1. Problem 1

Let \( \mathbf{v}_1(i) \) and \( \mathbf{v}_3(i) \) be the \( i \)th elements of \( \mathbf{v}_1 \) and \( \mathbf{v}_3 \) respectively, we have

\[
d(\mathbf{v}_1, \mathbf{v}_3) = w(\mathbf{v}_1 + \mathbf{v}_3) = \sum_{i=1}^{n} f(\mathbf{v}_1(i) \neq \mathbf{v}_3(i)),
\]

in which \( f(\mathbf{v}_1(i) \neq \mathbf{v}_3(i)) = 1 \) if \( \mathbf{v}_1(i) \neq \mathbf{v}_3(i) \) and \( f(\mathbf{v}_1(i) \neq \mathbf{v}_3(i)) = 0 \) if \( \mathbf{v}_1(i) = \mathbf{v}_3(i) \).

Similarly

\[
d(\mathbf{v}_1, \mathbf{v}_2) + d(\mathbf{v}_2, \mathbf{v}_3) = \sum_{i=1}^{n} f(\mathbf{v}_1(i) \neq \mathbf{v}_2(i)) + \sum_{i=1}^{n} f(\mathbf{v}_2(i) \neq \mathbf{v}_3(i))
\]

\[
= \sum_{i=1}^{n} [f(\mathbf{v}_1(i) \neq \mathbf{v}_2(i)) + f(\mathbf{v}_2(i) \neq \mathbf{v}_3(i))].
\]

We have two different cases:

- If \( f(\mathbf{v}_1(i) \neq \mathbf{v}_3(i)) = 0 \), this means \( \mathbf{v}_1(i) = \mathbf{v}_3(i) \). Then either \( \mathbf{v}_2(i) = \mathbf{v}_1(i) = \mathbf{v}_3(i) \), in which case \( f(\mathbf{v}_1(i) \neq \mathbf{v}_2(i)) + f(\mathbf{v}_2(i) \neq \mathbf{v}_3(i)) = 0 \), or \( \mathbf{v}_2(i) \neq \mathbf{v}_1(i) = \mathbf{v}_3(i) \), in which case \( f(\mathbf{v}_1(i) \neq \mathbf{v}_2(i)) + f(\mathbf{v}_2(i) \neq \mathbf{v}_3(i)) = 2 \). As the result \( f(\mathbf{v}_1(i) \neq \mathbf{v}_2(i)) + f(\mathbf{v}_2(i) \neq \mathbf{v}_3(i)) \geq f(\mathbf{v}_1(i) \neq \mathbf{v}_3(i)) \).

- If \( f(\mathbf{v}_1(i) \neq \mathbf{v}_3(i)) = 1 \), this means \( \mathbf{v}_1(i) \neq \mathbf{v}_3(i) \). As the result, either \( \mathbf{v}_2(i) \neq \mathbf{v}_1(i) \) or \( \mathbf{v}_2(i) \neq \mathbf{v}_3(i) \), but not both. Hence, \( f(\mathbf{v}_1(i) \neq \mathbf{v}_2(i)) + f(\mathbf{v}_2(i) \neq \mathbf{v}_3(i)) = 1 = f(\mathbf{v}_1(i) \neq \mathbf{v}_3(i)) \).

In both cases, we have \( f(\mathbf{v}_1(i) \neq \mathbf{v}_2(i)) + f(\mathbf{v}_2(i) \neq \mathbf{v}_3(i)) \geq f(\mathbf{v}_1(i) \neq \mathbf{v}_3(i)) \), hence

\[
d(\mathbf{v}_1, \mathbf{v}_2) + d(\mathbf{v}_2, \mathbf{v}_3) \geq d(\mathbf{v}_1, \mathbf{v}_3).
\]

2. Problem 2

\[
G = \begin{pmatrix}
0 & 1 & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
1 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 \\
1 & 0 & 1 & 1 & 0 & 0 & 0 & 1
\end{pmatrix}
\]

\[
H = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 1 & 0 & 0 & 1 & 1 & 1 & 0 \\
0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 & 1 & 0 & 1 & 1
\end{pmatrix}
\]

There are no 3 or less columns in \( H \) that add up to zero.

There are 4 columns in \( H \) that add up to zero. Hence, \( d_{\text{min}} = 4 \).
3. Problem 3

\[ S = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \end{pmatrix} \]

4. Problem 4

The code has 16 codewords obtained by considering all linear combinations if the rows of G. The codewords and their weights are given by

<table>
<thead>
<tr>
<th>codeword</th>
<th>weight</th>
<th>codeword</th>
<th>weight</th>
</tr>
</thead>
<tbody>
<tr>
<td>00000000</td>
<td>0</td>
<td>10110001</td>
<td>4</td>
</tr>
<tr>
<td>01111000</td>
<td>4</td>
<td>11001001</td>
<td>4</td>
</tr>
<tr>
<td>11100100</td>
<td>4</td>
<td>01010101</td>
<td>4</td>
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<td>00101101</td>
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</tr>
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<td>11010010</td>
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<td>01100011</td>
<td>4</td>
</tr>
<tr>
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<td>00011011</td>
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</tr>
<tr>
<td>00110110</td>
<td>4</td>
<td>10000111</td>
<td>4</td>
</tr>
<tr>
<td>01001110</td>
<td>4</td>
<td>11111111</td>
<td>8</td>
</tr>
</tbody>
</table>

Hence, the weight distribution of the code is given by:

\[ A_0 = 1, A_4 = 14, A_8 = 1. \]

The probability of an undetected error is

\[ \sum_{1}^{n} A_i p^i (1-p)^{n-i} = 14(0.01)^4(0.99)^4 + (0.01)^8 = 1.34 \times 10^{-7}. \]

5. Problem 4
<table>
<thead>
<tr>
<th>correctable error patterns</th>
<th>Syndromes</th>
</tr>
</thead>
<tbody>
<tr>
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</tr>
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<td>0011</td>
</tr>
<tr>
<td>10001000</td>
<td>1111</td>
</tr>
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