Error Correcting Codes

Lecture 1

Chap 1 Introduction

1. Block diagram

- Notation:
  - $x^k$: Vector
  - $x^k = (x_1, x_2, ..., x_k)$
  - $\hat{x}$: Estimation of $x$

2. Two types of codes:
   - Block code
   - Convolutional code

   a. Divide source sequence into block of $k$ bits.
      Denote each block as $\overrightarrow{u} = (u_0, u_1, ..., u_{k-1})$, called message.
      Each block is independent with each other.

   b. Encode $\overrightarrow{u}$ to $\overrightarrow{v} = (v_0, v_1, ..., v_{n-1})$, called codeword.

   c. Code rate $R = \frac{k}{n}$.
      $R_t$: high efficiency; $R_r$: high reliability.

   d. Decoding for block code: $\overrightarrow{F} = (v_0, ..., v_{n-1})$
      $\left(\overrightarrow{u} = (\hat{u}_0, \hat{u}_1, ..., \hat{u}_{k-1})
      \overrightarrow{v} = (\hat{v}_0, \hat{v}_1, ..., \hat{v}_{n-1})\right)$.
2. Convolutional Code:

Encoding of each block will depend on the value of previous \( m \) blocks.

3. Optimal Decoding

\[ \hat{v} \rightarrow \hat{u} \quad \begin{array}{c}
\text{\( k \) bits} \\
\text{\( n \) bits}
\end{array} \]

2^k message \( \rightarrow 2^n \) codewords \( \hat{v} \)

**Goal:**

\[ \min P(\hat{u} \neq \hat{u}) = P(\hat{v} \neq \hat{v}) = P(E) \quad \text{probability of Error} \]

\[ \min P(E) = \sum_{\hat{v}} P(E|\hat{v})P(\hat{v}) \]

If we can make \( P(E|\hat{v}) \) small for all \( \hat{v} \), then \( P(E) \) will be minimized

\[ P(E|\hat{v}) = P(\hat{v} \neq \hat{v}|\hat{v}) \]

\[ = 1 - P(\hat{v} = \hat{v}|\hat{v}) \]

**Decoding Rule:**

For a given \( \hat{v} \), we should choose \( \hat{v} \) as the one with the largest probability \( P(\hat{v}|\hat{v}) \)

Decoding steps when given \( \hat{v} \):

1. Compute \( P(\hat{v}|\hat{v}) \) for all possible values of \( \hat{v} \)
2. Set \( \hat{v} \) as the one that has the largest posterior probability \( P(\hat{v}|\hat{v}) \), i.e. \( P(\hat{v}|\hat{v}) > P(\hat{v}|\hat{v}) \) for all \( \hat{v} \neq \hat{v} \).

4. Maximum Likelihood Decoding (MLD)

\[ P(\hat{v}|\hat{v}) = \frac{P(\hat{v})P(\hat{v}|\hat{v})}{P(\hat{v})} \quad \text{Bayesian Rule} \]

maximize
If \( p(v) \) is uniformly distributed over all possible \( v \), then \( \max p(v|r) \) is the same as \( \max p(r|v) \), which means we pick the codeword that has the largest likelihood.

5. Error Performance Measure

1) Block (word or frame) error Prob, \( P_w \), the probability that a codeword is in error.

\[ e_{g1} : (0000) \rightarrow (0001) \rightarrow (0100) \]
Message codeword decoded codeword
One error happens, take the whole codeword as one error.

2) Bit (symbol) error Prob, \( P_b \).

\[ e_{g2} : \text{two bits of error happened in } e_{g1} \]
\[ e_{g3} : \begin{array}{c}
(0001) \\
\downarrow
\end{array}, \begin{array}{c}
(0100) \\
\downarrow
\end{array}, \begin{array}{c}
(010), \begin{array}{c}
111 \\
\downarrow
\end{array}
\end{array} \]
\[ P_w = \frac{2}{3} \]
\[ P_b = \frac{3}{12} = \frac{1}{4} \]

6. Code Gain

1) unencoded system: System without channel coding/decoding.

2) code gain: Compare coded system with unencoded system.

3) \( Eb/N0 \): Signal to noise ratio (SNR).

Given \( (Eb/N0) \) coded, \( (Eb/N0) \) unencoded:

Code gain: \( \eta = 10 \log_{10} \left( \frac{Eb}{N0} \text{ coded} \right) - 10 \log_{10} \left( \frac{Eb}{N0} \text{ un-coded} \right) \)

\[ = 10 \log_{10} \left( \frac{Eb}{N0} \text{ coded} \right) \]