has occurred, and a power allocation rule \( \mu \) that jointly minimize the detection delay. We consider three problem setups. The first one is Lorden’s quickest change detection problem with an algorithm level ARL constraint, which is formulated as
\begin{equation}
\begin{aligned}
\text{(P1)} \quad \min_{\mu \in \mathcal{U}, T \in \mathcal{T}} & \quad d(\mu, T), \\
\text{s.t.} & \quad E^\infty_{\text{tr}}[N] \geq \eta,
\end{aligned}
\end{equation}

where \( \mathcal{T} \) is the set of all stopping time with \( E^\mu_{\text{tr}}[T] < \infty \), \( N \) is the total number of non-trivial observations taken by the sensor before it claims that the change has happened and
\begin{equation}
\begin{aligned}
d(\mu, T) &= \sup_{t \geq 1} d_t(\mu, T), \\
d_t(\mu, T) &= \exp(\sup_{i \geq 1} E^\mu_{\text{tr}}[(T - t + 1)^+ F_t \cdot 1]),
\end{aligned}
\end{equation}

where \( F_t = \sigma\{Z_1, \ldots, Z_t\} \). Unlike the standard Lorden’s setup, here the worst case delay \( d(\mu, T) \) is a function of observations \( \{Z_1, \ldots, Z_{t-1}\} \) controlled by \( \mu \); hence the expectation used in (4) is \( E^\mu_{\text{tr}} \) rather than \( E_{\text{tr}} \). The algorithm level ARL constraint uses expectation \( E^\infty_{\text{tr}} \) rather than \( E_{\text{tr}} \) because all the non-trivial observations are i.i.d. with pdf \( f_0 \) under probability measure \( P^\infty_{\text{tr}} \). Hence, the distribution law of \( N \) is independent of the power allocation scheme \( \mu \) and the energy arriving sequence \( \nu \). As the result, this problem setup is robust against the variation of the ambient environment.

The second problem considered in this paper is Lorden’s quickest change detection problem with a system level ARL constraint, which is formulated as
\begin{equation}
\begin{aligned}
\text{(P2)} \quad \min_{\mu \in \mathcal{U}, \nu \in \mathcal{N}, T \in \mathcal{T}} & \quad d(\mu, T), \\
\text{s.t.} & \quad E^\nu_{\text{tr}}[\mathcal{T}] \geq \gamma,
\end{aligned}
\end{equation}

(P2) and (P1) have the same objective function, but their constraints are quite different. For the system level constraint, a lower bound is set on the expected duration to a false alarm. The stopping time \( T \) not only depends on the number of non-trivial observations, but also relies on the time interval between each two successive observations, hence the system level ARL constraint depends on the power allocation \( \mu \) and uses expectation \( E^\nu_{\text{tr}} \). Since the power allocation \( \mu \) is further related to the energy arriving process \( \nu \), this setup is more sensitive to the environment.

In some applications, Pollak’s formulation is of interest since its delay metric is less conservative than that of Lorden’s formulation. In our context, Pollak’s formulation can be written as
\begin{equation}
\begin{aligned}
\text{(P3)} \quad \min_{\mu \in \mathcal{U}, T \in \mathcal{T}} & \quad \sup_{t \geq 1} E^\mu_{\text{tr}}[T - t | \mathcal{T} \geq t], \\
\text{s.t.} & \quad E^\infty_{\text{tr}}[\mathcal{T}] \geq \gamma.
\end{aligned}
\end{equation}

Even without the additional energy causality constraint, the optimal solution for Pollak’s formulation is still open [11], [12]. Therefore, in this paper, we discuss only the asymptotic solution for Pollak’s formulation. In the sequel, we will see that the proposed asymptotically optimal solution under the system level ARL constraint is also asymptotically optimal under the algorithm level ARL constraint. Hence, in the paper, we discuss only the system level ARL constraint for Pollak’s formulation in detail.

III. OPTIMAL SOLUTION FOR LORDEN’S FORMULATION WITH THE ALGORITHM LEVEL ARL CONSTRAINT

In this section, we study the optimal solution for (P1) under the binary energy arriving model with \( E_0 = 0 \). We use \( L(\cdot) \) to denote the likelihood ratio (LR), and use \( l(\cdot) = \log L(\cdot) \) to denote the log likelihood ratio (LLR). For the observation sequence \( \{Z_k\} \), LR is defined as
\begin{equation}
L(Z_k) = \begin{cases} \frac{l(Z_k)}{l(0)} & \text{if } \mu_k = 1 \\ 1 & \text{if } \mu_k = 0 \end{cases},
\end{equation}

The CUSUM statistic and Page’s stopping time can be written as [16]
\begin{equation}
S_k = \max_{1 \leq q \leq k} \left[ \prod_{i=q}^{k} L(Z_i) \right] = \max_{1 \leq q \leq k} S_{k-1} \cdot \max[1, L(Z_k)],
\end{equation}

and
\begin{equation}
T_k = \inf\{k \geq 0 | S_k \geq B\}
\end{equation}

for some constant threshold \( B \), respectively. In order to characterize the energy arriving and spending time, for an arbitrary realization of the power allocation scheme
\( \mu \) and energy arriving process \( \nu \), we use the following notations throughout this section:

1) \( \{a_k, k = 1, 2, \ldots\} \) to denote the time instants at which the sensor harvests a unit of energy, i.e., \( \nu_{a_k} = 1 \);

2) \( \{b_k, k = 1, 2, \ldots\} \) to denote the time instants at which the sensor takes observations, i.e., \( \mu_{b_k} = 1 \).

Since \( F_0 = 0 \), using above notations, the energy causality constraint indicates the following inequality:

\[
 b_k \geq a_k, \quad k = 1, 2, \ldots \tag{8}
\]

In this section we also use \( \{X_k^{(a_k, b_k)} = 1, 2, \ldots\} \) to denote the non-trivial observation sequence. Specifically, \( \bar{X}_k \) and \( X_k^{(a_k, b_k)} \) are used interchangeably, but \( X_k^{(a_k, b_k)} \) will be used when we want to emphasize the sampling time. In particular, \( X_k^{(a_k, b_k)} \) is the \( k^{th} \) non-trivial observation taken by the sensor at time \( b_k \) using the energy collected at time \( a_k \).

We emphasize that the above notations are related to the realizations of the stochastic processes \( \mu \) and \( \nu \). Specifically, \( \{a_k\} \) depends on the realization of \( \nu \), \( \{b_k\} \) depends on the realization of \( \mu \), and \( \{X_k^{(a_k, b_k)}\} \) depends on both of them.

Generally, for a given detection strategy \( (\mu, T_p) \), the detection delay \( d_1(\mu, T) \) in (4) varies from different change point \( \tau \), hence the worst case delay takes the supreme over \( \tau \). If there is an equalizer strategy which makes \( d_1(\mu, T) \) be a constant over \( \tau \), it might be a good candidate for the optimal strategy for the minimax problem. Similar to the conclusion that Page’s stopping time is an equalizer rule for the classic Lorden’s problem [10], we have the following proposition:

**Proposition 3.1:** From any sequential probability ratio test, the power allocation scheme \( \mu^* = \nu \) and Page’s stopping time \( T_P \) together achieve an equalizer rule, i.e.,

\[
 d_1(\mu^*, T_P) = \frac{1 - P(0; F_0)}{P(1; F_0)} = 1. \tag{9}
\]

**Proof:** Since \( \mu^* = \nu \) indicates that \( \rho_0^* \)'s are i.i.d. over \( k \), \( \{Z_k\} \)'s are conditionally i.i.d. given the change point \( \tau \).

Let \( W_k = \max[S_k, 1] \). On the event \( \{T_p \geq t\}, T_p \) is a non-increasing function of \( W_{t-1} \). Since \( W_{t-1} \geq 1 \) and event \( \{W_{t-1} = 1\} \) \( \in F_{t-1} \), the worst case of \( T_p \) happens at \( W_{t-1} = 1 \), that is

\[
 d_1(\mu^*, T_p) = \sup_{t} E_{\mu^*}^{W_t} [T_p - t + \sum_{k=1}^{t} \bar{X}_k] \bigg| T_p - t + \sum_{k=1}^{t} \bar{X}_k. \tag{10}
\]

Since \( \{Z_k\} \)'s are conditionally i.i.d. under \( \mu^* \), \( \{W_k\} \) is a homogeneous Markov chain, then, \( d_1(\mu^*, T_p) = d_1(\mu^*, T_p) \).

**Remark 3.2:** \( \mu^* = \nu \) indicates that \( \mu_0^* = \nu_k \) (or \( b_k = a_k \)) for every \( k \), that is, the sensor spends the energy on taking observation immediately when it obtains an energy from the environment. Therefore, we call \( \mu^* \) the *immediate power allocation scheme* in the sequel.

**Remark 3.3:** The equalizer property plays a critical role in the proof of (asymptotic) optimality and the performance analysis in the sequel. From this property, we have \( d(\mu^*, T_p) = d_1(\mu^*, T_p) - F_{\mu^*}^{T_p} \), which can greatly simplify the analysis. Since the proof of Proposition 3.1 holds regardless of the ARL constraint, we can conclude that \( \{\mu^*, T_p\} \) is also an equalizer rule for (P2).

The optimal solution of (P1) is described in the following lemma.

**Lemma 3.4:** With zero initial energy, the optimal power allocation strategy for (P1) is \( \mu^* \), and the optimal stopping time is \( T_p \) with the threshold \( B \) being a constant such that \( E_\infty N | = \eta \).

**Proof:** The proof consists of two steps. The first step is to show that for an arbitrary but given power allocation \( \mu, T_p \) is the optimal stopping time. The second step is to show that under \( T_p \), \( \mu^* \) is the optimal power allocation scheme. A detailed proof is provided in Appendix A.

**Remark 3.5:** We emphasize that \( E_0 = 0 \) is an important assumption for proving the optimality. Technically, the optimality of \( \mu^* \) relies on the inequality \( b_k \geq a_k \) for every \( k \), which is only true under \( E_0 = 0 \). If \( E_0 \neq 0 \), the optimal power allocation is difficult to find, but \( \mu^* \) is still a good strategy since it is asymptotically optimal as \( \eta \to \infty \). As will be shown in Proposition 3.7, the detection delay \( d(\mu^*, T_p) \) scales linearly with \( \log \eta \); hence, the contribution of a finite initial energy \( E_0 \) is negligible when \( \eta \to \infty \).

In the following, we analyze the performance of \( (\mu^*, T_p) \) by determining the detection delay and the algorithm level ARL. We notice that the strategy \( (\mu^*, T_p) \) is independent of \( E_0 \), hence the following propositions hold for any initial energy level. Since \( \{Z_k\} \) is a conditionally i.i.d. sequence under \( \mu^* \), we can apply Wald’s identity in our analysis. We first have the following proposition:

**Proposition 3.6:** Suppose \( B > 1 \), then for any initial energy \( E_0 \), we have

\[
 E_{\infty} [N] = \frac{E_{\infty} [\kappa]}{1 - P(0; F_0)} \tag{12}
\]

**Proof:** The proof follows closely that of Theorem 6.2 in [10]. A detailed proof is given in Appendix B.

In Proposition 3.6, ARL and \( d(\mu^*, T_p) \) are given as functions of \( P_\infty(F_0) \) and \( P_1(F_0) \), whose precise values are difficult to evaluate. The following result, which is an extension of Lorden’s asymptotical result [16], shows \( d(\mu^*, T_p) \) scales linearly with \( \log \eta \) when \( \eta \to \infty \).

**Proposition 3.7:** As \( \eta \to \infty \), then for any initial energy \( E_0 \), we have

\[
 d(\mu^*, T_p) \sim \frac{1}{\eta^\kappa} \tag{12}
\]

in which \( I = I(f_1, f_0) \) is the KL divergence of \( f_1 \) and \( f_0 \).

**Proof:** This statement can be shown by discussing the relationship between the one-sided sequential probability ratio test...
(SPRT) and the CUSUM test. The discussion is similar to the proof of Lemma 4.2, therefore, we omit the proof for brevity.

IV. ASYMPTOTICALLY OPTIMAL SOLUTION UNDER THE SYSTEM LEVEL ARL CONSTRAINT

In this section, we consider (P2) and (P3) for any value of $E_0$ under the binary energy arriving model. Inspired by the previous section, we propose to use the simple detection strategy $(\mu^*, T_p)$. We will show that this simple strategy is asymptotically optimal for (P2) and (P3) as $\gamma \to \infty$.

The asymptotic optimality of $(\mu^*, T_p)$ in the rare false alarm region ($\gamma \to \infty$) can be shown by two steps. In the first step, we derive a lower bound on the detection delay for any power allocation and detection scheme. In the second step, we show that $(\mu^*, T_p)$ achieves this lower bound, which then implies that $(\mu^*, T_p)$ is asymptotically optimal.

The following lemma presents our lower bound on the detection delay.

Lemma 4.1: For any initial energy $E_0$, as $\gamma \to \infty$,

$$\inf\{d(\mu, T) : E_\infty[T] \geq \gamma\} \geq \inf\left\{\sup_{t \geq 1} \frac{b^\mu_t}{I} \mid T \geq t \right\} : E_\infty[T] \geq \gamma \geq \frac{1}{p} \frac{\log \gamma}{I} (1 + o(1)).$$

Proof: Please see Appendix C.

This lower bound $\log \gamma |(pI)^{-1} + o(1)|$ can be obtained by $(\mu^*, T_p)$ for both (P2) and (P3), which is specified in Lemma 4.2 and Lemma 4.4.

Lemma 4.2: $(\mu^*, T_p)$ is asymptotically optimal for (P2) as $\gamma \to \infty$. Specifically, for any initial energy $E_0$,

$$d(\mu^*, T_p) \sim \frac{1}{p} \frac{\log \gamma}{I}.$$

Proof: As discussed in Remark 3.3, $(\mu^*, T_p)$ is an equalizer rule for (P2), i.e., $d(\mu^*, T_p) = d_1(\mu^*, T_p) = E_{T_p}^\mu[T]$. The statement can be shown by discussing the relationship between the CUSUM and the one-sided SPRT. Denote the SPRT statistic as

$$A_{1:k} = \prod_{i=1}^{k} I_i(Z_i),$$

and define the stopping time

$$T_{s,1} = \inf\{k > 1 \mid A_{1:k} > B\}.$$

Since

$$S_k = \max_{1 \leq q \leq k} \left[\prod_{i=q}^{k} L(Z_i)\right] \geq \prod_{i=1}^{k} L(Z_i) = A_{1:k},$$

we always have

$$E_{T_p}^\mu[T] \leq E_{T_{s,1}}^\mu[T].$$

Let $B = \gamma$, by the performance of the SPRT (Proposition 4.11 in [10]), we have

$$E_{T_p}^\mu[T] \sim \frac{\log \gamma}{pI}.$$ 

Moreover, by (10) in Theorem 2 of [16], the threshold $B = \gamma$ will guarantee

$$E_{T_p}^\mu[T] \geq \gamma.$$ 

Then, by Lemma 4.1 we have

$$d(\mu^*, T_p) \sim \frac{1}{p} \frac{\log \gamma}{I}.$$

Remark 4.3: Although $(\mu^*, T_p)$ is shown to be asymptotically optimal for (P2), we were not able to show the optimality of $(\mu^*, T_p)$. For a general power allocation $\mu \neq \mu^*$, the observation sequence $\{Z_k\}$ is not necessarily conditionally i.i.d. anymore. This is one of the main challenges. In addition, the technique used in the proof of Lemma 3.4 cannot be applied here. Although the non-trivial observation sequence $\{X_k\}$ is relatively easy to handle, it is difficult to evaluate the detection delay from $\{X_k\}$ since the detection delay is also related to the time intervals between two successive non-trivial observations, which are not necessarily i.i.d. under a general power allocation strategy $\mu$.

Lemma 4.4: $(\mu^*, T_p)$ is asymptotically optimal for (P3) as $\gamma \to \infty$. Specifically, for any initial energy $E_0$,

$$\sup_{t \geq 1} \frac{E_{T_p}^\mu[T] - t + T_p \geq t}{t} \sim \frac{1}{p} \frac{\log \gamma}{I}.$$ 

Proof: We consider the one-sided SPRT with threshold $B = \gamma$, which will guarantee $E_{T_p}^\mu[T] \geq \gamma$. Let $T_{\gamma,t}$ denote the stopping time of the SPRT starting at time instant $t$, i.e.,

$$T_{\gamma,t} = \inf\left\{m \geq 1 \prod_{i=t}^{t+m-1} L(Z_i) \geq B\right\},$$

then Page’s stopping time can be written as

$$T_p = \inf\{T_{s,t} + t - 1 \mid t = 1, 2, \ldots\}.$$ 

Note that

$$\{T_p < t\} = \{T_{s,1} < t\} \cup \cdots \cup \{T_{s,t-1} < t\} \in \mathcal{F}_{t-1}.$$

therefore, $\{T_p < t\} \in \mathcal{F}_{t-1}$. Then, for an arbitrary $t$,

$$E_{T_p}^\mu[T_{p} - t + T_p \geq t] \leq E_{T_p}^\mu[T_{s,1} - 1].$$

Here, (a) is due to (17), (b) is due to the fact that $T_{s,t}$ is independent of $\mathcal{F}_{t-1}$, and (c) is true because $\{Z_k\}$’s are conditionally i.i.d. under $\mu^*$, hence $T_{s,t}$ has the same distribution under $P_{T_p}^\mu$.
as $T_{s1}$ does under $P^*_{\mu}$. Since $E^{\mu*}_{T_{s1}} \sim \log \gamma / p I$, combining this with Lemma 4.1, we have

$$\sup_{t \geq 1} E^{\mu*}_{T_{s1}} [T_{s1} - t | T_{s1} \geq t] \sim \frac{1}{p} \frac{\log \gamma}{I}.$$  

As we mentioned in Section II, although we consider Pollak’s formulation only under the system level ARL constraint in detail in this paper, the proposed strategy $(\mu^*, T_{s1})$ is also asymptotically optimal for the formulation under the algorithm level ARL constraint, which is stated in the following proposition:

**Proposition 4.5:** For any initial energy $E_0$, $(\mu^*, T_{s1})$ is asymptotically optimal for Pollak’s formulation under the algorithm level ARL constraint as $\gamma \to \infty$, and we have

$$\sup_{t \geq 1} E^{\mu*}_t [T_{s1} - t | T_{s1} \geq t] \sim \frac{1}{p} \frac{\log \gamma}{I}.$$  

**Proof:** Following the similar argument used in (27), we have

$$E^{\mu*}_t [T_{s1}] = E^{\mu*}_t [\sigma_N] = E^{\mu*}_N \left[ \sum_{i=1}^{N} \tau_i \right] = \frac{1}{p} E^{\mu*}_N \left[ \sum_{i=1}^{N} \tau_i \right].$$

That is, under the immediate power allocation $\mu^*$, the algorithm level ARL constraint $E^{\mu*}_N$ is equivalent to $\gamma$ and can be equivalently converted into a system level ARL constraint $E^{\mu*}_{\infty} [T_{s1}]$. Setting $\gamma = \eta / p$ for a given $\eta$, $\gamma \to \infty$ is equivalent to $\gamma \to \infty$. By Lemma 4.4, $(\mu^*, T_{s1})$ is asymptotically optimal under the system level ARL constraint, hence it is asymptotically optimal under the algorithm level ARL constraint.

**V. EXTENSION**

In this section, we extend the original problem setup by assuming that the energy harvester can receive more than one unit energy at each time slot. Specifically, we assume that the energy arriving sequence $\nu = \{\nu_1, \ldots, \nu_k, \ldots\}$ is i.i.d. over $k$. $\nu_k \in \mathcal{V} = \{0, 1, 2, \ldots\}$, in which $\{\nu_k = 0\}$ means that the energy harvester collects nothing at time $k$ and $\{\nu_k = 1\}$ means that the energy harvester collects $i$ units of energy at time $k$. We use $\nu_k = P^*(\nu_k = i)$ to denote its probability mass function (pmf). Then the energy left in the battery at the end of time slot $k$ is updated by

$$E_k = \min \{C, E_{k-1} + \nu_k - \mu_k\}.$$  

The sensor has an initial energy level $E_0$, and the energy causality constraint indicates that $E_k \geq 0$ for $k = 0, 1, \ldots$.  

Under this setup, we consider (P2) and (P3). We propose to use a generalized immediate power allocation strategy:

$$\hat{\mu}^* = \begin{cases} 1 & \text{if } E_{k-1} + \nu_k \geq 1 \\ 0 & \text{if } E_{k-1} + \nu_k = 0 \end{cases}.$$  

That is, the sensor keeps taking observations as long as the battery is not empty.

In the following, we show that this generalized immediate power allocation $\hat{\mu}^*$ combined with Page’s stopping time $T_{\hat{\mu}}$ is asymptotically optimal for (P2) and (P3) in this random energy arriving case. Corresponding to Lemma 4.1, Lemma 4.2 and Lemma 4.4, we have Lemma 5.1 and Lemma 5.2.

**Lemma 5.1:** For any initial energy $E_0$, as $\gamma \to \infty$, \n
$$\inf \{d(\mu, T) : E^{\mu*}_T[T_{s1} \geq \gamma] \} \geq \inf \left\{ \sup_{t \geq 1} E^{\mu*}_t [T_{s1} - t | T_{s1} \geq t] : E^{\mu*}_T[T_{s1} \geq \gamma] \right\} \geq \frac{1}{p} \frac{\log \gamma}{I} (1 + o(1)).$$  

where $p = E^\mu [\nu_k]$.  

**Proof:** We first show that $E^\mu [\nu_k]$ exists, and $0 < E^\mu [\nu_k] < 1$.  

We show that $E_k$ is a regular and finite Markov chain. At each time $k$, $E_k$ has only $C + 1$ possible states. If $E_k = 0$, the transition probability is given as

$$P^\mu(E_{k+1} = 0 | E_k = 0) = p_0 + p_1,$$

$$P^\mu(E_{k+1} = j - 1 | E_k = 0) = p_j, \text{ for } 1 \leq j \leq C,$$

$$P^\mu(E_{k+1} = C | E_k = 0) = \sum_{j=C+1}^{\infty} p_j.$$  

If $E_k = i$ with $1 \leq i \leq C$, the transition probability is given as

$$P^\mu(E_{k+1} = i - 1 | E_k = i) = p_0,$$

$$P^\mu(E_{k+1} = j - 1 | E_k = i) = p_j, \text{ for } 1 \leq j \leq C - i,$$

$$P^\mu(E_{k+1} = C | E_k = i) = \sum_{j=C+1}^{\infty} p_j.$$  

The above transition probabilities indicate that $E_k$ is a regular Markov chain. We denote the stationary distribution as $\tilde{\nu} = [\tilde{\nu}_0, \tilde{\nu}_1, \ldots, \tilde{\nu}_C]^T$, where $\tilde{\nu}_i$ is the stationary probability for the state $E_k = i$. Since $\tilde{\nu}_C = 0$ only happens when $E_{k-1} = 0$ and $\nu_k = 0$, then we have

$$E^\mu [\tilde{\nu}_C] = P^\mu [\tilde{\nu}_C = 1] = 1 - P^\mu [\tilde{\nu}_C = 0] = 1 - P^\mu [\nu_k = 0 | E_{k-1} = 0] = 1 - p_0 = \tilde{\nu}_0 \quad \text{as } k \to \infty.$$  

Hence, $E^\mu [\tilde{\nu}_C]$ exists, and $0 < E^\mu [\tilde{\nu}_C] < 1$.  

We denote $\tilde{p} = E^\mu [\tilde{\nu}_C]$. The rest of the proof follows the one in Appendix C by replacing $p$ with $\tilde{p}$.  

**Lemma 5.2:** $(\hat{\mu}^*, T_{\hat{\mu}})$ is asymptotically optimal for (P2) and (P3) as $\gamma \to \infty$. Specifically, for any initial energy $E_0$, \n
$$d(\mu_0, T_{\hat{\mu}}) \sim 1 \frac{\log \gamma}{p} I, $$  

and

$$\sup_{t \geq 1} E^{\hat{\mu}^*}_t [T_{\hat{\mu}} - t | T_{\hat{\mu}} \geq t] \sim \frac{1}{p} \frac{\log \gamma}{I}.$$  

**Proof:** Please see Appendix D.  

**Remark 5.3:** The above lemmas indicate that $E_0$ does not affect the asymptotic optimality. Since the detection delay goes to infinity as $\gamma \to \infty$, a finite initial energy $E_0$, which only
contributes a finite number of observations, does not decrease the detection delay significantly. However, the battery capacity \( C \) would affect the detection delay since the parameter \( \bar{p} \) is a function of \( C \) and \( \nu \).

VI. NUMERICAL SIMULATION

In this section, we give some numerical examples to illustrate the analytical results obtained in this paper. In these numerical examples, we assume that the pre-change distribution \( f_0 \) is \( \mathcal{N}(0, \sigma^2) \) and the post-change distribution \( f_1 \) is \( \mathcal{N}(0, P + \sigma^2) \). In this case, the KL divergence is
\[
I(f_1, f_0) = \frac{1}{2} \left[ \log \frac{P}{1 + P/\sigma^2} + \frac{P}{\sigma^2} \right],
\]
and the signal-to-noise ratio is defined as \( SNR = 10 \log P/\sigma^2 \).

In the first example, we illustrate the equalizer property of \( (\mu^*, T^*_p) \) under Lorden’s formulation. As we mentioned, the equalizer property plays a critical role in the performance analysis, since it allows us to study \( d(\mu^*, T^*_p) \) through a relatively simple expression \( E_{\mu^*} T^*_p \). In this example, we compare our optimal strategy with a seemingly reasonable strategy: a save-test power allocation scheme combined with the CUSUM test. The save-test power allocation \( \mu^t \) is described as follows:
\[
\mu^t_k = \begin{cases} 
0 & \text{if } p_k < c_1 \text{ and } S_{k-1} < c_2 \\
1 & \text{otherwise}
\end{cases}
\]
That is, the \( \mu^t \) is a two-threshold strategy: 1) The sensor saves the collected energy for future use if the energy stored in the battery is less than a threshold \( c_1 \) and the CUSUM statistic is less than a threshold \( c_2 \); and 2) the sensor takes an observation when either of these two thresholds is exceeded. This rule says that if the CUSUM statistic is low (suggesting that a change has not happened yet) and the energy stored in the sensor is low, the sensor saves its energy. On the other hand, if either the sensor has enough energy, or the CUSUM statistic is high, the sensor should take an observation. In this simulation, we set \( v_0 = 0 \), \( \sigma^2 = 1 \), \( SNR = 0dB \), \( p = 0.5 \) and \( \gamma = 500 \). The simulation result is shown in Fig. 1. In the figure, the blue line with circles is the performance of \( (\mu^*, T^*_p) \), the green dash line with stars is the performance of \( (\mu^t, T^*_p) \). This simulation confirms our analysis that \( (\mu^*, T^*_p) \) is an equalizer rule, i.e., \( d(\mu^*, T^*_p) = d(\mu^t, T^*_p) \). However, \( (\mu^t, T^*_p) \) is not an equalizer rule. Actually, in the save-test power allocation scheme, \( d(\mu^t, T^*_p) \) is larger than others. This is due to the fact that in the first time slot, both the CUSUM statistic and the energy stored in the sensor are zero, hence the sensor chooses to store its energy. The sensor will not take observations until the stored energy exceeds \( c_2 \). The duration of this energy collection period is independent of the change point. Then, the worst case happens at \( t = 1 \), and the detection delay caused by the energy collection period is larger than that caused by the immediate power allocation. Since Lorden’s performance metric focuses on the worst case, the save-test power allocation is not as good as the immediate power allocation.

In the second example, we illustrate the relationship between the detection delay and the expected number of observations to false alarm with respect to the energy arriving probability \( p \) under setup (P1). In this simulation, we set \( \sigma^2 = 1 \), \( SNR = 0dB \). The simulation result is shown in Fig. 2. In this figure, the blue line with circles is the simulation result for \( p = 0.2 \), the green line with stars and the red line with squares are results for \( p = 0.5 \) and \( p = 0.8 \), respectively. The black dash line is the performance of the classical Lorden’s problem, which serves as a lower bound since in this case the sensor can take observations at every time slot. As we can see, for a given \( \eta \), the detection delay is in inverse proportion to the energy arriving probability \( p \). The larger \( p \) is, the closer the performance is to the lower bound.

In the third scenario, we examine the asymptotic optimality of \( (\mu^*, T^*_p) \) for (P2) and (P3). In this simulation, we set \( p = 0.3 \), \( \sigma^2 = 1 \) and \( SNR = 5dB \). In this case, we have \( I(f_1, f_0) = 0.8681 \). The simulation result is shown in Fig. 3. In this figure, the blue line with circles is the performance of (P2). The red
line with squares is the performance of (P3), and the black dash is calculated by \( \log \gamma / p I \). Along all the scales, the red curve is below the blue one, which indicates that Pollak’s detection delay is smaller than Lorden’s detection delay. We also notice that these three curves are parallel to each other, which confirms that the proposed strategy, \((\mu^*, T_p)\), is asymptotically optimal since the difference between them is negligible as \( \gamma \to \infty \).

In the fourth scenario, we examine the asymptotic optimality of \((\tilde{\mu}^*, T_p)\) for (P2) and (P3) in the extension case that the energy arrives randomly both in amount and in time. In the simulation, we use \( C = 3 \), and we assume that the amount of energy is valued in the set \( \mathcal{V} = \{0, 1, \ldots, 4\} \). In this case, the probability transition matrix is given as

\[
P = \begin{bmatrix}
p_0 + p_1 & p_2 & p_3 & p_4 \\
p_0 & p_1 & p_2 & p_3 + p_4 \\
0 & p_0 & p_1 & \sum_{i=2}^4 p_i \\
0 & 0 & p_0 & \sum_{i=1}^4 p_i 
\end{bmatrix}
\]

(22)

In this simulation, we set \( p_0 = 0.8, p_1 = 0.1, p_2 = 0.05, p_3 = 0.025, p_4 = 0.025 \), then the stationary distribution is \( \mathbf{w} = [0.182, 0.545, 0.200, 0.7273]^T \) and \( \tilde{p} = 1 - p_0 \tilde{w}_0 = 0.9964 \).

In this simulation, \( \sigma^2 = 1 \) and \( SNR = 5dB \). The simulation result is shown in Fig. 4. In this figure the blue line with circles is the performance of (P2). The red line with squares is the performance of (P3), and the black dash is calculated by \( |\log \gamma| / \tilde{p} I \). Similar to the results obtained in the third simulation, along all the scales, Pollak’s detection delay is smaller than Lorden’s detection delay, and these three curves are parallel to each other, which confirms that the proposed strategy, \((\tilde{\mu}^*, T_p)\), is asymptotically optimal as \( \gamma \to \infty \).

In the last scenario, we compare our proposed strategy \((\tilde{\mu}^*, T_p)\) with the seemingly reasonable strategy \((\mu^*, T_p)\) discussed in the first simulation. In this simulation, the energy arriving process is the same as that in the fourth simulation. Moreover, we set \( C = 7, \sigma^2 = 1, E_0 = 0 \). For \( \mu^* \), we set \( r_1 = 5 \) and \( r_2 = 1 \). In the simulation, we consider Lorden’s detection delay, and we adjust the SNR from \( 0dB \) to \( 20dB \) by keeping the system level ARL around 1100. The simulation result is shown in Fig. 5. In this figure, the blue line with circles is the performance of our proposed strategy \((\tilde{\mu}^*, T_p)\), the red line with squares is the performance of \((\mu^*, T_p)\). From the figure, we can see our proposed strategy has a smaller detection delay than \((\mu^*, T_p)\) in all parameter range.

Another similar simulation is also conducted under a fixed \( SNR = 0dB \) with varying system level ARL. By keeping the rest of simulation parameters same as before, the simulation result is listed in Table I. This simulation result also shows that \((\tilde{\mu}^*, T_p)\) outperforms \((\mu^*, T_p)\).

### Table I

<table>
<thead>
<tr>
<th>System level ARL</th>
<th>log ARL</th>
<th>Lorden’s detection delay</th>
</tr>
</thead>
<tbody>
<tr>
<td>(2.28 \times 10^2)</td>
<td>2.275</td>
<td>41.5, 57.8</td>
</tr>
<tr>
<td>(7.13 \times 10^2)</td>
<td>2.704</td>
<td>72.8, 82.9</td>
</tr>
<tr>
<td>(2.56 \times 10^3)</td>
<td>3.375</td>
<td>142.7, 148.1</td>
</tr>
<tr>
<td>(5.01 \times 10^3)</td>
<td>3.709</td>
<td>178.1, 187.1</td>
</tr>
</tbody>
</table>

VII. CONCLUSION

In this paper, we have studied the non-Bayesian quickest change detection problem using a sensor powered by energy harvested from the environment. Since the energy harvester collects energy randomly, the quickest change detection problem is subjected to a casual energy constraint. Three non-Bayesian quickest change detection problem setups, namely Lorden’s problem under the algorithm level ARL, Lorden’s problem under the system level ARL and Pollak’s problem under the system level ARL, have been considered. For the binary energy arriving model, we have shown that the immediate power allocation scheme coupled with the CUSUM procedure is optimal for the first setup when the initial energy level is zero, and is asymptotically optimal for the second and the third setup as ARL goes to infinity for any initial energy level. For the more general energy arriving model, we have shown that the proposed generalized immediate power allocation coupled with the CUSUM procedure is still asymptotically optimal for the second and the third setups. In terms of future work, similar to [23], [24], it will be interesting to include the cost of using energy in the problem formulation. It will also be interesting to consider the Bayesian version of the problem. We will also consider the problem with multiple distributed sensors.
APPENDIX A

PROOF OF LEMMA 3.4

We first show that $T_p$ is optimal for any $\mu$. For any path of any power utility process $\mu$, the quasi change point of the non-trivial observation sequence is defined as

$$n = \inf \{ k : \tilde{X}_k \sim f_1 \} = \inf \{ k : b_k \geq t \}. \quad (23)$$

This implies that $n$ can be viewed as the change point happening in the non-trivial observation sequence $\{ \tilde{X}_k \}$. Moreover, $N$ can be viewed as a stopping time on the non-trivial observation sequence. Therefore, a rule minimizing the detection delay $(T - t)^+$ among $\{ Z_k \}$ is the same as the one minimizing $(N - n)^+$ among $\{ \tilde{X}_k \}$. Specifically, the stopping rule is decided by

$$\min_N \sup_{n \geq 1} \mathbb{E}_N \left[ (N - n + 1)^+ \tilde{X}_1, \ldots, \tilde{X}_{n - 1} \right], \quad \text{s.t. } \mathbb{E}_\infty[N] \geq \eta.$$ 

Since $\{ \tilde{X}_k \}$ is a conditionally i.i.d. (conditioned on $n$) sequence with pre-change distribution $f_0$ and post-change distribution $f_1$ under any path of the power utility process $\mu$, the above problem is the classical Lorden’s quickest change detection problem [16], and the CUSUM test with threshold $B$, which is a constant such that $\mathbb{E}_\infty[N] = \eta$, is optimal. Since the CUSUM test is path-wise optimal, it is optimal for any power utility $\mu$.

To prove the optimality of $\mu^*$ under $T_p$, we examine the following problem:

$$\min_{\mu \in \mathcal{U}} \mathbb{E}_\mu[T_p], \quad \text{s.t. } \mathbb{E}_\infty[N] = \eta. \quad (24)$$

Notice that the objective function is the same as $d_1(\mu, T_p)$. Since

$$\mathbb{E}_\mu[T_p] = \mathbb{E}_\mu[b_N] \overset{(a)}{\geq} \mathbb{E}_\mu[\alpha_N] - \mathbb{E}_1[\alpha]_{\mu^*} [T_p],$$

in which inequality (a) is due to (8), and equality (b) is true because $T_p = \alpha_N$ under $\mu^* = \nu$. Therefore, $\mu^*$ is optimal for the problem (24).

Since

$$\min_{\mu, T} d_1(\mu, T) = d_1(\mu^*, T_p) = d_1(\mu^*, T_p),$$

in which the last equality is due to Proposition 3.1, we have

$$d(\mu^*, T_p) = d_1(\mu^*, T_p).$$

Combining this with the fact that

$$d(\mu, T) \geq d_1(\mu, T),$$

we know that $(\mu^*, T_p)$ is the optimal solution for (P1).

APPENDIX B

PROOF OF PROPOSITION 3.6

We first examine the quantity $\mathbb{E}_\infty[N]$. Since $\{ \tilde{X}_k \}$ is i.i.d. under $P_{\infty}$, $N$ can be viewed as a renewal process, with renewals occurring whenever the sum of LLR is less than or equal to zero, and with a termination when the sum is larger than or equal to the upper threshold, that is,

$$N = \sum_{j=1}^{J} \kappa_j,$$

where $\kappa_1, \kappa_2, \ldots$ are i.i.d. repetitions of $\kappa$, and $J$ is the number of repetitions before the termination. Let $M_j$ denote the indicator of the event that the $j^{th}$ repetition exits at the upper boundary. That is $M_j = 1$ if the $j^{th}$ repetition exits at the upper boundary, and $M_j = 0$ if the $j^{th}$ repetition exits at the lower boundary, then $J = \inf \{ j : M_j = 1 \}$. Hence, under $P_{\infty}$, $J$ is a geometric random variable with

$$P_{\infty}(J = j) = (1 - P_{\infty}(F_0)^j P_{\infty}(F_0))^{j-1}, \quad j = 1, 2, \ldots.$$ 

Then, we have

$$\mathbb{E}_\infty[J] = \frac{1}{1 - P_{\infty}(F_0)}. \quad (25)$$

Since $\mathbb{E}_\infty[J] < \infty$, and $\{ \kappa_j \}$ is i.i.d., we can apply Wald’s identity:

$$\mathbb{E}_\infty[N] = \mathbb{E}_\infty \left[ \sum_{j=1}^{J} \kappa_j \right] = \mathbb{E}_\infty[J] \mathbb{E}_\infty[\kappa]. \quad (26)$$

Substituting (25) into (26), we have (10).

Following the similar argument as above, we get

$$\mathbb{E}_1[N] = \frac{\mathbb{E}_1[\kappa]}{1 - P(1)}.$$

Denote $\tau_i = a_i - a_{i-1}$ as the time interval between two successive observations, the pmf of $\tau_i$ is

$$P(\tau_i = j) = (1 - p)^{j-1} p,$$

and the average of the time interval between two successive observations is

$$\mathbb{E}^\nu[\tau] = \frac{1}{p}.$$ 

For the average detection delay, we have

$$d(\mu^*, T_p) \overset{(a)}{=} \mathbb{E}_\mu[\nu]$$

$$= \mathbb{E}_1[\nu]$$

$$= \mathbb{E}_1[\sum_{k=1}^{N} \tau_k]$$

$$\overset{(b)}{=} \mathbb{E}^\nu[\tau] \mathbb{E}_1[N]$$

$$= \frac{1}{p} \mathbb{E}_1[N]. \quad (27)$$
Here, (a) is due to equalizer property, (b) is the Wald’s identity. Then (11) follows.

## APPENDIX C

### PROOF OF LEMMA 4.1

This proof relies on several supporting propositions and Theorem 1 of [19].

**Proposition C.1:** For an arbitrary but given power allocation \( \mu \), let \( I_1 = pI \), we have

\[
\lim_{m \to \infty} \text{esssup}_{P_{\mu}^T} \left\{ \frac{1}{m} \max_{0 < q \leq m} \sum_{i=t}^{t+q} l(Z_i) \geq (1 + \varepsilon) I_1 \right\} = 0
\]

\[
\forall \varepsilon > 0.
\]

**Proof:** We first show that the inequality

\[
\frac{1}{m} \sum_{i=t}^{t+m-1} l(Z_i) \leq I_1, \text{as } m \to \infty,
\]

holds almost surely under \( P_{\mu}^T \) for any \( t \geq 1 \).

To show this, we first consider the immediate power allocation \( \mu^* \), by the strong law of large numbers, we have

\[
\frac{1}{m} \sum_{i=t}^{t+m-1} \mu_i = \frac{\hat{m}}{m} \overset{a.s.}{\to} p, \text{as } m \to \infty,
\]

where \( \hat{m} \) is the amount of energy collected from \( t \) to \( t+m-1 \). In the immediate power allocation \( \mu^* \), \( \hat{m} \) is equal to the number of nonzero elements in \( \{\mu^*_i, \ldots, \mu^*_{t+m-1}\} \). We also have

\[
\frac{1}{m} \sum_{i=n}^{n+m-1} l(X_i) \overset{a.s.}{\to} pI = I_1,
\]

in which \( n \) is the quasi change point defined in (23). Therefore, under \( \mu^* \), as \( m \to \infty \), we have

\[
\frac{1}{m} \sum_{i=t}^{t+m-1} l(Z_i) = \frac{\hat{m}}{m} \sum_{i=n}^{n+m-1} l(X_i) \overset{a.s.}{\to} pI = I_1. \tag{30}
\]

For an arbitrary power allocation \( \mu \) with \( \lim \sup_{k \to \infty} \mu_k = 1 \), the amount of energy allocated from \( t \) to \( t+m-1 \) is bounded by the amount of energy collected in this period plus the amount of energy stored in the battery at time \( t \). That is, \( \hat{m} \leq \hat{m} + E \leq \hat{m} + C \), where \( \hat{m} \) denotes the number of nonzero elements in \( \{\mu_t, \ldots, \mu_{t+m-1}\} \). Therefore, as \( m \to \infty \),

\[
\frac{1}{m} \sum_{i=t}^{t+m-1} l(Z_i) \leq \frac{\hat{m} + C}{m} \sum_{i=n}^{n+m-1} l(X_i) \overset{a.s.}{\to} pI.
\]

For the power allocation scheme \( \mu \) with \( \lim \sup_{k \to \infty} \mu_k = 0 \), we have

\[
\lim_{m \to \infty} \frac{1}{m} \sum_{i=t}^{t+m-1} l(Z_i) = 0 \leq pI.
\]

Therefore, inequality (29) holds for any arbitrary \( \mu \). Notice that i) (29) holds in the almost sure sense, since (30) converges in the almost sure sense; and ii) (29) holds for any realization of \( Z_1, \ldots, Z_{t-1} \).

For any \( \varepsilon > 0 \), define

\[
T_{\varepsilon}^T = \sup \left\{ m > 1 \left\{ \frac{1}{m} \sum_{i=t}^{t+m-1} l(Z_i) \geq (1 + \varepsilon) I_1 \right\} \right\}.
\]

Due to (29), we have

\[
\text{essinf}_{P_{\mu}^T} \{ T_{\varepsilon}^T < \infty \} = 1,
\]

which indicates that

\[
\lim_{m \to \infty} \text{esssup}_{P_{\mu}^T} \left\{ \frac{1}{m} \max_{0 < q \leq m} \sum_{i=t}^{t+q} l(Z_i) \geq (1 + \varepsilon) I_1 \right\} = 0.
\]

Note that Proposition C.1 holds for every \( t \geq 1 \), therefore

\[
\lim_{m \to \infty} \sup_{t \geq 1} \text{esssup}_{P_{\mu}^T} \left\{ \frac{1}{m} \max_{0 < q \leq m} \sum_{i=t}^{t+q} l(Z_i) \geq (1 + \varepsilon) I_1 \right\} = 0. \tag{31}
\]

To prove Lemma 4.1, we need Theorem 1 in [19], which is restated as follows:

**Theorem C.2:** ([19]) Let \( \{Z_k\} \) be a random variable sequence with a deterministic but unknown change point \( t \). Under probability measure \( P_t \), the conditional distribution of \( Z_k \) is \( f_t(\cdot | Z_i^{k-1}) \) for \( k < t \) and is \( f_t(\cdot | Z_i^{k-1}) \) for \( k \geq t \). Let

\[
l(Z_k) = \log \frac{f_t(Z_k | Z_i^{k-1})}{f_t(Z_i^{k-1})}.
\]

If the condition

\[
\lim_{m \to \infty} \sup_{t \geq 1} \text{esssup}_{P_{\mu}^T} \left\{ \max_{0 < q \leq m} \sum_{i=t}^{t+q} l(Z_i) \geq (1 + \varepsilon) I_1 \right\} = 0,
\]

\[
\forall \varepsilon > 0 \]  \tag{32}

holds for some constant \( I_1 \). Then, as \( \gamma \to \infty \),

\[
\inf \{ d(\mu, T) : E_{\infty} [T] \geq \gamma \}
\]

\[
\geq \inf \{ \sup_{t \geq 1} E_t [T - t] : E_{\infty} [T] \geq \gamma \}
\]

\[
\geq (I_1^{-1} + o(1)) \log \gamma.
\]

**Proof:** Please refer to [19].

In our case, for any arbitrary but given power allocation \( \mu \), the pre-change conditional density of \( Z_k \) is given as

\[
f_t^\mu(Z_k | Z_i^{k-1}) = f_0(X_k) P \{ (\mu_k = 1) | Z_i^{k-1} \} + \delta(\phi) P \{ (\mu_k = 0) | Z_i^{k-1} \},
\]

\[
E_{\infty} [T] \geq \gamma
\]

\[
\inf \{ d(\mu, T) : E_{\infty} [T] \geq \gamma \}
\]

\[
\geq \inf \{ \sup_{t \geq 1} E_t [T - t] : E_{\infty} [T] \geq \gamma \}
\]

\[
\geq (I_1^{-1} + o(1)) \log \gamma.
\]

**Proof:** Please refer to [19].
where \( \delta(\phi) \) is the Dirac delta function. Similarly, the post-change conditional density is
\[
\begin{align*}
    f_{\delta}^p(Z_k | Z_1^{k-1}) &= f_1(X_k)P \left\{ \{ \mu_k = 1 \} | Z_1^{k-1} \right\} \\
    &+ \delta(\phi)P \left\{ \{ \mu_k = 0 \} | Z_1^{k-1} \right\}.
\end{align*}
\]
Therefore, the log likelihood ratio in Theorem C.2 can be written as
\[
    l(Z_k) = \log \frac{f_{\delta}^p(Z_k | Z_1^{k-1})}{f_0^p(Z_k | Z_1^{k-1})} = \left\{ \begin{array}{ll}
        \log f_1(X_k), & \text{if } \mu_k = 1 \\
        0, & \text{if } \mu_k = 0
    \end{array} \right.
\]
which is consistent with the definition in (7). Moreover, (31) indicates that, for any arbitrary power allocation, (32) holds for the constant \( I_1 = pI \). Therefore, the conclusion in Theorem C.2 indicates the result for our case:

APPENDIX D
PROOF OF LEMMA 5.2

We first prove the asymptotic optimality of \((\hat{\mu}^*, T_p)\) for problem (P2). The proof relies on some supporting propositions and Theorem 4 of [19].

**Proposition D.1:** For the power allocation \( \hat{\mu}^* \), we have
\[
    \lim_{m \to \infty} \sup_{k \geq 1} \text{esssup} P_k^{\hat{\mu}^*} \left\{ \frac{1}{m} \sum_{i=k}^{k+m-1} l(Z_i) \leq \hat{\mu}I - \epsilon \mid Z_1, \ldots, Z_{k-1} \right\} \to 0 \\
    \forall \epsilon > 0. \tag{33}
\]

**Proof:** As we have shown in Proposition C.1, for any realization of \( Z_1, \ldots, Z_{k-1} \), and \( \forall k \geq t \), under the power allocation scheme \( \hat{\mu}^* \), we have
\[
    \frac{1}{m} \sum_{i=k}^{k+m-1} l(Z_i) \to \hat{\mu}I, \quad m \to \infty.
\]
Then for all \( k \geq t \),
\[
    \lim_{m \to \infty} \text{esssup} P_k^{\hat{\mu}^*} \left\{ \frac{1}{m} \sum_{i=k}^{k+m-1} l(Z_i) \leq \hat{\mu}I - \epsilon \mid Z_1, \ldots, Z_{k-1} \right\} \to 0 \quad \forall \epsilon > 0.
\]
Therefore
\[
    \lim_{m \to \infty} \text{esssup} P_k^{\hat{\mu}^*} \left\{ \frac{1}{m} \sum_{i=k}^{k+m-1} l(Z_i) \leq \hat{\mu}I - \epsilon \mid Z_1, \ldots, Z_{k-1} \right\} \to 0. \tag{34}
\]

The proposition follows from the fact that (34) holds for all \( k \geq t \)

**Proposition D.2:** Under the power allocation scheme \( \hat{\mu}^* \), Page’s stopping time \( T_p \) satisfies
\[
    \sup_{k \geq 1} P_k^{\hat{\mu}^*} (k \leq T_p < k + m_\alpha) \leq \alpha, \tag{35}
\]
where
\[
    \lim_{m_\alpha} \frac{m_\alpha}{\log \alpha} > (\hat{\mu}I)^{-1},
\]
but
\[
    \log m_\alpha = o(\log \alpha) \Rightarrow \alpha \to 0.
\]

**Proof:** For any \( k \),
\[
    P_k^{\hat{\mu}^*} (k \leq T_p < k + m_\alpha) \leq \sum_{k=k}^{k+m_\alpha-1} P_k^{\hat{\mu}^*} \left\{ \prod_{i=k}^{k} l(Z_i) \geq B, \exists 0 \leq j \leq k - 1 \right\} \\
    \leq \sum_{k=k}^{k+m_\alpha-1} P_k^{\hat{\mu}^*} \left\{ \prod_{i=k}^{k} l(Z_i) \geq B, \exists 0 \leq j \leq k' - 1 \right\} \\
    \leq \sum_{k=k}^{k+m_\alpha-1} P_k^{\hat{\mu}^*} \left\{ \prod_{i=k}^{k} l(Z_i) \geq B, \exists 0 < k'' < k' \right\} \\
    \leq \sum_{k=k}^{k+m_\alpha-1} \exp(-\log B) \\
    = m_\alpha \exp(-\log B). \tag{36}
\]

Here, (a) is true because the likelihood ratio of \( \{Z_i\} \) and that of \( \{\hat{X}_i\} \) are the same. Then we substitute \( \{Z_i\} \) with \( \{\hat{X}_i\} \), and change the probability measure correspondingly. \( i', k' \) and \( j' \) are the new indices in \( \{\hat{X}_i\} \) corresponding to the original \( i, k \) and \( j \) in \( \{Z_i\} \). (b) holds because under \( P_\infty \), \( \{\hat{X}_i\} \) are i.i.d., then we reverse the sequence. (c) is due to Doob’s martingale inequality, since \( \{l(\hat{X}_i)\} \) is a martingale under \( P_\infty \) with expectation 1. By (36), we can simply choose \( m_\alpha = \log \alpha (\hat{\mu}I)^{-1} + \epsilon \), and choose \( B \), the threshold of the CUSUM test, such that \( m_\alpha \exp(-\log B) = \alpha \).

To prove Lemma 5.2, we need Theorem 4 ii) of [19], which is restated as follows:

**Theorem D.3:** ([19]) Let \( \{Z_k\} \) be a random variables sequence with a deterministic but unknown change point \( t \). Under probability measure \( P_t \), the conditional distribution of \( Z_k \) is \( f_1(Z_k | Z_1^{k-1}) \) for \( k < t \) and is \( f_0(Z_k | Z_1^{k-1}) \) for \( k \geq t \). Denote \( l(Z_k) \) as
\[
    l(Z_k) = \frac{f_1(Z_k | Z_1^{k-1})}{f_0(Z_k | Z_1^{k-1})}.
\]
Denote \( c^\ast \) as the threshold used in Page’s stopping time. Then

\[
E_\infty [T_p] \geq c^\ast.
\]

Denote \( \bar{E}_t[T] \) as Lorden’s detection delay, i.e.,

\[
\bar{E}_t[T] = \sup_{t \geq 1} \text{ess sup} \mathbb{E}_t \{ (T - t + 1)^+ Z_0, \ldots, Z_{t-1} \}.
\]

If \( \forall \delta > 0 \), the condition

\[
\lim_{m \to \infty} \sup_{k \geq 1} \text{ess sup} P_t \left\{ \left\{ \frac{1}{m} \sum_{i=k}^{k+m} l(Z_i) \leq l_1 + \delta Z_0, \ldots, Z_{k-1} \right\} \right\} \to 0
\]

holds for some constant \( l_1 \), and as \( \alpha \to 0 \), there exists some \( m_\alpha \) which depends on only \( \alpha \) such that

\[
\sup_{k \geq 1} P_\infty (k \leq T_\leq k + m_\alpha) \leq \alpha,
\]

where

\[
\lim \inf \frac{m_\alpha}{\log \alpha} > l_1
\]

but,

\[
\log m_\alpha - o(\log \alpha) \text{ as } \alpha \to 0.
\]

Then,

\[
\bar{E}_t[T_p] \leq (l_1 + 1 + o(1))c \text{ as } c \to \infty.
\]

**Proof:** Please refer to [19].

By Proposition D.1 and D.2, \((\hat{\mu}^*, T_p)\) is a strategy that satisfies the conditions in Theorem D.3. Hence, if we choose \( c = \log \gamma \) and \( l_1 = pI \) in the theorem, it is easy to verify that \( d(\hat{\mu}^*, T_p) \leq ((pI)^{-1} + o(1)) \log \gamma \) with \( \mathbb{E}_{\hat{\mu}^*} E_{T_p} \geq \gamma \). Therefore, \((\hat{\mu}^*, T_p)\) is asymptotically optimal for (P2).

In the rest of this appendix, we show the asymptotic optimality of \((\hat{\mu}^*, T_p)\) for problem (P3).

**Lemma D.4:**

\[
\sup_{t \geq 1} \mathbb{E}_{\hat{\mu}^*} [T_p - t | T_p \geq t] \sim \frac{1}{pI} \frac{\log \gamma}{I}.
\]

**Proof:** Follow the similar argument in the proof of Lemma 4.4, we have

\[
\mathbb{E}_{\hat{\mu}^*} [T_p - t | T_p \geq t] \leq \mathbb{E}_{\hat{\mu}^*} [T_{s+1, t} - 1 | T_p \geq t] = \mathbb{E}_{\hat{\mu}^*} [T_{s+1, t}].
\]

We claim that

\[
\mathbb{E}_{\hat{\mu}^*} [T_{s+1, t} | E_t = 0] = \mathbb{E}_{\hat{\mu}^*} [T_{s+1, t} | E_t = 0], \text{ for } i = 1, \ldots, \theta,
\]

that is, at the change point \( t \), if there is energy left in the battery, the average detection delay tends to be smaller than that of the case with an empty battery. Hence we have \( \mathbb{E}_{\hat{\mu}^*} [T_{s+1, t} | E_t = 0] \geq \mathbb{E}_{\hat{\mu}^*} [T_{s+1, t} | E_t | E_t = 0] \).

Let \( B = \gamma \), we have

\[
T_{s,t} = \inf \left\{ m \geq 1 \left| \sum_{i=t}^{t+m} l(Z_i) \geq \log \gamma \right. \right\}.
\]

We define a sequence of stopping times \( \{T_{s,1}^{(n)}, \ldots, T_{s,1}^{(n)} \} \) in the following manner:

1) Set \( E_t = 0 \). Define

\[
T_{s,1} = \inf \left\{ m \geq 1 \sum_{i=t}^{t+m} l(Z_i) \geq \log \gamma \right\}.
\]

2) Set \( E_{t \mid n-1} = 0 \). Define

\[
T_{s,n} = \inf \left\{ m \geq 1 \sum_{i=t}^{t+m} l(Z_i) \geq \log \gamma \right\}.
\]

That is, at change point \( t \), we discard all the energy left in the battery and then start a SPRT under the power allocation \( \hat{\mu}^* \). When the previous SPRT stops, we empty the battery again, and start a new SPRT immediately. Then, this sequence of stopping time \( \{T_{s,1}^{(n)}, \ldots, T_{s,n}^{(n)} \} \) are independent with the same distribution of \( T_{s,t} \) under \( E_t = 0 \). Therefore, by the strong LLN, for an \( N \) that large enough, we have

\[
\frac{M}{N} = \frac{T_{s,1}^{(1)} + T_{s,1}^{(2)} + \ldots + T_{s,1}^{(N)}}{N} \xrightarrow{a.s.} \mathbb{E}_{\hat{\mu}^*} [T_{s,t} | E_t = 0].
\]

where \( M = \sum_{i=1}^{N} T_{s,1}^{(i)} \). Since we have

\[
\sum_{i=t}^{t+M} l(Z_i) \geq N \log \gamma,
\]

as \( \gamma \to \infty \), \( M \to \infty \), then

\[
\frac{1}{M} \sum_{i=t}^{t+M} l(Z_i) \geq \frac{N}{M} \log \gamma,
\]

that is

\[
\hat{p} I \geq \frac{N}{M} \log \gamma \text{ or } \frac{M}{N} \geq \frac{\log \gamma}{\hat{p} I}.
\]

If we ignore the overshoot, we will have

\[
\mathbb{E}_{\hat{\mu}^*} [T_{s,t} | E_t = 0] \sim \frac{\log \gamma}{\hat{p} I}.
\]

Then, we have

\[
\mathbb{E}_{\hat{\mu}^*} [T_p - t | T_p \geq t] \leq \frac{\log \gamma}{\hat{p} I} (1 + o(1)).
\]

**REFERENCES**

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