On the Capacity Bounds for Poisson Interference Channels

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Abstract—The Poisson interference channel, which models optical communication systems with multiple transceivers in the short-noise-limited regime, is investigated. Conditions for the strong interference regime are characterized and the corresponding capacity region is derived, which is the same as that of the compound Poisson multiple-access channel with each receiver decoding both messages. For the cases when the strong interference conditions are not satisfied, inner and outer bounds on the capacity region are derived. The inner bound is derived via approximating the Poisson interference channel by a binary interference channel and then evaluating the corresponding Han–Kobayashi region. The outer bounds are obtained via various techniques, including noise reduction, genie-aided scheme, and channel transformation. The Poisson Z-interference channel is then studied. The sum rate capacity is obtained when the cross link coefficient is either sufficiently small or sufficiently large.

Index Terms—Capacity region, interference channel, Poisson channel, sum rate capacity, Z-interference channel.

I. INTRODUCTION

IN OPTICAL communications, intensity modulation and direct detection (IM/DD) are widely used. In the thermal-noise-limited regime, one can use Gaussian distribution to model the noise. However, in the short-noise-limited regime, the Poisson channel is a suitable model [3]. The point-to-point Poisson channel has been intensively studied both in continuous time [4]–[7] and discrete time [8]–[10]. However, the study of multiuser Poisson channels is limited, with few exceptions on the multiple-access channel (MAC) [11], on the broadcast channel [12] and on the Poisson channel with side information at the transmitter [13]. The optical interference channel was discussed in [14], in which several open questions were posted. In this paper, we study the Poisson interference channel that models simultaneous transmission of multiple optical communication links.

The Poisson interference channel differs from the standard Gaussian interference channel well-studied in [15]–[18], in the following aspects. Unlike the Gaussian channel, the Poisson channel is not scale-invariant. Hence, one cannot arbitrarily multiply the input signal with a constant and later on divide the output with the same constant while preserving the mutual information between the input and the output. Furthermore, in the Gaussian interference channel, once the receiver decodes a signal, it can completely cancel the effects of this signal. This is not the case for the Poisson channel anymore. As a result, certain techniques used to derive various capacity bounds for the Gaussian interference channel are not applicable for the Poisson interference channel. On the other hand, the Poisson channel still has the properties of infinite divisibility and independent increments that the Gaussian channel has [19]. This similarity allows us to apply certain ideas designed for the Gaussian interference channel to the Poisson interference channel with appropriate modifications.

In this paper, we first identify the conditions for the strong interference regime and characterize the corresponding capacity region, which is the same as that of the compound Poisson MAC with each receiver decoding both messages. The basic idea involves carefully constructing a new Poisson process based on the decoded signal to enhance the receivers and then performing a proper thinning operation, in which we randomly and independently erase points from the Poisson process.

We then study the case in which the strong interference conditions are not satisfied. We derive an inner bound on the capacity region by first approximating the Poisson interference channel with a binary interference channel via the approach introduced by Wyner [4], and then evaluating the Han–Kobayashi [20] region for the binary interference channel under certain input distributions. Furthermore, we derive several outer bounds on the capacity region. These bounds are derived based on reducing noise level at the receiver, providing various side-information to receivers, and channel transformation.

We further study the Poisson Z-interference channel, in which one of the cross link coefficients is zero. For this channel, we determine the sum rate capacity when the remaining cross link coefficient is either sufficiently small or sufficiently large.

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sufficient large. We show that when the remaining cross link coefficient is sufficiently small, the sum capacity is achieved by treating the interference (if any) as noise. In this case, to maximize the sum rate capacity, the transmitter whose corresponding receiver does not experience the interference does not necessarily transmit at the maximum possible rate. This is in contrast to the corresponding case of the Gaussian Z-interference channel, in which the user not experiencing the interference always transmits at the full rate in order to achieve the sum rate capacity. When the cross link coefficient is sufficiently large, the sum capacity is achieved by letting the receiver experiencing interference decode both messages.

The remainder of the paper is organized as follows. In Section II, we introduce the Poisson interference channel model. In Section III, we derive conditions for the strong interference channel. In Section IV, we show that the capacity region of the Poisson interference channel remains unchanged if we approximate it with a binary input binary output discrete memoryless channel. We derive inner bound and outer bounds for the capacity region for the general Poisson interference channel in Section V. In Section VI, we study the sum capacity for the Poisson Z-interference channel. We present numerical examples in Section VII. Finally, we conclude the paper with a few remarks in Section VIII.

II. SYSTEM MODEL

A. Poisson Interference Channel

We study the two-user Poisson interference channel, in which inputs $X_1(t)$ and $X_2(t)$ from the transmitters satisfy the following conditions:

$$0 \leq X_i(t) \leq A_i, \quad \text{for all } t \text{ and for } i = 1, 2,$$

which means that the input of the Poisson channel is non-negative and less than a threshold due to safety considerations. The output $Y_1(t)$ for $0 \leq t \leq T$ at receiver 1 is a doubly stochastic Poisson process with the instantaneous rate $aX_1(t) + bX_2(t) + \lambda_1$, and the output $Y_2(t)$ for $0 \leq t \leq T$ at receiver 2 is another doubly stochastic Poisson process with the instantaneous rate $cX_1(t) + dX_2(t) + \lambda_2$. More precisely, for all $0 \leq t < s \leq T$,

$$\Pr \{Y_1(s) - Y_1(t) = k|X_1(0, T), X_2(0, T)\} = \frac{1}{k!} \Lambda_1^k e^{-\Lambda_1}, \quad k = 0, 1, \ldots,$$

$$\Pr \{Y_2(s) - Y_2(t) = k|X_1(0, T), X_2(0, T)\} = \frac{1}{k!} \Lambda_2^k e^{-\Lambda_2}, \quad k = 0, 1, \ldots,$$

in which

$$\Lambda_1 = \int_t^s (aX_1(r) + bX_2(r) + \lambda_1)dr,$$

$$\Lambda_2 = \int_t^s (cX_1(r) + dX_2(r) + \lambda_2)dr,$$

and $X_i(0, T)$ denotes $\{X_i(t) : 0 \leq t \leq T\}$. We use $Y_1 = \mathcal{P}(aX_1(0, T) + bX_2(0, T) + \lambda_1)$ and $Y_2 = \mathcal{P}(cX_1(0, T) + dX_2(0, T) + \lambda_2)$ to denote the output processes at receivers 1 and 2, respectively. Figure 1 illustrates the Poisson interference channel under consideration. It is easy to see that the received signals at receivers are non-decreasing step functions.

We use $M_i \in \mathcal{M}_i$ to denote the message of transmitter $i$. A code $(T, \mathcal{M}_1, \mathcal{M}_2, \mu_1, \mu_2)$ is defined to include

1) Two message sets $\mathcal{M}_i = \{1, \ldots, \lfloor |\mathcal{M}_i| \rfloor \}$ for $i = 1, 2$;

2) Two sets of waveforms $X_{i,m_i}(\cdot)$, for $m_i \in \{1, \ldots, \lfloor |\mathcal{M}_i| \rfloor \}$ and $i = 1, 2$, each of which satisfies the power constraint (1);

3) Two encoders, $f_i$ at encoder $i$ for $i = 1, 2$, each of which maps each message $m_i$ to a waveform $X_{i,m_i}(\cdot)$;

4) Two decoders, $D_i$ at receiver $i$ for $i = 1, 2$: $Y_i(0, T) \rightarrow \hat{M}_i \in \{1, \ldots, \lfloor |\mathcal{M}_i| \rfloor \}$, each of which maps a received signal $Y_i(0, T)$ to a message $\hat{m}_i$;

5) Two average error probabilities: $\mu_i = \Pr(\hat{M}_i \neq M_i)$ for $i = 1, 2$.

A rate pair $(R_1, R_2)$ is said to be achievable if for any $\epsilon > 0$, there exists a sequence of codes $(T, \mathcal{M}_1, \mathcal{M}_2, \mu_1, \mu_2)$ with $|\mathcal{M}_i| \geq 2^{T R_i}$ and $\mu_i \leq \epsilon$ for $i = 1, 2$. The capacity region $C$ is the closure of the set of all achievable rate pairs. We are also interested in the sum rate capacity defined as

$$C_{sum} = \max_{(R_1, R_2) \in C} \{R_1 + R_2\}.$$

In order to simplify the notation, we define $\tilde{p} = 1 - p$ for $0 \leq p \leq 1$ and define the following functions that are used throughout the paper:

$$\psi(\alpha, \beta; \gamma, \lambda) := (\alpha + \beta + \lambda) \ln(\alpha + \beta + \lambda) - (\gamma + \beta + \lambda) \ln(\gamma + \beta + \lambda),$$

$$\phi(p, A, \lambda) := \tilde{p} \lambda A (A + \lambda) \ln(A + \lambda) - (Ap + \lambda) \ln(Ap + \lambda).$$

In addition, we use

$$\hat{Y} = Y \downarrow p,$$

to denote the thinning operation with the parameter $p$, in which we obtain a process $\hat{Y}$ by randomly and independently erasing points from a Poisson process $Y$ with the probability $\tilde{p}$ (thus keeping each point of $Y$ with the probability $p$).

In the definition above, if we require both decoders to decode both messages, we then have the Poisson compound MAC channel, whose capacity region is clearly outer-bounded by that of the Poisson interference channel.

![Fig. 1. The Poisson interference channel.](Image)
B. Poisson Z-Interference Channel

In this paper, we will also consider the Poisson Z-interference channel shown in Figure 2. In the Poisson Z-interference channel, one of the receivers (say receiver 1) does not experience any interference. One can get the Poisson Z-interference channel model by setting $b = 0$ in the Poisson interference channel discussed in Section II-A. For the Poisson Z-interference channel, we will focus on the sum rate capacity.

C. Differences Between Poisson and Gaussian Interference Channels

In the following, we summarize the differences between the Poisson and Gaussian interference channels, which helps to explain why the analysis of the Poisson interference channel requires development of new techniques.

There are two main differences between the Poisson interference channel and the Gaussian interference channel. Firstly, the Poisson channel is not scale-invariant. More specifically, for a constant $a \neq 1$, $aP(X_1(0, T) + \lambda_1/a)$ is different from $P(aX_1(0, T) + \lambda_1)$, and as a result, $I(X_1(0, T); aP(X_1(0, T) + \lambda_1/a))$ is different from $I(X_1(0, T); P(aX_1(0, T) + \lambda_1))$. Secondly, in the Poisson interference channel, even if $X_1(0, T)$ is known, it is not possible to construct $P(bX_2(0, T) + \lambda_1)$ from $P(aX_1(0, T) + bX_2(0, T) + \lambda_1)$, i.e., it is not possible to completely cancel the effects of $X_1(0, T)$ from the output even if its value is known. Hence, complete interference cancellation is not possible in the Poisson interference channel. On the other hand, the Poisson channel also has the infinite divisibility and independent increments properties that the Gaussian channel has. This similarity allows us to apply certain ideas from the Gaussian interference channels to the Poisson interference channel.

III. Capacity Region for the Strong Interference Channel

In this section, we characterize the capacity region of the Poisson interference channel in the so-called strong interference regime. We obtain the capacity region result by showing that if the channel parameters satisfy certain conditions, the capacity region of the Poisson interference channel is the same as that of a compound Poisson MAC. The central argument of our proof is to show that, if certain conditions are satisfied, any rate pair $(R_1, R_2)$ achievable for the Poisson interference channel is also achievable for the Poisson compound MAC.

Since it is clear that rate pairs achievable for the compound MAC is achievable for the interference channel, the equivalence of the two capacity regions is established. The argument for the equivalence of these two channels is slightly different from that of the Gaussian interference channel [16], due to the differences between the Gaussian channel and the Poisson channel discussed in Section II. As detailed in the proof, it involves enhancement and thinning arguments.

**Theorem 3.1:** If

$$\frac{b}{d} \geq \frac{\lambda_1}{\lambda_2} \geq \frac{a}{c}, \quad (4)$$

and

$$\frac{b}{d} \geq 1 \geq \frac{a}{c}, \quad (5)$$

the capacity region of the Poisson interference channel is the convex hull of $\bigcup_{0 \leq p, q \leq 1} R_{p,q}$ with $R_{p,q}$ being:

$$R_1 \leq \min \{ \mathbb{E}[\psi(aA_1^bX_1^b, bA_2X_2^b; aA_1p, \lambda_1), \mathbb{E}[\psi(cA_1X_1^b, dA_2X_2^b; cA_1p, \lambda_2)] \},$$

$$R_2 \leq \min \{ \mathbb{E}[\psi(bA_2X_2^b, aA_1X_1^b; bA_2q, \lambda_1), \mathbb{E}[\psi(dA_2X_2^b, cA_1X_1^b; dA_2q, \lambda_2)] \},$$

$$R_1 + R_2 \leq \min \{ \mathbb{E}[\psi(aA_1X_1^b + bA_2X_2^b, 0; aA_1p + bA_2q, \lambda_1), \mathbb{E}[\psi(cA_1X_1^b + dA_2X_2^b, 0; cA_1p + dA_2q, \lambda_2)] \},$$

(6)

in which the expectation is computed for binary input distributions with $P[X_1^b = 1] = 1 - P[X_1^b = 0] = p$ and $P[X_2^b = 1] = 1 - P[X_2^b = 0] = q$.

**Proof:** As discussed above, it suffices to show that, if the conditions in the theorem are satisfied, any rate pair achievable in the Poisson interference channel is also achievable in the Poisson compound MAC. We show this by certain enhancement and thinning arguments. Figure 3(a) shows the operations involved at receiver 1. Suppose that a rate pair $(R_1, R_2)$ is achievable for the Poisson interference channel. Hence, there exists a code $(T, M_1, M_2, \mu_1, \mu_2, \mu),$ such that

$$\frac{1}{T} \log |M_1| \geq R_1 - \epsilon \text{ for any } \epsilon > 0 \text{ and sufficiently large } T.$$  

Furthermore, decoder 1 can decode $X_1(0, T)$ with an average error probability less than $\mu_1.$ Once $X_1(0, T)$ is successfully decoded, decoder 1 generates a doubly stochastic Poisson process $P((bc/d-a)X_1(0, T) + b\lambda_2/d - \lambda_1),$ which is independent of all other processes considered in the problem given $X_1(0, T),$ with the instantaneous rate $(bc/d-a)X_1(t) + b\lambda_2/d - \lambda_1.$ We can do this as long as $bc/d-a \geq 0$ and $b\lambda_2/d - \lambda_1 \geq 0.$ Decoder 1 then adds this process to
Y_1 and obtain
\[
\hat{Y}_1 = Y_1 + \mathcal{P}(bc/d - a)X_1(0, T) + b\lambda_2/d - \lambda_1).
\]
Due to the properties of the Poisson process [21], \(\hat{Y}_1\) is a doubly stochastic Poisson process with the instantaneous rate \(bcX_1(t)/d + bX_2(t) + b\lambda_2/d\). Finally, decoder 1 constructs a process \(\hat{Y}_2 = \hat{Y}_1 \downarrow (d/b)\). This can be done if \(d/b \leq 1\).

Again due to the properties of the Poisson process [21], \(\hat{Y}_2\) is a doubly stochastic Poisson process with the instantaneous rate \(d/b(bcX_1(t)/d + bX_2(t) + b\lambda_2/d) = cX_1(t) + dX_2(t) + \lambda_2\).

Hence, the joint statistic of \((\hat{Y}_2(t), X_1(t), X_2(t))\) is the same as that of \((Y_2(t), X_1(t), X_2(t))\). Since decoder 2 is able to decode \(X_2(0, T)\) based on \(\hat{Y}_2\) with an error probability less than \(\mu_2\), decoder 1 is able to decode \(X_2(0, T)\) based on \(\hat{Y}_2\) with an error probability less than \(\mu_2\) as well, as long as it decodes \(X_1\) successfully. Hence, receiver 1 is able to decode \(X_1\) based on \(\hat{Y}_1\) with an error probability less than \(\mu_1 + \mu_2\).

To carry out the operations above, we require that
\[
\frac{bc}{a} \geq a, \quad \frac{b\lambda_2}{a} - \lambda_1 \geq 0, \quad \frac{d}{b} \leq 1. \tag{7}
\]
Similarly, Figure 3 (b) illustrates the operations involved at receiver 2. If the following conditions are satisfied, decoder 2 is able to decode both \(X_1\) with an error probability less than \(\mu_1 + \mu_2\) and \(X_2\) with an error probability less than \(\mu_2\) based on \(\hat{Y}_2\):
\[
\frac{bc}{a} \geq d, \quad \frac{c\lambda_1}{a} - \lambda_2 \geq 0, \quad \frac{a}{c} \leq 1. \tag{8}
\]
It is clear that conditions (7) and (8) are equivalent to (4) and (5). Hence, if these conditions are satisfied, the capacity region of the Poisson interference channel is the same as that of the compound MAC. The Poisson MAC has been studied in [11]. Using [11, eq. (2.6)], we know the capacity of the compound MAC is contained in the union of all rate pairs that satisfy the following conditions:
\[
R_1 \leq \min \frac{1}{T} \left\{ \int_0^T \mathbb{E}\psi \left( aA_1X_1(t), bA_2X_2(t); aA_1p, \lambda_1 \right) dt, \right.
\]
\[
\left. \int_0^T \mathbb{E}\psi \left( cA_1X_1^b(t), dA_2X_2^b(t); cA_1p, \lambda_2 \right) dt \right\},
\]
\[
R_2 \leq \min \frac{1}{T} \left\{ \int_0^T \mathbb{E}\psi \left( bA_2X_2(t), aA_1X_1(t); bA_2q, \lambda_1 \right) dt, \right.
\]
\[
\left. \int_0^T \mathbb{E}\psi \left( cA_1X_1(t), dA_2X_2(t); cA_1p, \lambda_2 \right) dt \right\},
\]
\[
R_1 + R_2 \leq \min \frac{1}{T} \left\{ \int_0^T \mathbb{E}\psi \left( aA_1X_1(t) + bA_2X_2(t), 0; aA_1p + bA_2q, \lambda_1 \right) dt, \right.
\]
\[
\left. \int_0^T \mathbb{E}\psi \left( cA_1X_1(t) + dA_2X_2(t), 0; cA_1p + dA_2q, \lambda_2 \right) dt \right\}.
\]

Furthermore, following [11], we know that binary “quickly varying” inputs can achieve the equalities above. We use \(X_i^b\) to denote the input to the binary MAC obtained by this approximation. Using [11, eqs. (3.4)-(3.9)], we conclude that the capacity region of a compound MAC is given by (6).

IV. DISCRETE-TIME BINARY CHANNEL APPROXIMATION

In this section, we show that a Poisson interference channel can be approximated by a discrete-time binary input binary output (BIBO) memoryless interference channel that has the same capacity region (although other quantities, such as the error exponent, might be different). This result will be used for deriving capacity bounds in the following sections.

Following [4], we approximate Poisson interference channels by BIBO discrete memoryless channels as follows. We fix some small constant \(\Delta > 0\), and divide the transmission time into intervals with each having duration \(\Delta\). We further require that
\begin{itemize}
  \item The transmitted signal \(X_i(t)\) takes only values 0 and \(A_i\), and is constant over the interval \((n\Delta, (n+1)\Delta]\), for every \(n = 1, 2, \ldots\);
  \item Receiver \(i\) observes only samples \(Y_i(n\Delta), n = 1, 2, \ldots\); or alternatively, the increments \(\hat{Y}_{i,n} = Y_i(n\Delta) - Y_i((n-1)\Delta)\). Furthermore, receiver \(i\) interprets \(\hat{Y}_{i,n} \geq 2\) (a rare event when \(\Delta\) is small) as being the same as \(\hat{Y}_{i,n} = 0\). Thus receiver \(i\) obtains an output
\end{itemize}

\[
Y_{i,n} = \begin{cases} 
1 & \hat{Y}_{i,n} = 1 \\
0 & \hat{Y}_{i,n} \neq 1.
\end{cases}
\]
V(1|1, 1) = 1 − V(0|1, 1) = 1 − exp(-(aA + bA + λ2)Δ) = (aA + bA + λ2)Δ + O(Δ2),
V(1|0, 0) = 1 − V(0|0, 0) = 1 − exp(-(cA + dA + λ2)Δ) = (cA + dA + λ2)Δ + O(Δ2).

V. CAPACITY BOUNDS

In this section, we derive inner and outer bounds on the capacity region for general Poisson interference channels that do not satisfy the strong interference conditions (4) and (5).

A. Inner Bound for General Poisson Interference Channels

After the binary approximation, we can obtain an achievable region for the Poisson interference channel by evaluating the corresponding binary discrete-time interference channel based on the Han-Kobayashi region [20] for certain input distributions.

We evaluate the simplified Han-Kobayashi region [22] of the binary interference channel using |Q| = 1 (i.e., no time-sharing), W1 and W2 as binary random variables with P(W1) = 1 − P(W2) = p1 and P(W2) = 1 − P(W2) = p2. We further assume that Xb1 is generated from W1 via a testing binary symmetric channel with parameter q1, and Xb2 is generated from W2 via a testing binary symmetric channel with parameter q2. Using this input distribution, we can compute various mutual information terms involved in the Han-Kobayashi region. To obtain the achievable rate region of the original Poisson interference channel from the achievable region from this binary interference channel, we then need a proper normalization 1/Δ. The computation of various mutual information term using this input distribution can be found in Appendix C.

B. Outer Bounds for General Poisson Interference Channels

In the section, we derive several outer bounds on the capacity region for the Poisson interference channel when one or both of the strong interference conditions (4) and (5) are not satisfied. The following table lists the main propositions, the corresponding scenarios and the main techniques used for each case.

<table>
<thead>
<tr>
<th>Proposition</th>
<th>Condition</th>
<th>Technique</th>
</tr>
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<tbody>
<tr>
<td>Proposition 5.1</td>
<td>(4) does not hold, (5) holds</td>
<td>noise reduction</td>
</tr>
<tr>
<td>Proposition 5.2</td>
<td>(4) holds, (5) does not hold</td>
<td>channel transformation</td>
</tr>
<tr>
<td>Proposition 5.4</td>
<td>neither (4) nor (5) holds</td>
<td>genie aided</td>
</tr>
<tr>
<td>Proposition 5.5</td>
<td>both (4) and (5) hold partially</td>
<td>genie aided</td>
</tr>
</tbody>
</table>

1) Outer Bound When (5) Holds But (4) Does Not Hold:
The first outer bound is derived to be applicable when (5) is satisfied, but (4) is not satisfied.

In this case, we have either λ1/λ2 > b/d or a/c > λ1/λ2. We consider the case λ1/λ2 > b/d, the other case is similar. We derive an outer bound based on reducing the intensity of the noise. Consider an enhanced Poisson interference channel with output process at receiver 1 being Y1 = P(aX10(T) + bX20(T) + b/λ2) and the output process at receiver 2 being the same as Y2. Since λ1/λ2 > b/d, we have b/λ2 < λ1. It is clear that any rate pair (R1, R2) achievable for the original Poisson interference channel is also achievable for the channel specified by (X1, X2) → (Y1, Y2), because one can generate (Y1, Y2) whose joint statistics with (X1, X2) is the same as that of (Y1, Y2, X1, X2) by adding Y1 with a randomly generated process P(λ1 − b/λ2). Now, the channel (X1, X2) → (Y1, Y2) is a strong interference channel, since the condition in (4) is satisfied with the reduced noise. As a result, we have the following proposition.

Proposition 5.1: When (5) is satisfied and λ1/λ2 > b/d, the capacity region of the Poisson interference channel is inside of the convex hull of \( \bigcup \{ R_{p,q} \} \) with \( R_{p,q} \) being:

\[
R_1 \leq \min \left\{ \mathbb{E}[aA] \rho X_1^b, \mathbb{E}[aA] \rho X_2^b, 0, aA p + bA q, bA2q, bA2q ; \right\},
\]

with the expectation being computed for binary input distributions with \( P[X_1 = 1] = 1 - P[X_1 = 0] = p \) and \( P[X_2 = 1] = 1 - P[X_2 = 0] = q \).

2) Outer Bound When (4) Holds But (5) Does Not Hold:
The second outer bound is applicable when (4) is satisfied but (5) is not satisfied.

In this case, we have either 1 > b/d ≥ a/c or b/d ≥ a/c > 1. We consider the case 1 > b/d ≥ a/c, and the other case is similar. We derive an outer bound using arguments similar to that of Theorem 3.1. If the condition (4) and 1 > b/d ≥ a/c are satisfied, we have that all operations in Figure 3 (b) are still valid. And hence, under these conditions, receiver 2 is able to decode both X1 and X2. As a result, the capacity region of the Poisson interference channel remains the same if we require receiver 2 decode both messages. Hence, we have the following result.

Proposition 5.2: When (4) is satisfied and 1 > b/d, the capacity region of the original Poisson interference channel is inside of the convex hull of \( \bigcup \{ R_{p,q} \} \) with \( R_{p,q} \) being:

\[
R_1 \leq \min \left\{ \mathbb{E}[aA] \rho X_1^b, 0, aA p, \lambda_1 \right\},
\]

\[
\mathbb{E}[cA] \rho X_1^b, \mathbb{E}[cA] \rho X_2^b, 0, cA p + dA q, \lambda_2 \right\},
\]
to receiver 1. Suppose a genie generates a doubly stochastic and hence the joint statistics of Fig. 4. The channel transformation at receiver 1.

We first consider the situation when \( X \) and provides this information to receiver 1 (See Figure 4 for an illustration.). Now, based on \( Y \) and this additional information, receiver 1 can generate a Poisson process

\[
\tilde{Y}_2 = (Y_1 + Y'_1) \downarrow \frac{c}{a}.
\]

It is easy to verify that the process \( \tilde{Y}_2 \) is a doubly stochastic Poisson process with instantaneous rate \( cX_1(t) + dX_2(t) + \lambda_2 \) and hence the joint statistics of \( (\tilde{Y}_2, X_1, X_2) \) is the same as that of \( (Y_2, X_1, X_2) \). Since receiver 2 can decode \( X_2 \) from \( Y_2 \) with an error probability less than \( \mu_2 \), receiver 1 is also able to decode \( X_2 \) from \( \tilde{Y}_2 \). As a result, the capacity region of the original Poisson interference channel should be inside of the capacity region of the new MAC with receiver 1 having two observation processes \( (Y_1, Y'_1) \).

We first have the following lemma.

**Lemma 5.3:** The capacity region of the MAC \( (X_1, X_2) \to (Y_1, Y'_1) \) is the convex hull of \( \bigcup_{0 \leq p,q \leq 1} R_{p,q} \) with \( R_{p,q} \) being:

\[
R_1 \leq \mathbb{E} \psi(aA_1 X^b_1, bA_2 X^b_2; aA_1 p, \lambda_1),
\]

\[
R_2 \leq \mathbb{E} \psi(bA_2 X^b_2, aA_1 X^b_1; bA_2 q, \lambda_1),
\]

\[
+ \mathbb{E} \psi((ad/c-b) A_2 X^b_2, 0; (ad/c-b) A_2 q, \lambda_2-c\lambda_1/a),
\]

\[
R_1 + R_2 \leq \mathbb{E} \psi(aA_1 X^b_1 + bA_2 X^b_2, 0; aA_1 p + bA_2 q, \lambda_1)
\]

\[
+ \mathbb{E} \psi((ad/c-b) A_2 X^b_2, 0; (ad/c-b) A_2 q, \lambda_2-c\lambda_1/a),
\]

with the expectation being computed for binary input distributions with \( P[X^b_1 = 1] = 1 - P[X^b_1 = 0] = p \) and \( P[X^b_2 = 1] = 1 - P[X^b_2 = 0] = q \).

**Proof:** The proof follows closely from that of [11] and [23], and details are provided in Appendix B.

Based on the above lemma, we have the following proposition.

**Proposition 5.4:** When \( a/c > 1 \) and \( a/c \geq \lambda_1/\lambda_2 \), the capacity region of the Poisson interference channel is inside of the convex hull of \( \bigcup_{0 \leq p,q \leq 1} R_{p,q} \) with \( R_{p,q} \) being:

\[
R_1 \leq \mathbb{E} \psi(aA_1 X^b_1, bA_2 X^b_2; aA_1 p, \lambda_1),
\]

\[
R_2 \leq \min \{\mathbb{E} \psi(dA_2 X^b_2, 0; dA_2 q, \lambda_2),
\]

\[
\mathbb{E} \psi(bA_2 X^b_2, aA_1 X^b_1; bA_2 q, \lambda_1)
\]

\[
+ \mathbb{E} \psi((ad/c-b) A_2 X^b_2, 0; (ad/c-b) A_2 q, \lambda_2-c\lambda_1/a)\},
\]

with the expectation being computed for binary input distributions with \( P[X^b_1 = 1] = 1 - P[X^b_1 = 0] = p \) and \( P[X^b_2 = 1] = 1 - P[X^b_2 = 0] = q \).

**Proof:** As \( a/c > 1 \) and \( a/c \geq \lambda_1/\lambda_2 \), all bounds in Lemma 5.3 hold. In addition,

\[
R_2 \leq \mathbb{E} \psi(dA_2 X^b_2, 0; dA_2 q, \lambda_2),
\]

as \( R_2 \) has to be smaller than the capacity of the interference free point-to-point channel from transmitter 2 to receiver 2, i.e., the channel \( X_2 \to \mathcal{P}(dX_2 + \lambda_2) \).

For the case when \( 1 \geq a/c > b/d \), i.e., \( d/b > c/a \geq 1 \), we reverse the roles of receiver 1 and receiver 2 in the argument above. Namely, a genie provides receiver 2 with extra information for it to decode \( X_1 \). Then the capacity region of the original Poisson interference channel is contained in the capacity region of the enhanced MAC.

**4) Outer Bound When (5) and (4) Hold Partially:** The genie aided outer bound developed in Section V-B3 can be extended to be applicable for when (5) and (4) hold partially. Suppose a genie provides a doubly stochastic Poisson process \( \hat{Y}_1 = \mathcal{P}(\delta_1 X_2(0, T) + \delta_2) \) to receiver 1. Here, given \( X_2(0, T) \), \( \hat{Y}_1 \) is independent of all other processes of interest to this problem. Obviously, this does not decrease the capacity region of the Poisson interference channel. Since receiver 1 can decode \( X_1(0, T) \) with an error probability less than \( \mu_1 \) from the process \( Y_1 \), receiver 1 can also generate a process \( \hat{Y}_1 = \mathcal{P}(\delta_1 X_1(0, T)) \) that is independent of all other processes given \( X_1(0, T) \). Now, we construct a process \( \hat{Y}_2 \) by setting

\[
\hat{Y}_2 = (Y_1 + \hat{Y}_1) \downarrow \frac{1}{\delta}.
\]

Then \( \hat{Y}_2 \) is a doubly stochastic Poisson process with instantaneous rate

\[
\frac{a + \delta_3}{\delta} X_1(t) + \frac{b + \delta_1}{\delta} X_2(t) + \frac{\lambda_1 + \delta_2}{\delta}.
\]
Now if we set
\[
\delta_3 = c \delta - a, \\
\delta_1 = d \delta - b, \\
\delta_2 = \lambda_2 \delta - \lambda_1,
\]
then the process \( \hat{Y}_2 \) has the same joint statistics with \((X_1, X_2)\) as that of \((Y_2, X_1, X_2)\). Hence, from \( \hat{Y}_2 \), receiver 1 is able to decode \( X_2 \) with an error probability less than \( \mu_2 \). To carry out the operations above, we require that
\[
\delta \geq \max \left\{ 1, \frac{a}{c}, \frac{b}{d}, \frac{\lambda_1}{\lambda_2} \right\}.
\]
As a result, receiver 1 should be able to decode both \( X_1 \) and \( X_2 \) from \((Y_1, \hat{Y}_1)\). Using the result in Lemma 5.3, we have the following proposition.

**Proposition 5.5:** For general parameters, the capacity region of the Poisson interference channel is inside of the convex hull of \( \bigcup_{0 \leq p, q \leq 1} R_{p,q} \) with \( R_{p,q} \) being:
\[
R_1 \leq E \psi(aA_1X_1^b, bA_2X_2^b; aA_1p, \lambda_1),
\]
\[
R_2 \leq \min \left\{ E \psi(bA_2X_2^b, aA_1X_1^b; bA_2q, \lambda_1), \right.
\]
\[
E \psi((d\delta - b)A_2X_2^b, 0; (d\delta - b)A_2q, \lambda_2 \delta - \lambda_1), \left. \right\},
\]
\[
+ \left\{ E \psi((d\delta - b)A_2X_2^b, 0; (d\delta - b)A_2q, \lambda_2 \delta - \lambda_1) \right\},
\]
\[
R_1 + R_2 \leq E \psi(aA_1X_1^b + bA_2X_2^b, 0; aA_1p + bA_2q, \lambda_1)
\]
\[
+ E \psi((d\delta - b)A_2X_2^b, 0; (d\delta - b)A_2q, \lambda_2 \delta - \lambda_1),
\]
with the expectation being computed for binary input distributions with \( P[X_1^b = 1] = 1 - P[X_1^b = 0] = p \) and \( P[X_2^b = 1] = 1 - P[X_2^b = 0] = q \), and \( \delta \) is any parameter that satisfies \( \delta \geq \max \left\{ 1, \frac{a}{c}, \frac{b}{d}, \frac{\lambda_1}{\lambda_2} \right\} \).

One can follow the same steps above and derive another outer-bound by providing genie-aided information to receiver 2.

**VI. POISSON Z-INTERFERENCE CHANNEL.**

In this section, we study the Poisson Z-interference channel illustrated in Figure 2. The sum capacity of the Gaussian Z-interference channel has been determined in [24] for all parameter ranges. The argument relies on the ability to completely cancel the interference once the signal is decoded in the Gaussian channel. However, such an argument does not work for the Poisson channel. In the following, we characterize the sum capacity of the Z-interference channel in certain parameter range by studying the corresponding discrete-time memoryless binary Z-interference channel and thus leveraging the capacity results for the discrete-time memoryless Z-interference channel [25].

We first determine the sum capacity of the Poisson Z-interference channel when \( c \) is sufficiently small, i.e., in the weak interference regime.

**Theorem 6.1:** For the Poisson Z-interference channel, if
\[
c < \min \{ a, a \lambda_2/\lambda_1 \},
\]
then the sum capacity is given by
\[
C_{sum} = \lim_{0 \leq p \leq 1} \left[ \phi(p, aA_1, \lambda_1) + \frac{u + dA_2}{e} (1 + dA_2/u) u/(dA_2) \right] + u (1 + u/(dA_2)) \ln (1 + dA_2/u)],
\]
with \( u = pcA_1 + \lambda_2 \). Furthermore, the sum capacity is achieved by receiver 1 treating the signal of transmitter 1 as noise.

**Proof:** First, following from Theorem 4.1, the capacity region of the Poisson Z-interference is the same as the capacity region of the following discrete-time memoryless binary Z-interference channel (after normalizing by \( 1/\Delta \)) when \( \Delta \) tends to zero. Again, we use \( X_2^b \) and \( Y_1^b \) to denote the input and output of the corresponding binary Z-interference channel. Using the binary approximation discussed in Section V-A, the transition probabilities
\[
V_1(Y_1^b|X_1^b) = \text{(to receiver 1, as it is independent of } X_2^b \text{ and } Y_2^b|X_2^b, X_1^b \text{ (to receiver 2)} \text{ of the binary interference channel are given by:})
\]
\[
\begin{align*}
V_1(1|0) &= 1 - V_1(0|0) = \lambda_1 \Delta + O(\Delta^2), \quad (15) \\
V_1(1|1) &= 1 - V_1(0|1) = (aA_1 + \lambda_1) \Delta + O(\Delta^2), \quad (16) \\
V_2(1|0, 0) &= 1 - V_2(0|0, 0) = \lambda_2 \Delta + O(\Delta^2), \quad (17) \\
V_2(1|0, 1) &= 1 - V_2(0|0, 1) = (dA_2 + \lambda_2) \Delta + O(\Delta^2), \quad (18) \\
V_2(1|1, 0) &= 1 - V_2(0|1, 0) = (cA_1 + \lambda_2) \Delta + O(\Delta^2), \quad (19) \\
V_2(1|1, 1) &= 1 - V_2(0|1, 1) = (cA_1 + dA_2 + \lambda_2) \Delta + O(\Delta^2). \quad (20)
\end{align*}
\]
Using these transition probabilities, in the following, we show that if
\[
c < \min \{ a, a \lambda_2/\lambda_1 \},
\]
then there exists a distribution \( W(Y_2^b|X_2^b, Y_1^b) \), such that
\[
V_2(Y_2^b|X_2^b, x_2^b) = \sum_{y_1^b} V_1(y_1^b|x_1^b) W(y_2^b|x_2^b, y_1^b), \forall (x_1^b, x_2^b, y_2^b).
\]
\[
(21)
\]
We can characterize \( W(Y_2^b|X_2^b, Y_1^b) \) by the following parameters
\[
q_1 = W(1|0, 0) = 1 - W(0|0, 0), \\
q_2 = W(1|1, 0) = 1 - W(0|1, 0), \\
q_3 = W(1|0, 1) = 1 - W(0|0, 1), \\
q_4 = W(1|1, 1) = 1 - W(0|1, 1).
\]
The construction is illustrated in Figure 5 (in the figure, we show only the transition probabilities to “1”). To satisfy
the conditions in (21), \( q_i, i = 1, \ldots, 4 \) should be chosen such that (ignoring \( O(\Delta^2) \) terms):

\[
\begin{align*}
V_2(1|0,0) &= \lambda_2 \Delta = q_1 (1 - \lambda_1 \Delta) + q_3 \lambda_1 \Delta, \\
V_2(1|0,1) &= (dA_2 + \lambda_2) \Delta = q_2 (1 - \lambda_1 \Delta) + q_4 \lambda_1 \Delta, \\
V_2(1|1,0) &= (cA_1 + \lambda_2) \Delta = q_1 (1 - (aA_1 + \lambda_1 \Delta) \\
&\quad + q_3 (aA_1 + \lambda_1) \Delta, \\
V_2(1|1,1) &= (cA_1 + dA_2 + \lambda_2) \Delta = q_2 (1 - (aA_1 + \lambda_1) \Delta \\
&\quad + q_4 (aA_1 + \lambda_1) \Delta.
\end{align*}
\]

From these equations, we obtain

\[
\begin{align*}
q_1 &= \frac{\Delta (a \lambda_2 - c \lambda_1)}{a}, \\
q_2 &= \frac{\Delta (adA_2 + a \lambda_2 - c \lambda_1)}{a}, \\
q_3 &= \frac{c + (a \lambda_2 - c \lambda_1) \Delta}{a}, \\
q_4 &= \frac{c + \Delta (adA_2 + a \lambda_2 - c \lambda_1)}{a}.
\end{align*}
\]

Now, one can verify that, if (14) holds, then we have \( 0 < q_1 < 1 \) (we need strict inequalities to take care of \( O(\Delta^2) \) terms), when \( \Delta \) is sufficiently small.

Since the capacity region of the Z-interference channel depends only on the marginal distributions of two receivers, the capacity region of the channel specified in (15)–(20) is the same as the capacity region of the channel specified in Figure 5. The channel specified in Figure 5 satisfies \( X_1^b \to (X_2^b, Y_1^b) \to Y_2^b \). The sum capacity of a discrete-time memoryless Z-interference channel that satisfies this Markov chain property has been determined in [25]. Applying the result in [25], we have the sum capacity of the Poisson Z-interference channel given by

\[
C_{\text{sum}} = \lim_{\Delta \to 0} \frac{1}{\Delta} \max_{\rho \in \mathcal{D}} \left\{ I(X_1^b; Y_1^b) + I(X_2^b; Y_2^b) \right\}
= \max_{\rho \in \mathcal{D}} \left\{ I(X_1^b; Y_1^b) + I(X_2^b; Y_2^b) \right\}
\]

in which \( p = \Pr(X_1^b = 1) \) and \( q = \Pr(X_2^b = 1) \), and

\[
\begin{align*}
I_{X_1; Y_1} &\triangleq \lim_{\Delta \to 0} \frac{1}{\Delta} I(X_1^b; Y_1^b), \\
I_{X_2; Y_2} &\triangleq \lim_{\Delta \to 0} \frac{1}{\Delta} I(X_2^b; Y_2^b).
\end{align*}
\]

This sum capacity is achieved by treating the other user’s signal (if any) as noise.

It is easy to verify that (22) can be written as

\[
C_{\text{sum}} = \lim_{0 < p, q \leq 1} \left[ \phi(p, aA_1, \lambda_1) + \phi(q, dA_2, pcA_1 + \lambda_2) \right].
\]

To solve this optimization problem, we observe that \( \phi(p, aA_1, \lambda_1) \) does not depend on \( q \). For any \( p \), by a simple calculation, it can be shown that the optimal value of \( q \) that maximizes \( \phi(q, dA_2, pcA_1 + \lambda_2) \) is given by

\[
q^* = \frac{u}{dA_2} \left[ \frac{1}{e} \left( 1 + \frac{dA_2}{u} \right)^{1 + \frac{dA_2}{u}} - 1 \right].
\]

After detailed calculation by plugging (24) in (23), we have

\[
C_{\text{sum}} = \lim_{0 < p, q \leq 1} \left[ \phi(p, aA_1, \lambda_1) + \frac{u + dA_2}{e} (1 + dA_2/u)^{u/(dA_2)} \right.
\]

\[
\left. + u (1 + u/(dA_2)) \ln (1 + dA_2/u) \right],
\]

which concludes the proof.

The condition in Theorem 6.1 can also be understood based on the transformation illustrated in Figure 6. From the figure, we can see that the capacity regions of these two channels are equivalent if the conditions in Theorem 6.1 are satisfied. Furthermore, we have \( X_1 \to (X_2, Y_1) \to Y_2 \) in the transformed channel.

In all numerical tests we have tried, the objective function in (25) is concave. However, we are not able to analytically show whether the function to be optimized is concave or not, because the form is too complicated.

For the Gaussian Z-interference channel with weak interference, in order to achieve the sum capacity, the transmitter whose corresponding receiver does not experience interference should transmit at the full possible rate [24]. The following example shows that this is not the case for the Poisson Z-interference channel anymore. In this example, we set \( A_1 = A_2 = c = d = \lambda_1 = \lambda_2 = 1 \) and \( a = 2.1 \). It is easy to verify that these choices of parameters satisfy the conditions in Theorem 6.1. Figure 7 plots the values of \( R_1 \), \( R_2 \) and \( R_{\text{sum}} \) as functions of \( p \). To maximize \( R_1 \), i.e., to maximize the rate of the user not experiencing interference,
we should choose \( p = 0.455 \). However, to maximize the sum rate, we should choose \( p = 0.439 \). Hence, for the Poisson Z-interference channel with weak interference, to achieve the sum capacity, the user not experiencing interference does not necessarily transmit at the maximum rate. It is also easy to show that the optimal value of \( p^* \) that maximizes the sum rate is always less or equal to the optimal value of \( p^*_1 \) that maximizes \( R_1 \) of user 1 only.

We next characterize the sum capacity when the interference is sufficiently strong, i.e., in the strong interference regime.

**Theorem 6.2:** Define \( c^* \) as the minimal value of \( c \) such that

\[
\min_{0 \leq p \leq 1} \{ \phi(p, cA_1, dA_2 + \lambda_2) - \phi(p, aA_1, \lambda_1) \} = 0. \tag{26}
\]

For the Poisson Z-interference channel, if \( c \geq c^* \), the capacity region is the convex hull of \( \bigcup_{p,q} R_{p,q} \) with

\[
R_{p,q} = (\phi(p, aA_1, \lambda_1), \tilde{p} \phi(q, dA_2, \lambda_2) + \phi(q, dA_2, cA_1 + \lambda_2)),
\]
and the sum capacity is given by

\[
C_{\text{sum}} = \max_{0 \leq p,q \leq 1} \{ \phi(p, aA_1, \lambda_1) + \tilde{p} \phi(q, dA_2, \lambda_2) + \phi(q, dA_2, cA_1 + \lambda_2) \}. \tag{27}
\]

**Remark 6.3:** We note that in (27), \( \phi(p, aA_1, \lambda_1) \) is a concave function of \( p \), \( \tilde{p} \phi(q, dA_2, \lambda_2) + \phi(q, dA_2, cA_1 + \lambda_2) \) is not a function of \( q \), while \( \phi(q, dA_2, \lambda_2) \) is a non-increasing function of \( q \). Hence the optimization problem in (27) can be converted to a one-dimensional optimization problem by first solving the optimal value of \( q^* \) as a function of \( p \) that maximizes \( \tilde{p} \phi(q, dA_2, \lambda_2) + \phi(q, dA_2, cA_1 + \lambda_2) \) and then optimizing the resulting objective function over \( p \).

**Proof:** We consider the binary Z-interference channel obtained via the binary approximation. From Theorem 4.1, the capacity region of the Poisson Z-interference channel is the same as the binary Z-interference channel. For any \( 0 \leq p, q \leq 1 \), we have

\[
I_{X_1;Y_1} := \lim_{\Delta \to 0} \frac{1}{\Delta} I(X_1^b; Y_1^b) = \phi(p, aA_1, \lambda_1),
\]

\[
I_{X_1;Y_2} := \lim_{\Delta \to 0} \frac{1}{\Delta} I(X_1^b; Y_2^b) = \phi(p, cA_1, qdA_2 + \lambda_2),
\]

\[
I_{X_1;Y_2|X_2} := \lim_{\Delta \to 0} \frac{1}{\Delta} I(X_1^b; Y_2^b|X_2^b)
= \tilde{q} \phi(p, cA_1, \lambda_2) + \phi(p, cA_1, dA_2 + \lambda_2), \tag{28}
\]

\[
I_{X_1;Y_2|X_1} := \lim_{\Delta \to 0} \frac{1}{\Delta} I(X_1^b; Y_2^b|X_1^b)
= \tilde{p} \phi(q, dA_2, \lambda_2) + \phi(q, dA_2, cA_1 + \lambda_2).
\]

**Converse:** We first have a simple outer bound on the capacity region. For any given \( 0 \leq p, q \leq 1 \), define \( R_{p,q} \) be the set of \( (R_1, R_2) \) satisfying the following conditions:

\[
R_1 \leq I_{X_1;Y_1}, \tag{29}
\]

\[
R_2 \leq I_{X_2;Y_2|X_1}, \tag{30}
\]

It is easy to see that the convex hull of \( \bigcup_{p,q} R_{p,q} \) is an outer bound of the capacity region.

**Achievability:** If we additionally require receiver 2 to decode messages from transmitter 1, then any rate pair achievable with this additional requirement is also achievable for the Poisson Z-interference channel. For any given \( 0 \leq p, q \leq 1 \), let \( \mathcal{R}_{p,q}^a \) be the set of \( (R_1, R_2) \) satisfying the following conditions:

\[
R_1 \leq I_{X_1;Y_1}, \tag{31}
\]

\[
R_1 \leq I_{X_1;Y_2|X_2}, \tag{32}
\]

\[
R_2 \leq I_{X_2;Y_2|X_1}, \tag{33}
\]

\[
R_1 + R_2 \leq I_{X_1;X_2}. \tag{34}
\]

As the first link is interference-free, if \( R_1 \) satisfies (31), receiver 1 is able to decode the first message. The channel between two transmitters and receiver 2 can be viewed as a multiple access channel. Therefore, as long as the bounds in (32)–(34) are satisfied, receiver 2 can decode both messages and hence the second message. Hence, the convex hull of \( \bigcup_{p,q} \mathcal{R}_{p,q}^a \) is an inner bound for the capacity region of the Poisson Z-interference channel.

Now, we simplify \( R_{p,q}^a \). First, we have

\[
\frac{\partial}{\partial \lambda} \phi(p, A, \lambda) = \tilde{p} \ln p + p \ln(A + \lambda) - \ln(Ap + \lambda) \leq 0. \tag{35}
\]

which implies that \( \phi(p, A, \lambda) \) is a non-increasing function of \( \lambda \). Hence

\[
\phi(p, cA_1, \lambda_2) \geq \phi(p, cA_1, dA_2 + \lambda_2).
\]

Coupled with (28), this implies that for any \( p \), we have

\[
\min_{0 \leq q \leq 1} I_{X_1;Y_2|X_2} = \phi(p, cA_1, dA_2 + \lambda_2).
\]

As a result, if

\[
\min_{0 \leq p \leq 1} [\phi(p, cA_1, dA_2 + \lambda_2) - \phi(p, aA_1, \lambda_1)] = 0,
\]

we have

\[
I_{X_1;Y_2|X_2} \geq I_{X_1;Y_1}, \text{ for all } 0 \leq p, q \leq 1, \tag{36}
\]

which implies that (32) is redundant. In addition, from (31) and (33), we have

\[
R_1 + R_2 \leq I_{X_1;Y_1} + I_{X_2;Y_2|X_1}
\]

\[
= I_{X_1;Y_2} + I_{X_2;Y_2|X_1}
\]

\[
= I_{X_1,X_2;Y_2},
\]

where (a) follows because

\[
I_{X_1;Y_2} = \phi(p, cA_1, qdA_2 + \lambda_2)
\]

\[
\geq \phi(p, cA_1, dA_2 + \lambda_2)
\]

\[
\geq \phi(p, aA_1, \lambda_1)
\]

\[
= I_{X_1;Y_1},
\]

in which (b) is true due to (35), (c) is true if (26) holds. This implies that (34) is redundant. As a result, the region \( R_{p,q}^a \) is the same as

\[
R_1 \leq I_{X_1;Y_1},
\]

\[
R_2 \leq I_{X_2;Y_2|X_1},
\]
Since the simplified $R_{p,q}^{2}$ is the same as $R_{p,q}$, the sum rate (actually the entire region) can be achieved by the scheme that transmitter 1 transmits at a rate that both receiver 1 and receiver 2 can decode (when receiver 2 decodes, it treats signals from transmitter 1 as interference) the message from transmitter 1, then transmitter 2 sends at the rate $I_{X_2;Y_2|X_1}$ and receiver 2 decodes the message from transmitter 2 after first decoding the message from transmitter 1.

Hence, the sum rate capacity is given by

$$C_{sum} = \max_{0 \leq p, q \leq 1} \{ \phi(p, aA_1, \lambda_1) + \bar{\phi}(q, dA_2, \lambda_2) + p\phi(q, dA_2, cA_1 + \lambda_2) \}.$$

Remark 6.4: The scenario is similar to that of the strong Poisson interference channel discussed in Theorem 3.1 in the sense that the cross link coefficient is sufficiently large. However, since $b = 0$, the construction in the proof of Theorem 3.1 does not work here anymore.

VII. NUMERICAL EXAMPLES

In this section, we provide several numerical examples to illustrate the results derived in the paper.

Figure 8 plots the bounds for the Poisson interference channel with $\lambda_1 = 1.5; \lambda_2 = 1; A_1 = 1; A_2 = 1; a = 1; b = 1.3; c = 1.2; d = 1$, which satisfy the condition that $\lambda_1/\lambda_2 \geq b/d \geq 1 \geq a/c$. In this case, the outer bound derived in Proposition 5.1 is applicable.

Figure 9 plots the bounds when $\lambda_1 = 1; \lambda_2 = 1.2; A_1 = 1; A_2 = 1; a = 1; b = 1; c = 1.3; d = 1.1$, which satisfy $1 > b/d \geq \lambda_1/\lambda_2 \geq a/c$. Hence, the outer bound derived in Proposition 5.2 is applicable.

Figure 10 plots the capacity region bounds when $a = 5, b = 1, c = 3, d = 1, \lambda_1 = 1, \lambda_2 = 2; A_1 = 1; A_2 = 1$.

Figure 11 plots the capacity region bounds when $a = 1.05, b = 1.1, c = 3, d = 1, \lambda_1 = 1, \lambda_2 = 2; A_1 = 1; A_2 = 1$.

Figure 12 illustrates the sum capacity of the Poisson Z-interference channel in the weak interference regime. In the simulation, we set $a = 2.1, b = 1, d = 1, \lambda_1 = 1, \lambda_2 = 1; A_1 = 1; A_2 = 1$. Therefore, as long as $c \leq \min\{a, a\lambda_2/\lambda_1\} = 2.1$, the channel is in the weak interference regime. From the figure, we can observe that the sum rate decreases as the cross-link coefficient increases.

Figure 13 illustrates the relationship between $a$ and the value $e^a$ defined in Theorem 6.2, i.e., the minimum value of $c$ so that the channel is in the strong interference regime. In the simulation, we set $b = 1, d = 1, \lambda_1 = 1, \lambda_2 = 1; A_1 = 1; A_2 = 1$, and let the value of $a$ change. One can observe that as $a$ increases, the value of $e^a$ increases. From the figure, we know that when
The relationship between $C_{\text{sum}}$ and the interference regime.

\[ \phi(p, aA_1, \lambda_1) + \bar{p}C_2, \]

in which $C_2$ is the capacity of the channel $X_2 \rightarrow Y_2$ when there is no interference. Clearly, this value is less than the sum of the capacity of $X_1 \rightarrow Y_1$ and the interference free channel $X_2 \rightarrow Y_2$.

\[ \text{VIII. Conclusions} \]

We have studied the Poisson interference channel, a suitable model for multiuser optical communications in the short-noise-limited regime. We have derived the conditions for the strong interference channel under which the capacity region remains the same even if we require both decoders to decode both messages. We have also derived several inner and outer bounds for the capacity region. We have also characterized the sum rate capacity for the Poisson Z-interference channel when the cross link coefficient is either sufficiently small or sufficiently large.

For the future work, it will be interesting to obtain finite gap results similar to that of the Gaussian interference channel [18]. It is also of interest to investigate band-limited channels, which have been studied in the point-to-point setup in [7], in this multi-user setup. Another direction to pursue is to exploit the relationship between the mutual information and conditional mean estimations in Poisson channels established in [19] and [28]. The relationship between mutual information and conditional mean estimations for the Gaussian channel [26], [28] has provided important insights on the Gaussian Z-interference channels [27]. Although Poisson channels and Gaussian channels are different in terms of neutralizing interference, the unifying framework proposed in [29] will be useful for this line of investigation. It is also of interest to extend the results to different optical communications models, as those accounting for optical amplifiers, and in general, to identify regions of similar asymptotic behavior, of different models, such as the Gaussian and Poisson channels.

\[ \text{Appendix A} \]

\[ \text{Proof of Theorem 4.1} \]

The proof follows that of [4, Th. 2.1] with adaptations. In particular, the proof first follows the construction steps in [4] until (39). However, to set up the notations for the entire proof and for the completeness of the proof, we repeat them here with slightly different notations.

Let \( \{ X_{i,m_l}(:, ) \}_{i \in [M_i]} \) be the code in the hypothesis of the theorem, and let \( D_i : \mathcal{F}(T) \rightarrow \{1, \ldots, |M_i|\} \) be the corresponding decoding mapping. Here \( \mathcal{F}(T) \) is the space of step functions \( Y_i(0, T) \) observed by receiver \( i \). Let \( \epsilon > 0 \) be given. For \( 1 \leq m_i \leq |M_i| \), let \( B_{i,m} \) be the set of sample paths mapped by \( D_i \) to \( m_i \), i.e.,

\[ B_{i,m_l} = \{ Y_i(\cdot) : D_i(Y_i) = m_l \}. \]

For any event \( B_i \) in \( \mathcal{F}(T) \), and \( 1 \leq m_i \leq M_i \), we write

\[ \Pr\{ B_i X_{i,m_l}(\cdot) \} = P_{m_l}(B_i). \]
Hence
\[ 1 - \mu_i = \frac{1}{|\mathcal{M}_i|} \sum_{m_i=1}^{|\mathcal{M}_i|} \Pr(D_i(Y_i) = m_i | X_{i,m_i}(\cdot)) \]
\[ = \frac{1}{|\mathcal{M}_i|} \sum_{m_i=1}^{|\mathcal{M}_i|} P_m(B_{i,m_i}). \]

Let \( \mathcal{A}_i(T) \) be the \( \sigma \)-field of events in \( \mathcal{H}(T) \) generated by \( \hat{A}_i(T) \), which are the class of events defined by finite sets of samples of \( Y_i(0, T) \), i.e., \( (Y_i(t_1), \ldots, Y_i(t_k)) \), where \( t_k \in [0, T] \) and \( 1 \leq k < K < \infty \). From the argument based on measure theory [30, p. 56], we have that for any \( B_i \in \mathcal{A}_i(T) \) and arbitrary \( \delta > 0 \), there exists a \( \hat{B}_i \in \hat{A}_i(T) \) such that the symmetric difference
\[ P_m(B_i \triangle \hat{B}_i) \leq \delta, \]
in which the symmetric difference is defined as
\[ B_i \triangle \hat{B}_i = (B_i \setminus \hat{B}_i) \cup (\hat{B}_i \setminus B_i). \]

It then follows that for a set of events \( \{B_{i,m_i}\} \) and arbitrary \( \epsilon > 0 \), there exist a corresponding set of events \( \{\hat{B}_{i,m_i}\} \) defined by a suitably rich set of samples such that for all \( m_i, m_i' \in [1, |\mathcal{M}_i|] \),
\[ P_m(B_{i,m_i'} \setminus \hat{B}_{i,m_i}) \leq \epsilon / (2|\mathcal{M}_i|). \]  

(37)

Corresponding to the events \( \{\hat{B}_{i,m_i}\} \), we define the disjoint sets \( \{\hat{B}_{i,m_i}\} \) by
\[ \hat{B}_{i,1} = \hat{B}_{i,1}, \]
\[ \hat{B}_{i,m_i} = \hat{B}_{i,m_i} - \bigcup_{m_i' < m_i} \hat{B}_{i,m_i'}, \quad 2 \leq m_i \leq |\mathcal{M}_i|. \]

The disjoint set of events \( \{\hat{B}_{i,m_i}\} \) define the decoder \( \hat{D}_i \), if \( \hat{D}_i(Y_i(0, T)) = m_i \), when \( Y_i(0, T) \in \hat{B}_{i,m_i} \). Furthermore, a finite set \( \{t_k\} \) exists with
\[ 0 \leq t_1 \leq \cdots \leq t_K \leq T \]
such that the \( B_{i,m_i} \) are determined by the samples \( \{Y_i(t_k)\}_{k=1}^K \).

We then have
\[ \hat{\mu}_i = 1 - \frac{1}{|\mathcal{M}_i|} \sum_{m_i=1}^{|\mathcal{M}_i|} P_m(\hat{B}_{i,m_i}) \]
\[ \leq 1 - \frac{1}{|\mathcal{M}_i|} \sum_{m_i=1}^{|\mathcal{M}_i|} P_m(B_{i,m_i} \triangle \hat{B}_{i,m_i}) \]
\[ = 1 - \frac{1}{|\mathcal{M}_i|} \sum_{m_i=1}^{|\mathcal{M}_i|} \{P_m(B_{i,m_i}) - P_m(B_{i,m_i} \triangle \hat{B}_{i,m_i})\} \]
\[ = \mu_i + \frac{1}{|\mathcal{M}_i|} \sum_{m_i=1}^{|\mathcal{M}_i|} P_m(B_{i,m_i} \triangle \hat{B}_{i,m_i}). \]

It is easy to check that
\[ \hat{B}_{i,m_i} = \hat{B}_{i,m_i} - \bigcup_{m_i' < m_i} \hat{B}_{i,m_i'} \]
\[ = \hat{B}_{i,m_i} - \bigcup_{m_i' < m_i} \hat{B}_{i,m_i'} \]
\[ = \hat{B}_{i,m_i} \bigcap \hat{B}_{i,m_i}' \]

(38)

Thus using (37), (38) and following [4], we have
\[ P_m(B_{i,m_i} \triangle \hat{B}_{i,m_i}) \leq \epsilon. \]

We now construct a new decoder \( \hat{D}_i' \). Let \( N = K L \), where \( L \) is a large integer, and let \( \Delta = T / N \). The domain of \( \hat{D}_i' \) is \( \{Y_i(t_k)\}_{k=1}^K \). For \( k = 1, \ldots, K \), let \( n_k \) be the integer which satisfies
\[ (n_k - 1)\Delta < t_k \leq n_k \Delta. \]

The domain of \( \hat{D}_i' \) is \( \{Y_i(n\Delta)\}_{n=1}^N \). Let
\[ \hat{D}_i'((Y_i(n\Delta))_{n=1}^N) = \hat{D}_i((Y_i(n\Delta))_{k=1}^K), \]

(39)

in which \( Y_{i,n} \) is defined in (9). For decoder \( 1, 1 \leq m_i \leq M_1 \),
\[ P_m(D_i'(Y_i(0, T)) \neq \hat{D}_i(Y_i(0, T))) \]
\[ \leq P_m \left( \bigcup_{k=1}^K \{Y_i(t_k) \neq Y_i(n_k \Delta)\} + P_m \left( \bigcup_{k=1}^K \{Y_i(n_k \Delta) \geq 2\} \right) \right) \]
\[ \leq \sum_{k=1}^K \{P_m(Y_i(t_k) \neq Y_i(n_k \Delta)) + P_m(Y_i(n_k \Delta) \geq 2)\} \]
\[ = \sum_{k=1}^K \left[ 1 - \exp \left\{ - \int_{t_k}^{n_k \Delta} (aX_1(t) + bX_2(t) + \lambda_1)dt \right\} \right] \]
\[ + \sum_{k=1}^K \left[ 1 - \exp \{a_{m_1,n_k} - a_{m_1,n_k} \exp \{a_{m_1,n_k}\} \} \right] \]

(40)

in which
\[ a_{m_1,n_k} = \int_{n_{k-1}\Delta}^{n_k \Delta} (aX_1(t) + bX_2(t) + \lambda_1)dt \]
\[ \leq (aA_1 + bA_2 + \lambda_1) \Delta. \]

(41)

Continuing from (40), we have
\[ P_m(D_i'(Y_i(0, T)) \neq \hat{D}_i(Y_i(0, T))) \]
\[ \leq \sum_{k=1}^K \left[ (aA_1 + bA_2 + \lambda_1) + (aA_1 + bA_2 + \lambda_1)^2 \Delta + O(\Delta^2) \right] \Delta \]
\[ = \left( (aA_1 + bA_2 + \lambda_1) + (aA_1 + bA_2 + \lambda_1)^2 \Delta + O(\Delta^2) \right) T / L, \]

which can be made arbitrarily small as \( L \to \infty \). The same holds for receiver 2. As a result, part a) of the theorem is proved.

Now, assume the decoders satisfy part a), which implies that decoder \( D_i(Y_i(0, T)) \) depends only on \( Y_{i,s} \), \( 1 \leq n \leq N \).

We note that the probability law defining \( Y_{i,s} \), \( 1 \leq n \leq N \) when \( X_{i,m_i}(t), \ i = 1, 2 \) are transmitted is unchanged if \( X_{i,m_i}(t) \) are replaced by \( \tilde{X}_{i,m_i}(t) \) such that
\[ \int_{(n-1)\Delta}^{n\Delta} X_{i,m_i}(t)dt = \int_{(n-1)\Delta}^{n\Delta} \tilde{X}_{i,m_i}(t)dt, \quad \text{for } 1 \leq n \leq N. \]

(42)
For $1 \leq m_i \leq M_i$, $1 \leq n \leq N$, $i = 1, 2$, we define

$$t_{m_i,n} = (n-1)\Delta + \frac{1}{A_i} \int_0^{\frac{n\Delta}{(n-1)\Delta}} X_{i,m_i}(t)dt,$$

and, for $1 \leq m_i \leq M_i$, let

$$\tilde{X}_{i,m_i}(t) = \begin{cases} A_i, & (n-1)\Delta \leq t < t_{m_i,n} \\ 0, & t_{m_i,n} \leq t < n\Delta \end{cases}.$$

Since the code $\tilde{X}_{i,m_i}(t)$ satisfies the condition (42), using the same decoder, the error probabilities at the receivers remain the same. However, $\hat{X}_{i,m_i}(t)$ do not make transitions on an evenly spaced grid. Let $N' = NL$, where $L$ is a large integer, and let $\Delta' = T/N'$. For $1 \leq m_i \leq M_i$, $1 \leq n' \leq N'$, if $\hat{X}_{i,m_i}(t)$ is constant on $((n'-1)\Delta', n'\Delta')$, then $\hat{X}_{i,m_i}(t) = \tilde{X}_{i,m_i}(t)$ for $t$ in that interval. Otherwise, $\hat{X}_{i,m_i}(t) = 0$ for $t \in ((n'-1)\Delta', n'\Delta')$. Note that $\tilde{X}_{i,m_i}$ differs from $\tilde{X}_{i,m_i}$ for at most $N$ intervals. Hence, $a\tilde{X}_{1,m_1} + b\tilde{X}_{2,m_2}$ differs from $a\tilde{X}_{1,m_1} + b\tilde{X}_{2,m_2}$ for most $2N$ intervals. Thus, for any $m_1$ and $m_2$,

$$||a\tilde{X}_{1,m_1} + b\tilde{X}_{2,m_2} - (a\tilde{X}_{1,m_1} + b\tilde{X}_{2,m_2})||_1 \leq 2N(aA_1 + bA_2)\Delta' = 2(aA_1 + bA_2)T/L \rightarrow 0. \ (43)$$

Similarly, we have

$$||c\tilde{X}_{1,m_1} + d\tilde{X}_{2,m_2} - (c\tilde{X}_{1,m_1} + d\tilde{X}_{2,m_2})||_1 \leq 2(cA_1 + dA_2)T/L \rightarrow 0.$$

Let $\hat{\mu}_1$ be the error probability corresponding to the code $\{\tilde{X}_{i,m_i}\}$. Then

$$|\mu_1 - \hat{\mu}_1| \leq \frac{1}{|M_1|} \sum_{m_1=1}^{|M_1|} \frac{1}{|M_2|} \sum_{m_2=1}^{|M_2|} |P(B_{1,m_1}|\tilde{X}_{1,m_1}, \tilde{X}_{2,m_2}) - P(B_{1,m_1}|\tilde{X}_{1,m_1}, \tilde{X}_{2,m_2})|.$$

Following from [4, Lemma 2.2], for any $\varepsilon > 0$, we have

$$|P(B_{1,m_1}|\tilde{X}_{1,m_1}, \tilde{X}_{2,m_2}) - P(B_{1,m_1}|\tilde{X}_{1,m_1}, \tilde{X}_{2,m_2})| \leq \varepsilon$$

for sufficiently large $L$ as long as (43) is satisfied. Thus part b) of the theorem is established.

**APPENDIX B**

**PROOF OF LEMMA 5.3**

The capacity region of the MAC $(X_1, X_2) \rightarrow (Y_1, Y'_1)$ is given by [11]:

$$R_1 \leq \frac{1}{T} I(X_1(0, T); Y_1(0, T), Y'_1(0, T)|X_2(0, T)),$$

$$R_2 \leq \frac{1}{T} I(X_2(0, T); Y_1(0, T), Y'_1(0, T)|X_1(0, T)),$$

$$R_1 + R_2 \leq \frac{1}{T} I(X_1(0, T), X_2(0, T); Y_1(0, T), Y'_1(0, T)).$$

The key part of the proof is to compute the above mutual information terms. After that, we can follow the steps in [11] to get the desired results.

In this proof, we compute $I(X_1(0, T), X_2(0, T); Y_1(0, T), Y'_1(0, T))$ in detail, and the other two terms can be computed similarly.

Let $N_1(t)$ be the number of points observed from the doubly stochastic Poisson point process $Y_1$ before time $t$, and $T(t) = \{T_1, \ldots, T_{N_1(t)}\}$ be the corresponding ordered arrival times. Obviously, $(N_1(T), T(T))$ is a complete description of $Y_1(0, T)$. Similarly, let $N'_1(t)$ be the number of points observed from the doubly stochastic Poisson point process $Y'_1$ before time $t$, and $T'(t) = \{T'_1, \ldots, T'_{N'_1(t)}\}$ be the corresponding ordered arrival times. And $(N'_1(T), T'(T))$ is a complete description of $Y'_1(0, T)$. We also define $N_\alpha(t) = \{N(t), T(t), N'(t), T'(t)\}$, which is a complete description of $Y_1$ and $Y'_1$.

We note that given $x_1(0, T)$ and $x_2(0, T)$, $Y_1$ and $Y'_1$ are two independent Poisson point processes with the rates $\lambda_{Y_1}(t) = ax_1(t) + bx_2(t) + \lambda_1$ and $\lambda_{Y'_1}(t) = (ad/c - b)x_2(t) + \lambda_2 - c\lambda_1/a$, respectively. Hence, given $x_1(t)$ and $x_2(t)$, the sample function density of $Y_\alpha(T)$ is [21]

$$f_{Y_\alpha(T)|X_1, X_2}(y_\alpha|x_1, x_2) = \begin{cases} f_1(\lambda_{Y_1}(t), \lambda_{Y'_1}(t)), & n_1(T) = 0, n'_1(T) = 0 \\ f_2(\lambda_{Y_1}(t), \lambda_{Y'_1}(t)), & n_1(T) \geq 1, n'_1(T) = 0 \\ f_3(\lambda_{Y_1}(t), \lambda_{Y'_1}(t)), & n_1(T) = 0, n'_1(T) \geq 1 \\ f_4(\lambda_{Y_1}(t), \lambda_{Y'_1}(t)), & n_1(T) \geq 1, n'_1(T) \geq 1 \end{cases}$$

with

$$f_1(\lambda_{Y_1}(t), \lambda_{Y'_1}(t)) = \exp\left(-\int_0^T \lambda_{Y_1}(t)dt\right) \exp\left(-\int_0^T \lambda_{Y'_1}(t)dt\right),$$

$$f_2(\lambda_{Y_1}(t), \lambda_{Y'_1}(t)) = \exp\left(-\int_0^T \lambda_{Y_1}(t)dt\right) \prod_{i=1}^{n_1(t)} \lambda_{Y_i}(t_i) \exp\left(-\int_0^T \lambda_{Y'_1}(t)dt\right),$$

$$f_3(\lambda_{Y_1}(t), \lambda_{Y'_1}(t)) = \exp\left(-\int_0^T \lambda_{Y_1}(t)dt\right) \exp\left(-\int_0^T \lambda_{Y'_1}(t)dt\right) \prod_{j=1}^{n'_1(t)} \lambda_{Y'_j}(t_j),$$

$$f_4(\lambda_{Y_1}(t), \lambda_{Y'_1}(t)) = \exp\left(-\int_0^T \lambda_{Y_1}(t)dt\right) \prod_{j=1}^{n'_1(t)} \lambda_{Y'_j}(t_j) \exp\left(-\int_0^T \lambda_{Y'_1}(t)dt\right).$$

Now, define $\hat{\lambda}_{Y_1}(t) = \hat{\lambda}_1 + \mathbb{E}(aX_1(t) + bX_2(t))Y_\alpha(t); 0 \leq \tau \leq t)$ and $\hat{\lambda}_{Y'_1}(t) = \hat{\lambda}_2 - c\hat{\lambda}_1/a + \mathbb{E}((ad/c - b)X_2(t))Y_\alpha(t); 0 \leq \tau \leq t)$. Then following from [21] (Chap. 6), $Y_1$ and $Y'_1$ are self-exciting counting processes with intensity processes $\hat{\lambda}_{Y_1}(t), 0 \leq t \leq T$ and $\hat{\lambda}_{Y'_1}(t), 0 \leq t \leq T$. Moreover the sample density function of $Y_\alpha(T)$ is given by
We then obtain
\[
I(1, T), X_z(1, T) ; Y, Y'_1
\]
\[
= h(Y_0) - h(Y_0 | X_1, X_2)
\]
\[
= \mathbb{E}\left[ \int_0^T (\hat{\lambda}_{Y_1}(t) - \hat{\lambda}_{Y_1}(t)) dt + \int_0^T (\hat{\lambda}_{Y'_1}(t) - \hat{\lambda}_{Y'_1}(t)) dt \right]
\]
\[
+ \mathbb{E}\left[ \ln \left( 1(n_1(T) = 0, n'_1(T) = 0) + \ln \left( 1(n_1(T) \geq 1, n'_1(T) = 0) \Pi_{i=1}^{n_1(T)} \hat{\lambda}_{Y_i}(t_i) \
+ \ln \left( 1(n_1(T) = 0, n'_1(T) \geq 1) \Pi_{j=1}^{n'_1(T)} \hat{\lambda}_{Y'_j}(t'_j) \right) \right) \right]
\]
\[
- \mathbb{E}\left[ \ln \left( 1(n_1(T) = 0, n'_1(T) = 0) + \ln \left( 1(n_1(T) \geq 1, n'_1(T) = 0) \Pi_{i=1}^{n_1(T)} \hat{\lambda}_{Y_i}(t_i) \
+ \ln \left( 1(n_1(T) = 0, n'_1(T) \geq 1) \Pi_{j=1}^{n'_1(T)} \hat{\lambda}_{Y'_j}(t'_j) \right) \right) \right]
\]
\[
\equiv \mathbb{E}\left[ \ln \left( 1(n_1(T) = 0, n'_1(T) = 0) + \ln \left( 1(n_1(T) \geq 1, n'_1(T) = 0) \Pi_{i=1}^{n_1(T)} \hat{\lambda}_{Y_i}(t_i) \
+ \ln \left( 1(n_1(T) = 0, n'_1(T) \geq 1) \Pi_{j=1}^{n'_1(T)} \hat{\lambda}_{Y'_j}(t'_j) \right) \right) \right]
\]
\[
\equiv 0,
\]
(44)
In the above equation, step (a) follows because
\[
\mathbb{E}\left[ \int_0^T (\hat{\lambda}_{Y_1}(t) - \hat{\lambda}_{Y_1}(t)) dt + \int_0^T (\hat{\lambda}_{Y'_1}(t) - \hat{\lambda}_{Y'_1}(t)) dt \right]
\]
\[
\equiv \int_0^T \mathbb{E}[\hat{\lambda}_{Y_1}(t) - \hat{\lambda}_{Y_1}(t)] dt + \int_0^T \mathbb{E}[\hat{\lambda}_{Y'_1}(t) - \hat{\lambda}_{Y'_1}(t)] dt
\]
\[
\equiv 0,
\]
where (b) follows because \( \hat{\lambda}_{Y_1}, \hat{\lambda}_{Y_1}, \hat{\lambda}_{Y'_1} \) and \( \hat{\lambda}_{Y'_1} \) are bounded, and hence the integral and expectation can be exchanged, and (c) follows because \( \mathbb{E}[\hat{\lambda}_{Y_1}(t)] = \mathbb{E}[\hat{\lambda}_{Y_1}(t)] \) and \( \mathbb{E}[\hat{\lambda}_{Y'_1}(t)] = \mathbb{E}[\hat{\lambda}_{Y'_1}(t)] \).
Using [23, Proposition 1], (45) is equal to
\[
I(X_1(0, T), X_2(0, T); Y_1, Y_1') = \mathbb{E} \int_{0}^{T} (\dot{\lambda}_1(t) \ln \dot{\lambda}_1(t) - \lambda_1(t) \ln \dot{\lambda}_1(t))dt \\
+ \mathbb{E} \int_{0}^{T} (\dot{\lambda}_1'(t) \ln \dot{\lambda}_1'(t) - \dot{\lambda}_1'(t) \ln \dot{\lambda}_1(t))dt.
\]
We can now follow the same argument as that in [11] and conclude that binary input is optimal. Thus, the capacity region of the MAC \((X_1, X_2) \to (Y_1, Y_1')\) is given by the Lemma 5.3.

APPENDIX C

MULTUAL INFORMATION TERMS IN THE SIMPLIFIED HAN-KOYABAYASHI REGION IN SECTION V-A

In the following, we use \(p * q\) to denote \((1 - p)q + p(1 - q)\). It is easy to check that the various mutual information terms in the simplified Han-Kobayashi region [22] when \(|Q| = 1\), \(W_1\) and \(W_2\) being binary random variables with \(P_{W_1}(1) = 1 - P_{W_1}(0) = p_1 = 1 - \tilde{p}_1\) and \(P_{W_2}(1) = 1 - P_{W_2}(0) = p_2 = 1 - \tilde{p}_2\), \(X_1^p\) being generated from \(W_1\) via a testing binary symmetric channel with parameter \(q_1 = 1 - \tilde{q}_1\), and \(X_2^q\) being generated from \(W_2\) via a testing binary symmetric channel with parameter \(q_2 = 1 - \tilde{q}_2\) are:

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} I(X_1^p; Y_1^p | W_2) = -\tilde{p}_1(\lambda_1 + q_1 a_1 + p_1 * q_1 a_1) \ln(\lambda_1 + q_2 b_2 + p_1 * q_1 a_1) \\
- p_1(1 - q_1) \tilde{p}_1(\lambda_1 + q_2 b_2) \ln(\lambda_1 + q_2 b_2) \\
+ p_1 \tilde{q}_1 \tilde{p}_2(\lambda_1 + q_1 a_1) \ln(\lambda_1 + q_2 b_2) \\
+ p_1 \tilde{q}_1 \tilde{p}_2(\lambda_1 + q_1 a_1) \ln(\lambda_1 + q_2 b_2),
\]

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} I(X_2^q; Y_1^p | W_1) = -\tilde{p}_1(\lambda_2 + q_2 d_2 + p_2 * q_2 d_2 + q_1 c_1) \ln(\lambda_2 + q_2 d_2 + q_1 c_1) \\
- p_2(1 - q_2) \tilde{p}_1(\lambda_2 + q_2 d_2 + q_1 c_1) \ln(\lambda_2 + q_2 d_2 + q_1 c_1) \\
+ \tilde{p}_1(1 - p_2 * q_2)(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ \tilde{p}_1(1 - p_2 * q_2)(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1 \tilde{q}_1 \tilde{p}_2(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1),
\]

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} I(X_1^p; Y_2^q | W_2) = -\tilde{p}_1(\lambda_1 + q_1 a_1 + p_1 * q_1 a_1) \ln(\lambda_1 + q_2 b_2 + p_1 * q_1 a_1) \\
+ (1 - p_2 * q_2) \tilde{p}_1(\lambda_1 + q_1 a_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_2 * q_2 \tilde{p}_1(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1(1 - p_2 * q_2)(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1),
\]

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} I(W_1 X_1^p; Y_1^p | W_2) = -\tilde{p}_1(\lambda_2 + p_1 * q_1 a_1 + q_2 d_2) \\
\times \ln(\lambda_2 + p_1 * q_1 a_1 + q_2 d_2) \\
+ (1 - p_2 * q_2) \tilde{p}_1(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_2 * q_2 \tilde{p}_1(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1(1 - p_2 * q_2)(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1 \tilde{q}_1 \tilde{p}_2(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1 \tilde{q}_1 \tilde{p}_2(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1),
\]

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} I(W_1 X_2^q; Y_2^q | W_2) = -\tilde{p}_1(\lambda_2 + p_1 * q_1 a_1 + q_2 d_2) \\
\times \ln(\lambda_2 + p_1 * q_1 a_1 + q_2 d_2) \\
+ (1 - p_2 * q_2) \tilde{p}_1(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_2 * q_2 \tilde{p}_1(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1(1 - p_2 * q_2)(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1 \tilde{q}_1 \tilde{p}_2(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1 \tilde{q}_1 \tilde{p}_2(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1),
\]

\[
\lim_{\Delta \to 0} \frac{1}{\Delta} I(W_1 X_1^p; Y_2^q | W_2) = -\tilde{p}_1(\lambda_2 + p_1 * q_1 a_1 + q_2 d_2) \\
\times \ln(\lambda_2 + p_1 * q_1 a_1 + q_2 d_2) \\
+ (1 - p_2 * q_2) \tilde{p}_1(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_2 * q_2 \tilde{p}_1(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1(1 - p_2 * q_2)(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1 \tilde{q}_1 \tilde{p}_2(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1) \\
+ p_1 \tilde{q}_1 \tilde{p}_2(\lambda_2 + q_1 c_1) \ln(\lambda_2 + q_1 c_1),
\]

REFERENCES
