Sum-Rate Capacity of Poisson MIMO Multiple-Access Channels

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Abstract

In this paper, we analyze the sum-rate capacity of two-user Poisson MIMO multiple-access channels (MAC), when both the transmitters and the receiver are equipped with multiple antennas. We first characterize the sum-rate capacity of the Poisson MAC when each transmitter has a single antenna and the receiver has multiple antennas. Although the sum-rate capacity of Poisson MAC when the receiver is equipped with a single antenna has been characterized in the literature, the inclusion of multiple antennas at the receiver makes the problem more challenging. We obtain the optimal input that achieves the sum-rate capacity by solving a non-convex optimization problem. We show that, for certain channel parameters, it is optimal for a single user to transmit to achieve the sum-rate capacity, and for certain channel parameters, it is optimal for both users to transmit. We then characterize the sum-rate capacity of the channel where both the transmitters and the receiver are equipped with multiple antennas. We show that the sum-rate capacity of the Poisson MAC with multiple transmit antennas is equivalent to a properly constructed Poisson MAC with a single antenna at each transmitter, and has thus been characterized by the former case.

I. Introduction

Free-space optical (FSO) and visible light communication (VLC) can be modeled using Poisson channel, in which photon sensitive devices, embedded in the receivers [1], record the

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arrival of photons. The point-to-point single-user Poisson channel has been investigated from various perspectives [2]–[11]. On the other hand, multi-user Poisson channels are not very well understood. Among limited existing work, [12], [13] focused on the Poisson broadcast channel, [14] studied the Poisson multiple-access channel (MAC), [15] considered the optimization of the capacity region of Poisson MAC with respect to different power constraints, and [16] investigated the Poisson channel with side information at the transmitter. Furthermore, [17], [18] studied the discrete-time Poisson channel and showed that sum-capacity achieving distributions of the Poisson MAC under peak amplitude constraints are discrete with a finite number of mass points. [19] discussed a discrete memoryless Poisson MAC with noiseless feedback.

Among papers mentioned above, [14] is particularly relevant to us. In [14], the authors considered the Poisson single-input single-output MAC (SISO-MAC) with each of the transmitters and the receiver having a single antenna. It is shown in [14] that the complex continuous-time continuous-input discrete-output Poisson channel can be approximated with a discrete-time binary-input binary-output channel without loss of optimality in terms of the transmission rate. [14] further characterized the sum-rate capacity for the symmetric case where each transmitter has the same power constraint and has the same channel condition to the receiver. By exploiting the symmetric nature of the channel, [14] showed that, to achieve the sum-rate capacity, the objective function is Schur-concave and hence the multi-dimensional convex optimization problem can be converted into a one-dimensional convex optimization problem.

More recently, [20], [21] extended the study of [14] to the case with multiple antennas at each transmitter and single antenna at the receiver (MISO-MAC). It was shown that for each MISO-MAC, a SISO-MAC can be constructed to have the same sum-rate capacity. Hence, the problem of characterizing the sum-rate capacity of MISO-MAC is reduced to the problem of characterizing the sum-rate capacity of SISO-MAC. However, differently from the case studied in [14], the constructed SISO-MAC may not be symmetric anymore, which makes the problem more challenging. In particular, the objective function in the non-symmetric channel is not a convex function, and hence the techniques used in [14] are not applicable. It was shown in [20], [21] that there are four possible candidates for the optimal solution for the non-convex optimization problem. Two possible candidates correspond to the scenario where one user is active while the other user stays inactive, which is in contrast to the Gaussian MAC where both users must transmit (either simultaneously or at different time) to achieve the sum-rate capacity. The other two candidates of the optimal solution correspond to the scenario where both transmitters transmit simultaneously to achieve the sum-rate capacity.

In this paper, we further extend the study in [20], [21] to the case where all users are equipped with multiple antennas (MIMO-MAC). Having multiple receive antennas makes the problem considerably more complex than that of MISO-MAC [20], [21]. In particular, in MISO-MAC [20], [21], although the objective function is not convex, the set of nonlinear equations corresponding to KKT conditions, which are necessary but not sufficient conditions for optimality, can be converted into a set of linear equations along with a nonlinear but convex equation. This special structure of KKT conditions in MISO-MAC facilitates the further analysis. Unfortunately, such a conversion technique developed in [20], [21] for MISO-MAC is not applicable to MIMO-MAC anymore. As the result, we need to devise new techniques to analyze the MIMO-MAC. Despite this challenge, using a novel channel transformation argument, we show that characterizing the sum-rate capacity of MIMO-MAC can be reduced to characterizing the sum-rate capacity of the SIMO-MAC, in which each transmitter has only one antenna. Similarly to the SISO-MAC considered in [20], [21], the SIMO-MAC has a non-convex objective function. After analyzing the KKT conditions for the case with two transmitters, we draw a conclusion that there are three optimality scenarios for achieving the sum-rate capacity: 1) when only transmitter 1 is active and transmitter 2 is inactive; 2) when transmitter 2 is active and transmitter 1 is inactive; and 3) when both transmitters are active.

The remainder of this paper is organized as follows. Section II introduces the system model for this paper. Section III analyzes the SIMO-MAC and Section IV analyzes the MIMO-MAC. Section V provides several numerical examples. Finally, Section VI concludes the paper with some remarks.

II. System Model

In this section, we introduce the model studied in this paper. As shown in Fig. 1, we consider the two-user Poisson MIMO MAC. The analysis can be extended to the scenario with more than two transmitting users. Let J_n be the number of antennas at transmitter n, and M be the number of antennas at the receiver. Let $X_{nj_n}(t)$ be the input of the j_n^{th} transmitting antenna of transmitter n and $Y_m(t)$ be the doubly-stochastic Poisson process observed at the m^{th} receiving antenna. The input-output relationship can be described as:

$$Y_m(t) = \mathcal{P}\left(\sum_{n=1}^{2} S_{nj_nm}X_{nj_n}(t) + \lambda_m\right), \text{ for } m = 1, ..., M \tag{1}$$

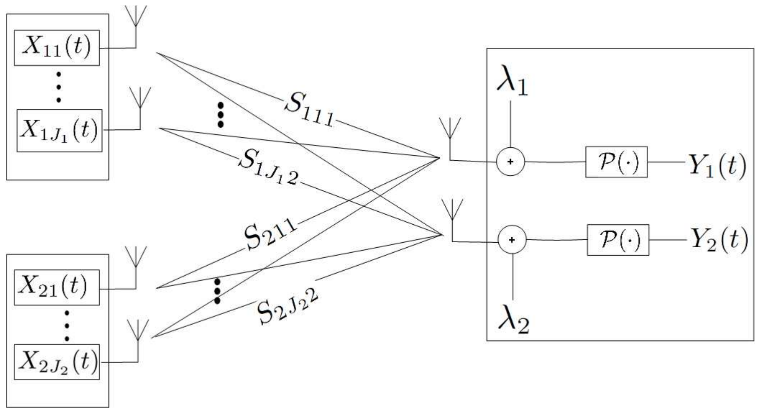

Fig. 1: The Poisson MIMO-MAC model.

in which S_{nj_nm} is the channel response between the j_n^{th} antenna of the transmitter n and the m^{th} receiving antenna, λ_m is the dark current at the m^{th} receive antenna, and $\mathcal{P}(\cdot)$ is the non-linear transformation converting the light strength to the doubly-stochastic Poisson process that records the timing and number of photons' arrivals. In particular, for any time interval $[t, t+\tau]$, the probability that there are k photons arriving at the m^{th} receiving antenna is

$$\Pr\{Y_m(t+\tau) - Y_m(t) = k\} = \frac{e^{-\Lambda_m}\Lambda_m^k}{k!}, \tag{2}$$

where

$$\Lambda_m = \int_t^{t+\tau} \left[\sum_{n=1}^{2} S_{nj_nm} X_{nj_n}(t') + \lambda_m\right] dt'. \tag{3}$$

We consider the peak power constraint, i.e., the transmitted signal $X_{nj_n}(t)$ must satisfy the following constraint:

$$0 \leq X_{nj_n}(t) \leq A_{nj_n}, \tag{4}$$

where A_{nj_n} is the maximum power allowed for antenna j_n of transmitter n.

We use μ_{nj} to denote the duty cycle of each transmit antenna, i.e., μ_{nj} is the percentage of time at which the j^{th} antenna of the transmitter n is on. We use $\boldsymbol{\mu}$ to denote the vector of all μ_{nj}s.

Throughout the paper, we use the following substitutions to simplify the notations:

$$a_{jm} \triangleq S_{1jm}A_{1j}, \tag{5}$$

$$b_{jm} \triangleq S_{2jm}A_{2j}, \tag{6}$$

$$\phi(x) \triangleq x\log(x), \tag{7}$$

$$\zeta(x,y) \triangleq \phi(x+y)-\phi(y). \tag{8}$$

The goal of this paper is to characterize the sum-rate capacity of the Poisson MIMO MAC.

III. SIMO-MAC Analysis

To simplify the presentation of main ideas, in this section, we focus on the case with $M=2$ and $J_1=J_2=1$. That is each transmitter has only one antenna while the receiver has 2 antennas. The solution of this special case is a building block to the solution to the general case where the transmitters also have multiple antennas. This general case is studied in Section IV. As $J_1=J_2=1$, to lighten the notation, we omit the subscript j_n in the remainder of this section.

A. Optimality Conditions

It has been shown in [14] that the continuous-input discrete-output Poisson MAC can be converted to a binary-input binary-output MAC. In particular, the input waveform can be limited to a two level waveform, i.e., 0 or A_n, for the transmitter n. Let μ_n be the duty cycle of each transmitter. Therefore, the sum-rate capacity of such a channel is given by

$$C_{sum} = \max_{0\le\mu_1,\mu_2\le 1} I_{X_1,X_2;Y}. \tag{9}$$

where

$$I_{X_1,X_2;Y} = \sum_{m=1}^{M} I_{X_1,X_2;Y_m} \tag{10}$$

with

$$\begin{aligned} I_{X_1,X_2;Y_m} = \; & (1-\mu_1)(1-\mu_2)\phi(\lambda_m)+\mu_1(1-\mu_2)\phi(a_m+\lambda_m) \\ & +\mu_2(1-\mu_1)\phi(b_m+\lambda_m)+\mu_1\mu_2\phi(a_m+b_m+\lambda_m) \\ & -\phi(a_m\mu_1+b_m\mu_2+\lambda_m). \end{aligned} \tag{11}$$

The problem to calculate the sum-rate capacity can be rewritten as

$$\begin{aligned} \textbf{P0:} \qquad & \max I_{X_1,X_2;Y}, \\ \text{s.t} \qquad & 0 \le \mu_1 \le 1, \\ & 0 \le \mu_2 \le 1. \end{aligned} \tag{12}$$

The problem **(P0)** has been solved for the special case of $M = 1$ (i.e., when the receiver has only one antenna) in [22]. The main idea in [22] is to convert a set of nonlinear equations that appear in the analysis into a set of linear equations and a convex function, which then can be solved. However, when $M > 1$ (i.e., when the receiver has multiple antennas) as considered in this paper, this technique is not applicable.

It can be easily shown that (10) is not a convex function. Accordingly, **(P0)** is a non-convex optimization problem. Therefore, KKT conditions, being necessary but sufficient conditions for non-convex optimization problem, can only be used to identify candidates for the optimal solution. In the following, we use I to denote $I_{X_1,X_2;Y}$.

The Lagrangian equation from (**P0**) is

$$\mathcal{L} = -I + \eta_1(\mu_1 - 1) - \eta_2\mu_1 + \eta_3(\mu_2 - 1) - \eta_4\mu_2,$$

where η_i, for $i = 1, \cdots, 4$ are Lagrangian multipliers.

These KKT conditions can be written as

$$\begin{aligned} \frac{\partial I}{\partial \mu_1}\Big|_{(\hat{\mu}_1,\hat{\mu}_2)} - \eta_1 + \eta_2 &= 0, \\ \frac{\partial I}{\partial \mu_2}\Big|_{(\hat{\mu}_1,\hat{\mu}_2)} - \eta_3 + \eta_4 &= 0, \\ \eta_1(\hat{\mu}_1 - 1) &= 0, \\ \eta_2\hat{\mu}_1 &= 0, \\ \eta_3(\hat{\mu}_2 - 1) &= 0, \\ \eta_4\hat{\mu}_2 &= 0, \end{aligned}$$

where

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} = \sum_{m=1}^{2} \Big(& -(1-\mu_2)\phi(\lambda_m) + (1-\mu_2)\phi(a_m + \lambda_m) \\ & -\mu_2\phi(b_m + \lambda_m) + \mu_2\phi(a_m + b_m + \lambda_m) \\ & -a_m(\log(a_m\mu_1 + b_m\mu_2 + \lambda_m) + 1)\Big), \end{aligned} \tag{13}$$

and

$$\begin{aligned}\frac{\partial I}{\partial \mu_2} &= \sum_{m=1}^{2}\Big(-(1-\mu_1)\phi(\lambda_m) - \mu_1\phi(a_m+\lambda_m) \\ &\quad +(1-\mu_1)\phi(b_m+\lambda_m) + \mu_1\phi(a_m+b_m+\lambda_m) \\ &\quad -b_m(\log(a_m\mu_1+b_m\mu_2+\lambda_m)+1)\Big). \end{aligned} \tag{14}$$

To find the solution for this non-convex optimization problem, the KKT conditions can be solved by considering different combinations of constraints being active at a given time. This yields 16 possible cases, each corresponding to whether $\eta_i, i = 1, \cdots, 4$ being zero or not. Out of these 16 cases, in Appendix A, we show that 13 cases can not be the optimal solution. For example when $\eta_1 = 0, \eta_2 = 0, \eta_3 \neq 0, \eta_4 = 0$, we have

$$\begin{aligned}\frac{\partial I}{\partial \mu_1} &= 0, \\ \frac{\partial I}{\partial \mu_2} - \eta_3 &= 0, \\ \eta_3 \neq 0 \Rightarrow \mu_2 &= 1.\end{aligned}$$

Then the optimal solution must satisfy $\frac{\partial I}{\partial \mu_1}\Big|_{(\mu_1,1)} = 0$. In this case having $\mu_2 = 1$ means that transmitter 2 is transmitting constantly and just imposing interference for the transmitter 1. Hence, $I(\mu_1, 0) \geq I(\mu_1, 1)$. Therefore, we may conclude that this case does not result in a candidate for the optimal solution. Detailed analysis on how to exclude these 13 cases is provided in Appendix A.

The feasible candidates for the optimal solution are listed below.

Case-1: $\eta_1 = 0, \eta_2 \neq 0, \eta_3 = 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned}\frac{\partial I}{\partial \mu_1} + \eta_2 &= 0, \\ \frac{\partial I}{\partial \mu_2} &= 0, \\ \eta_2 \neq 0 \Rightarrow \mu_1 &= 0.\end{aligned}$$

The candidate for the optimal solution is $(0, \mu_2)$, where μ_2 satisfies $\frac{\partial I}{\partial \mu_2}\Big|_{(0,\mu_2)} = 0$, i.e.,

$$\sum_{m=1}^{2}\Big(b_m\log(b_m\mu_2+\lambda_m)\Big) = \sum_{m=1}^{2}\Big(-\phi(\lambda_m)+\phi(b_m+\lambda_m)-b_m\Big). \tag{15}$$

This case corresponds to the scenario when only transmitter 2 is active and transmitter 1 is inactive. Equation (15) shows that the optimal value of μ_2 that satisfies the KKT conditions is the intersection between the right side function of the equation and the left side of the equation. It is easy to check that the left side of (15) is a monotonically increasing function of μ_2, while the right side of (15) is a constant. Therefore, there can be at most one value of μ_2 that satisfies this equation. We use $\tilde{\mu}_2$ to denote the solution to (15). If such solution does not exist or if the solutions lies out of $[0, 1]$, we simply set $\tilde{\mu}_2 = 0$. Hence, a candidate obtained from this case is $(0, \tilde{\mu}_2)$.

Case-2: $\eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \eta_4 \neq 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} &= 0, \\ \frac{\partial I}{\partial \mu_2} + \eta_4 &= 0, \\ \eta_4 \neq 0 \Rightarrow \mu_2 &= 0. \end{aligned}$$

Therefore, the optimal pair must satisfy $\frac{\partial I}{\partial \mu_1}\Big|_{(\mu_1, 0)} = 0$, which yields

$$\sum_{m=1}^{2} \Big(a_m \log(a_m \mu_1 + \lambda_m) \Big) = \sum_{m=1}^{2} \Big(- \phi(\lambda_m) + \phi(a_m + \lambda_m) - a_m \Big). \tag{16}$$

This case corresponds to the scenario when only transmitter 1 is active and transmitter 2 is inactive. It is clear that, similarly to Case-1, μ_1 is the intersection point of a monotonically increasing function and a constant. Therefore there can only be at most one such value of μ_1 that satisfy this equation. Let $(\bar{\mu}_1, 0)$ be the obtained solution, with $\bar{\mu}_1$ setting to zero if such solution does not exist or the solution lies outside of $[0, 1]$.

Case-3: $\eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \eta_4 = 0 \Rightarrow$

$$\frac{\partial I}{\partial \mu_1} = 0, \tag{17}$$

$$\frac{\partial I}{\partial \mu_2} = 0. \tag{18}$$

This case corresponds to the scenario when both transmitters are active. The pair (μ_1, μ_2) must satisfy (17) and (18) simultaneously. Let (μ_1^*, μ_2^*) be the obtained solution.

From (13) and (14), we know that (17) and (18) are a pair of nonlinear equations. The solution can be obtained efficiently by numerical methods. Under certain conditions, we can analyze these nonlinear equations further and draw definite conclusions. These analysis will be presented below.

After obtaining solutions from these three cases, we can then compare the rate obtained from them and set the solution to be the one that results in the largest rate.

In summary, we have the following theorem.

Theorem 1. *The sum-rate capacity of the Poisson MAC,* $C_{sum}^{SIMO-MAC} = I_{X_1X_2;Y}(\hat{\mu}_1, \hat{\mu}_2)$ *where* $(\hat{\mu}_1, \hat{\mu}_2)$ *is given by*

$$(\hat{\mu}_1, \hat{\mu}_2) = \begin{cases} (0, \tilde{\mu}_2) & \textit{if} \quad I(0, \tilde{\mu}_2) \geq \max\left(I(\bar{\mu}_1, 0), I(\mu_1^*, \mu_2^*)\right) \\ (\bar{\mu}_1, 0) & \textit{if} \quad I(\bar{\mu}_1, 0) \geq \max\left(I(0, \tilde{\mu}_2), I(\mu_1^*, \mu_2^*)\right) \\ (\mu_1^*, \mu_2^*) & \textit{otherwise} \end{cases}. \tag{19}$$

From this Theorem, we see that there are three optimality scenarios for achieving the sum-rate capacity: 1) when only transmitter 1 is active and transmitter 2 is inactive; 2) when transmitter 2 is active and transmitter 1 is inactive; and 3) when both transmitters are active. In the following, we will show analytically that single user transmission is indeed optimal for certain scenarios. This is in contrast to the Gaussian MAC where both users must transmit (either simultaneously or at different time) to achieve the sum-rate capacity.

B. Special Cases

In this subsection, we further analyze (17) and (18) and analytically show that there are finite number of solutions for some interesting scenarios.

1) Asymptotic Analysis: In this subsection, we show that when the transmission power of one transmitter is sufficiently higher than the other, then there is no solution to (17) and (18), and hence single user transmission is optimal.

Proposition 2. *For any* $\mu_1 \in (0, 1)$ *and* $\mu_2 \in (0, 1)$*, if* $a_m \to \infty$ *for any* $m \in \{1, 2\}$*, then* $\frac{\partial I}{\partial \mu_2} \to -\infty$.

Proof.

$$
\begin{aligned}
\lim_{a_m\to\infty}\frac{\partial I}{\partial \mu_2} &= \lim_{a_m\to\infty}\sum_{m=1}^{2}\Big(-(1-\mu_1)\phi(\lambda_m)-\mu_1\phi(a_m+\lambda_m)\\
&\quad +(1-\mu_1)\phi(b_m+\lambda_m)+\mu_1\phi(a_m+b_m+\lambda_m)\\
&\quad -b_m(\log(a_m\mu_1+b_m\mu_2+\lambda_m)+1)\Big)\\
&= \lim_{a_m\to\infty}\sum_{m=1}^{2}(1-\mu_1)\Big(\phi(b_m+\lambda_m)+\phi(\lambda_m)\Big)\\
&\quad +\lim_{a_m\to\infty}\sum_{m=1}^{2}\Big(\mu_1(a_m+\lambda_m)\log\left(\frac{b_m}{a_m+\lambda_m}+1\right)+\mu_1 b_m\log(a_m+b_m+\lambda_m)\\
&\quad -b_m\log(a_m\mu_1+b_m\mu_2+\lambda_m)+1\Big)\\
&\overset{(a)}{\le} c+\lim_{a_m\to\infty}\sum_{m=1}^{2}\left(\mu_1 b_m+b_m\mu_1\log(a_m)+\mu_1 b_m\log\left(\frac{b_m+\lambda_m}{a_m}+1\right)\right.\\
&\quad \left.-b_m\log(a_{1m}\mu_1)-b_m\log\left(\frac{b_m+\lambda_m}{a_m}+1\right)-b_m\right)\\
&= c+\lim_{a_m\to\infty}\sum_{m=1}^{2}\Big(\mu_1 b_m-b_m+b_m\mu_1\log(a_m)-b_m\log(a_m\mu_1)\Big)\\
&= c+\lim_{a_m\to\infty}\sum_{m=1}^{2}\Big(-b_m(1-\mu_1)+b_m(\mu_1\log(a_m)-\log(\mu_1)-\log(a_m))\Big)\\
&= c+\lim_{a_m\to\infty}\sum_{m=1}^{2}\Big(-b_m(1-\mu_1)-b_m(1-\mu_1)\log(a_m)-b_m\log(\mu_1)\Big)\\
&= -\infty.
\end{aligned}
$$

where (a) follows because $\ln(1+x)\le x$ and c is a positive constant. □

From Proposition 2, we may conclude that when $a_m\to\infty$ for any $m\in\{1,2\}$, there is no solution for Case-3. Similarly, when $b_m\to\infty$ for any $m\in\{1,2\}$, Case-3 does not lead to any solution. Hence in these scenarios, single user transmission is optimal.

2) Symmetric Channel: Here we show that for the symmetric channel where $a_m=b_m,\forall m\in[1,M]$, there are at most 2 solutions to (17) and (18).

Proposition 3. *If the channel is symmetric,* (17) *and* (18) *have at most two solutions. Furthermore, in these solutions,* $\mu_1=\mu_2$.

Proof. Let $S_mA \triangleq a_m = b_m = S_mA$. Then $\frac{\partial I}{\partial \mu_1} = 0$ yields

$$\sum_{m=1}^{2}\Big(-(1-\mu_2)\phi(\lambda_m) + (1-\mu_2)\phi(S_mA+\lambda_m)$$
$$-\mu_2\phi(S_mA+\lambda_m) + \mu_2\phi(2S_mA+\lambda_m)$$
$$-S_mA(\log(S_mA(\mu_1+\mu_2)+\lambda_m)+1)\Big) = 0.$$

This implies

$$\begin{aligned} &\sum_{m=1}^{2}\Big(S_mA\log(S_mA(\mu_1+\mu_2)+\lambda_m)+S_mA\Big) \\ =\ &\sum_{m=1}^{2}\Big(-(1-\mu_2)\phi(\lambda_m) + (1-\mu_2)\phi(S_mA+\lambda_m) \\ &-\mu_2\phi(S_mA+\lambda_m) + \mu_2\phi(2S_mA+\lambda_m)\Big). \end{aligned} \tag{20}$$

After plugging the value of $\sum_{m=1}^{2}\Big(S_mA\log(S_mA(\mu_1+\mu_2)+\lambda_m)+S_mA\Big)$ from (20) into $\frac{\partial I}{\partial \mu_2} = 0$ and rearranging terms, we obtain

$$\begin{aligned} &\sum_{m=1}^{2}\Big(-(1-\mu_1)\phi(\lambda_m) - \mu_1\phi(S_mA+\lambda_m) \\ &+(1-\mu_1)\phi(S_mA+\lambda_m) + \mu_1\phi(2S_mA+\lambda_m)\Big) \\ =\ &\sum_{m=1}^{2}\Big(-(1-\mu_2)\phi(\lambda_m) + (1-\mu_2)\phi(S_mA+\lambda_m) \\ &-\mu_2\phi(S_mA+\lambda_m) + \mu_2\phi(2S_mA+\lambda_m)\Big). \end{aligned} \tag{21}$$

The equation (21) implies that $\mu_1 = \mu_2$, as the left side and the right side of (21) are the same linear functions of μ_1 and μ_2, respectively.

We can now replace the value of $\mu_1 = \mu_2 = \mu$ in (20) and obtain:

$$\begin{aligned} &\sum_{m=1}^{2}\Bigg[\phi(\lambda_m) - 2\phi(S_mA+\lambda_m) + \phi(2S_mA+\lambda_m) \\ &-S_mA\log(2S_mA\mu+\lambda_m) - S_mA\Bigg] = 0. \end{aligned} \tag{22}$$

It is easy to verify that the left side of (22) is a strictly convex function of μ, while the right side is a constant. Therefore, there can be at most two values of $\mu \in (0,1)$ that satisfies the above equation. □

IV. MIMO-MAC ANALYSIS

In this section, using the results obtained in the SIMO-MAC case presented in Section III, we study the general case of the MIMO-MAC where both transmitters and the receiver are equipped with multiple antennas.

Similarly to the SIMO-MAC, the continuous-time continuous-input discrete-output Poisson MIMO-MAC can be converted to the discrete-time binary-input binary-output MAC. In particular, the input waveform of each antenna can be limited to be piecewise constant waveforms with two levels 0 or A_{nj} for the j^{th} antenna of the transmitter n. Depending on the on-off states of each antenna of transmitter n, there are 2^{J_n} states at transmitter n. In the following, we use $i_n \in [1, \cdots, 2^{J_n}]$ to index each of these 2^{J_n} states at transmitter n. We will use $P_n(i_n)$ to denote the probability that transmitter n lies in state i_n and $\mathbf{p}_n \triangleq [P_n(1), \cdots, P_n(2^{J_n})]$ to denote the vector of probabilities of states at transmitter n. We use the binary variable $b_{nj}(i_n)$ to indicate whether the j^{th} antenna of the transmitter n is on or off at state i_n, i.e., $b_{nj}(i_n) = 1$ if the j^{th} antenna of transmitter n is on for state i_n and is 0 otherwise. The sum-rate achievable using $(\mathbf{p}_1, \mathbf{p}_2)$ is given by

$$I_{\mathbf{X}_N;Y}(\mathbf{p}_1, \mathbf{p}_2) = \sum_{i_1=1}^{2^{J_1}} \sum_{i_2=1}^{2^{J_2}} \left[P_1(i_1) P_2(i_2) \sum_{m=1}^{M} \left[\zeta \left(\sum_{n=1}^{2} \sum_{j=1}^{J_n} S_{njm} A_{nj} b_{nj}(i_n), \lambda_m \right) \right] \right]$$
$$- \sum_{m=1}^{M} \left[\zeta \left(\sum_{n=1}^{2} \sum_{j=1}^{J_n} S_{njm} A_{nj} \mu_{nj}, \lambda_m \right) \right]. \tag{23}$$

It is easy to see that

$$\mu_{nj} = \sum_{i_n=1}^{2^{J_n}} P_n(i_n) b_{nj}(i_n). \tag{24}$$

To characterize the sum-rate capacity, we need to solve the following optimization problem:

$$\textbf{(P1):} \quad C_{sum}^{MIMO-MAC} = \max_{\mathbf{p}_1, \mathbf{p}_2} I_{\mathbf{X}_N;Y}(\mathbf{p}_1, \mathbf{p}_2), \tag{25}$$

$$\text{s.t} \quad 0 \le P_n(i_n) \le 1, \tag{26}$$

$$i_n = 1, \cdots, 2^{J_n}, n = 1, 2, \tag{27}$$

$$\sum_{i_n=1}^{2^{J_n}} P_n(i_n) = 1, n = 1, 2. \tag{28}$$

Problem (**P1**) is a complex non-convex optimization problem with a large number of variables. In particular, the number of variables $2^{J_1} + 2^{J_2}$ increases exponentially with the number of antennas. The main result of this section is the following theorem.

Theorem 4. *Solving problem (**P1**) is equivalent to solving the following problem*

$$\textbf{(P2)}\textit{:} \quad C_{sum}^{MIMO-MAC} \;=\; \max_{0\leq\mu_1,\mu_2\leq 1} I(\mu_1,\mu_2), \tag{29}$$

where

$$\begin{aligned} I(\mu_1,\mu_2) \;=\; & \sum_{m=1}^{M}\Big[(1-\mu_1)(1-\mu_2)\varphi(\lambda_m) \\ & +\mu_1(1-\mu_2)\varphi(a_m+\lambda_m)+(1-\mu_1)\mu_2\varphi(b_m+\lambda_m) \\ & +\mu_1\mu_2\varphi(a_m+b_m+\lambda)-\varphi(a_m\mu_1+b_m\mu_2+\lambda_m)\Big], \end{aligned} \tag{30}$$

with

$$a_m \triangleq \sum_{j_1}^{J_1} S_{1j_1m}A_{1j_1}, \tag{31}$$

$$b_m \triangleq \sum_{j_2}^{J_2} S_{2j_2m}A_{2j_2}. \tag{32}$$

Note that the right hand side of problem **(P2)** has the same form as **(P0)** solved in the SIMO-MAC case presented in Section III. Theorem 4 states that the sum-rate capacity of MIMO-MAC is the same as the sum-rate capacity of a properly constructed SIMO-MAC. The enabling element of our result is that in the MIMO-MAC, we show that to achieve the sum-rate capacity, all antennas of the same transmitter must be simultaneously on or off. This enables us to view these antennas of the same transmitter as one antenna with properly modified parameters.

Using Theorem 4 and results from the SIMO case, we know that three are different cases for the optimal inputs to achieve the sum-rate capacity of the Poisson MIMO-MAC. The three optimal solutions correspond to 1) the scenario where only transmitter 2 is active with both antennas being simultaneously on or off; 2) the scenario where only transmitter 1 is active with both antennas being simultaneously on or off; and 3) the scenario where both transmitters are active with antennas at transmitter 1 being simultaneously on or off.

A. Proof of Theorem 4

The proof of Theorem 4 follows a two-step structure and relies on Propositions 5 and 6 presented below. The proof of proposition 5 is similar to the proof of Proposition 7 in [20]. The proof of proposition 6, however, is significantly different because the proof method used in [20] is not applicable when there are multiple antennas at the receiver.

For the presentation convenience, we focus on the case $M = 2$ in this section. The proof for the cases with $M > 2$ is the same.

In Step-1, we prove the following proposition that simplifies the optimization problem from $(\mathbf{p}_1, \mathbf{p}_2)$ to $\boldsymbol{\mu}$.

Proposition 5. *At the optimality, for each transmitter, if the antenna with a smaller duty cycle is on then all antennas with a larger duty cycle must also be on.*

Proof. See Appendix B. □

In Step-2, we show the following proposition that characterizes the optimal value of $\boldsymbol{\mu}$.

Proposition 6. *At the optimality, for each transmitter, all antennas must have the same duty cycle and they must be on or off simultaneously.*

Proof. See Appendix C. □

From these two propositions, we prove Theorem 4, which shows that a MIMO-MAC channel can be converted to a SIMO-MAC channel with appropriate channels.

V. Numerical Results

In this section, we present numerical examples to illustrate the results obtained in the previous sections.

In the first example, we set $a_1 = a_2 = 5, b_1 = b_2 = 5, \lambda = 0.25$. The sum-rate achieved by three different cases are

Case - 1: $\tilde{\mu}_1 = 0, \tilde{\mu_2} = 0.25 \rightarrow I(\tilde{\mu}_1, \tilde{\mu_2}) = 2.616$.

Case - 2: $\bar{\mu}_1 = 0.25, \bar{\mu}_2 = 0 \rightarrow I(\bar{\mu}_1, \bar{\mu}_2) = 2.616$.

Case - 3: $\mu_1 = 0.2987, \mu_2 = 0.2987 \rightarrow I(\mu_1, \mu_2) = 3.6057$.

Therefore, for this channel, Case-3 where both transmitters are active achieves the sum-rate capacity.

For the next example, we set $a_1 = a_2 = 4, b_1 = b_2 = 10, \lambda = 0.25$. The sum-rate achieved by three different cases are

Case - 1: $\tilde{\mu}_1 = 0, \tilde{\mu}_2 = 0.3888 \rightarrow I(\tilde{\mu}_1, \tilde{\mu_2}) = 6.3720$.

Case - 2: $\bar{\mu}_1 = 0.4041, \bar{\mu}_2 = 0 \rightarrow I(\bar{\mu}_1, \bar{\mu}_2) = 2.276$.

Case - 3: Does not result in a solution, therefore $I_(\mu_1, \mu_2) = 0$.

Therefore, for this channel, Case-1 where only transmitter 2 is active achieves the sum-rate capacity. This confirms our conclusion that if the power of one transmitter is relatively high, it is optimal for only this user to be active to achieve the sum-rate capacity.

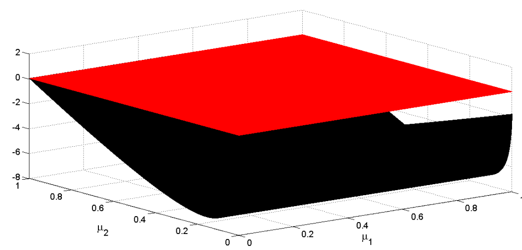

Fig. 2: $\frac{\partial I}{\partial \mu_2}$ and the zero plane.

Fig. 2 illustrates the Proposition 2. In this figure, the red surface is the zero plane, and the black surface is $\frac{\partial I}{\partial \mu_2}$ for different values of μ_1 and μ_2. In generating this figure, we set $a_1 = a_2 = 450$, $b_1 = b_2 = 1$ and $\lambda_1 = \lambda_2 = 0.25$. From the figure, we can see that, for this set of parameters, $\frac{\partial I}{\partial \mu_2}$ has no intersection with the zero plane (except on the boundary). This confirms our conclusion that, if the transmitter of one transmitter is relatively high, Case-3 does not yield a solution.

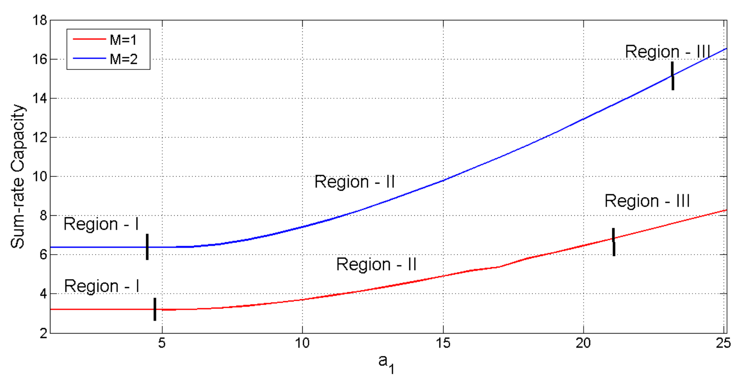

Fig. 3: Sum-rate capacity with respect to transmission power at transmitter 1 when $M = 1$ and $M = 2$. Region I corresponds to the case when only transmitter 2 is transmitting, Region II is when both transmitters are transmitting and Region III is when only transmitter 1 is transmitting.

Fig. 3 illustrates how the sum-rate capacities for the channels with different number of receiving antennas increases as the transmission power increases. In generating Fig. 3, we set $b_1 = b_2 = 10$ and $\lambda_1 = \lambda_2 = 0.25$, while increasing the value of $a_1 = a_2$. The sum-rate capacity increases as the available transmission power at transmitter 1 increases but this slope is larger when the receiver has multiple receiving antennas. In the figure, we also mark three regions corresponding to different input scenarios that achieve the sum-rate capacity. In Region-I, when the value of a_1 is small, it is optimal to allow only transmitter 2 to be active. When $a_1 = 4.7$ for $M = 1$ and $a_1 = 4.5$ for $M = 2$, both curves transit into Region-II where both transmitters must be active to achieve the sum-rate capacity and when the value of a_1 is high enough ($a_1 = 21.1$ for $M = 1$ and $a_1 = 23$ for $M = 2$), the curves are in Region-III where it is optimal for only transmitter 1 to be active.

VI. Conclusion

In this paper, we have characterized the sum-rate capacity of the Poisson MIMO-MAC. We have shown that the sum-rate capacity of a Poisson MIMO MAC can be characterized by studying a carefully constructed Poisson SIMO MAC. We have also shown that there are three possible operating scenarios for achieving the sum-rate capacity of the Poisson SIMO MAC. We have shown that it is optimal for either a single user to transmit or both transmitters to transmit depending on channel parameters. This is completely in contrast with the Gaussian channels where all of the users must transmit (either simultaneously or at different times) in order to achieve the sum-rate capacity.

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APPENDIX A
DETAILED ANALYSIS OF THE SIMO-MAC:

In order to find out the optimal solution to the given problem, we analyze the following 16 cases under different constraints.

Case-1: $\eta_1 = 0, \eta_2 \neq 0, \eta_3 = 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} + \eta_2 &= 0, \\ \frac{\partial I}{\partial \mu_2} &= 0 \\ \eta_2 \neq 0 \Rightarrow \mu_1 &= 0. \end{aligned}$$

Therefore the candidate for the optimal solution is $(0, \mu_2)$, where μ_2 satisfies $\frac{\partial I}{\partial \mu_2}\Big|_{(0,\mu_2)} = 0$. This case corresponds to the scenario when transmitter 1 is inactive and transmitter 2 is active.

Case-2: $\eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \eta_4 \neq 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} &= 0, \\ \frac{\partial I}{\partial \mu_2} + \eta_4 &= 0, \\ \eta_4 \neq 0 \Rightarrow \mu_2 &= 0. \end{aligned}$$

Therefore the optimal pair must satisfy $\frac{\partial I}{\partial \mu_1}\Big|_{(\mu_1,0)} = 0$. This case corresponds to the scenario when transmitter 1 is active and transmitter 2 is inactive.

Case-3: $\eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} &= 0, \\ \frac{\partial I}{\partial \mu_2} &= 0. \end{aligned}$$

This case corresponds to the scenario when it is optimal for both transmitters to transmit. The pair (μ_1, μ_2) must satisfy both of the equations simultaneously.

Case-4: $\eta_1 = 0, \eta_2 = 0, \eta_3 \neq 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} &= 0, \\ \frac{\partial I}{\partial \mu_2} - \eta_3 &= 0, \\ \eta_3 \neq 0 \Rightarrow \mu_2 &= 1. \end{aligned}$$

Then the optimal solution must satisfy $\frac{\partial I}{\partial \mu_1}\Big|_{(\mu_1,1)} = 0$. Since $I(\mu_1, 0) \geq I(\mu_1, 1)$, we conclude that this case does not result in a candidate for the optimal solution.

Case-5: $\eta_1 = 0, \eta_2 = 0, \eta_3 \neq 0, \eta_4 \neq 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} &= 0, \\ \frac{\partial I}{\partial \mu_2} - \eta_3 + \eta_4 &= 0, \\ \eta_3 \neq 0 \Rightarrow \mu_2 &= 1, \\ \eta_4 \neq 0 \Rightarrow \mu_2 &= 0. \end{aligned}$$

It is clear that this case is not possible.

Case-6: $\eta_1 = 0, \eta_2 \neq 0, \eta_3 = 0, \eta_4 \neq 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} + \eta_2 &= 0, \\ \frac{\partial I}{\partial \mu_2} + \eta_4 &= 0, \\ \eta_2 \neq 0 \Rightarrow \mu_1 &= 0, \\ \eta_4 \neq 0 \Rightarrow \mu_2 &= 0. \end{aligned}$$

It is clear that this case does not result in a candidate for the optimal solution.

Case-7: $\eta_1 = 0, \eta_2 \neq 0, \eta_3 \neq 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} + \eta_2 &= 0, \\ \frac{\partial I}{\partial \mu_2} - \eta_3 &= 0, \\ \eta_2 \neq 0 \Rightarrow \mu_1 &= 0, \\ \eta_3 \neq 0 \Rightarrow \mu_2 &= 1. \end{aligned}$$

It is clear that $\mu_1 = 0, \mu_2 = 1$ is not a candidate for the optimal solution.

Case-8: $\eta_1 = 0, \eta_2 \neq 0, \eta_3 \neq 0, \eta_4 \neq 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} + \eta_2 &= 0, \\ \frac{\partial I}{\partial \mu_2} - \eta_3 + \eta_4 &= 0, \\ \eta_2 \neq 0 \Rightarrow \mu_1 &= 0, \\ \eta_3 \neq 0 \Rightarrow \mu_2 &= 1, \\ \eta_4 \neq 0 \Rightarrow \mu_2 &= 0. \end{aligned}$$

It is clear that it is not a feasible case.

Case-9: $\eta_1 \neq 0, \eta_2 = 0, \eta_3 = 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} - \eta_1 &= 0, \\ \frac{\partial I}{\partial \mu_2} &= 0, \\ \eta_1 \neq 0 \Rightarrow \mu_1 &= 1. \end{aligned}$$

Similarly to Case-4, this case does not result in a candidate for optimal solution because $I(0,\mu_2) \geq I(1,\mu_1)$.

Case-10: $\eta_1 \neq 0, \eta_2 = 0, \eta_3 = 0, \eta \neq 0 \Rightarrow$

$$\begin{aligned}\frac{\partial I}{\partial \mu_1} - \eta_1 &= 0,\\ \frac{\partial I}{\partial \mu_2} + \eta_4 &= 0,\\ \eta_1 \neq 0 \Rightarrow \mu_1 &= 1,\\ \eta_4 \neq 0 \Rightarrow \mu_2 &= 0.\end{aligned}$$

It is clear that $I(1,0)$ does not result in a candidate for the optimal value of the sum-rate.

Case-11: $\eta_1 \neq 0, \eta_2 = 0, \eta_3 \neq 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned}\frac{\partial I}{\partial \mu_1} - \eta_1 &= 0,\\ \frac{\partial I}{\partial \mu_2} - \eta_3 &= 0,\\ \eta_1 \neq 0 \Rightarrow \mu_1 &= 1,\\ \eta_3 \neq 0 \Rightarrow \mu_2 &= 1.\end{aligned}$$

It is clear that this case does not result in the optimal candidate.

Case-12: $\eta_1 \neq 0, \eta_2 = 0, \eta_3 \neq 0, \eta_4 \neq 0 \Rightarrow$

$$\begin{aligned}\frac{\partial I}{\partial \mu_1} - \eta_1 &= 0,\\ \frac{\partial I}{\partial \mu_2} - \eta_3 + \eta_4 &= 0,\\ \eta_1 \neq 0 \Rightarrow \mu_1 &= 1,\\ \eta_3 \neq 0 \Rightarrow \mu_2 &= 1,\\ \eta_4 \neq 0 \Rightarrow \mu_2 &= 0.\end{aligned}$$

This case is not feasible due to the conflicting values of μ_2.

Case-13: $\eta_1 \neq 0, \eta_2 \neq 0, \eta_3 = 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned}\frac{\partial I}{\partial \mu_1} - \eta_1 + \eta_2 &= 0,\\ \frac{\partial I}{\partial \mu_2} &= 0,\\ \eta_1 \neq 0 \Rightarrow \mu_1 &= 1,\\ \eta_2 \neq 0 \Rightarrow \mu_1 &= 0.\end{aligned}$$

Clearly this case is not feasible.

Case-14: $\eta_1 \neq 0, \eta_2 \neq 0, \eta_3 = 0, \eta_4 \neq 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} - \eta_1 + \eta_2 &= 0, \\ \frac{\partial I}{\partial \mu_2} + \eta_4 &= 0, \\ \eta_1 \neq 0 \Rightarrow \mu_1 &= 1, \\ \eta_2 \neq 0 \Rightarrow \mu_1 &= 0, \\ \eta_4 \neq 0 \Rightarrow \mu_2 &= 0. \end{aligned}$$

This case is not feasible.

Case-15: $\eta_1 \neq 0, \eta_2 \neq 0, \eta_3 \neq 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} - \eta_1 + \eta_2 &= 0, \\ \frac{\partial I}{\partial \mu_2} - \eta_3 &= 0, \\ \eta_1 \neq 0 \Rightarrow \mu_1 &= 1, \\ \eta_2 \neq 0 \Rightarrow \mu_1 &= 0, \\ \eta_3 \neq 0 \Rightarrow \mu_2 &= 1. \end{aligned}$$

This case is not feasible.

Case-16: $\eta_1 \neq 0, \eta_2 \neq 0, \eta_3 \neq 0, \eta_4 \neq 0 \Rightarrow$

$$\begin{aligned} \frac{\partial I}{\partial \mu_1} - \eta_1 + \eta_2 &= 0, \\ \frac{\partial I}{\partial \mu_2} + \eta_4 &= 0, \\ \eta_1 \neq 0 \Rightarrow \mu_1 &= 1, \\ \eta_2 \neq 0 \Rightarrow \mu_1 &= 0, \\ \eta_3 \neq 0 \Rightarrow \mu_2 &= 1, \\ \eta_4 \neq 0 \Rightarrow \mu_2 &= 0. \end{aligned}$$

This case is not feasible.

APPENDIX B
PROOF OF PROPOSITION 5

The proof of this proposition is similar to the proof of Proposition 7 in [20] with proper modification.

Proof. We prove Proposition 5 by characterizing the optimal value of $(\mathbf{p}_1, \mathbf{p}_2)$ for any given $\boldsymbol{\mu}$. Hence, in this subsection, $\boldsymbol{\mu}$ is fixed. More specifically, we show that, at the optimality in the MIMO-MAC, if the antenna with the smaller duty cycle is on, then the other antenna should also be on.

From (23), it is clear that to optimize over $(\mathbf{p}_1, \mathbf{p}_2)$ for a given $\boldsymbol{\mu}$, we only need to focus on

$$\begin{aligned}\mathcal{H} &\triangleq \sum_{i_1=1}^{2^{J_1}} \sum_{i_2=1}^{2^{J_2}} P_1(i_1)P_2(i_2) \sum_{m=1}^{2} \zeta\left(\sum_{n=1}^{2}\sum_{j=1}^{J_n} S_{njm}A_{nj}b_{nj}(i_n), \lambda_m\right) \\ &= \sum_{i_1=1}^{2^{J_1}} P_1(i_1)\mathcal{H}_1(i_1), \end{aligned} \tag{33}$$

where

$$\mathcal{H}_1(i_1) = \sum_{i_2=1}^{2^{J_2}} P_2(i_2) \sum_{m=1}^{2} d(i_1, i_2), \tag{34}$$

and

$$d(i_1, i_2) \quad = \quad \zeta\left(\sum_{j=1}^{J_1} S_{1jm}A_{1j}b_{1j}(i_1) + \sum_{j=1}^{J_2} S_{2jm}A_{2j}b_{2j}(i_2), \lambda_m\right). \tag{35}$$

We focus on finding the optimal values of $\mathbf{p}_2$ first. To facilitate the understanding, we list the labeling of states of transmitter 2 and the corresponding values of b_{2j}s in Table I.

	b_{21}	b_{22}
$(i_1, 1)$	0	0
$(i_1, 2)$	0	1
$(i_1, 3)$	1	0
$(i_1, 4)$	1	1

TABLE I: The states of transmitter 2 and the corresponding values of b_{2j}s.

Using the definition of ζ function, we can easily check that

$$d(i_1, 1) < \min\{d(i_1, 2), d(i_1, 3)\} \leq \max\{d(i_1, 2), d(i_1, 3)\} < d(i_1, 4),$$

which is simultaneously true for any value of i_1. Since $\mathcal{H}_1(i_1)$ is simply a linear combination of $d(i_1, i_2)$s, for any given $\boldsymbol{\mu}$, maximizing $\mathcal{H}_1(i_1)$ is a linear programming problem, for which we have the following (assuming $\mu_{21} \geq \mu_{22}$, the other case being similar):

1) Since $d(i_1, 4)$ is the largest, $P_2(4)$ should be as large as possible. Therefore, we assign $P_2(4) = \mu_{22}$.
2) Since μ_{22} has been all used, we should set $P_2(2) = 0$.
3) Since $d(i_1, 3) > d(i_1, 1)$, we assign the remaining part of μ_{21} to state $(i_1, 3)$ and hence $P_2(3) = \mu_{21} - \mu_{22}$.
4) For the last state related to the term $d(i_1, 1)$, allot the remaining probability. Hence $P_2(1) = 1 - \mu_{21}$.

This assignment implies that if the antenna with a smaller duty cycle is on, the antenna with a larger duty cycle should also be on. Note that the above arguments are true for all i_1s, and hence this assignment maximizes $\mathcal{H}_1(i_1)$ for all i_1 simultaneously. Furthermore, this assignment is independent of $\mathbf{p}_1$.

Note that the above discussion for $J_2 = 2$ can be extended for $J_2 > 2$.

Similarly, by writing

$$\mathcal{H} = \sum_{i_2=1}^{2^{J_2}} P_2(i_2) \sum_{i_1=1}^{2^{J_1}} P_1(i_1) \sum_{m=1}^{2} \zeta \left(\sum_{j=1}^{J_1} S_{1jm} A_{1j} b_{1j}(i_1) + \sum_{j=1}^{J_2} S_{2jm} A_{2j} b_{2j}(i_2), \lambda_m \right),$$

and following the same procedure as above, we can calculaandte the optimal values of $\mathbf{p}_1$.

As the result, we know that (23) can be simplified to a function $\boldsymbol{\mu}$ depending on the relationships between the values of μ_{nj}s. For example, in the case of two transmitter antennas, we have four symmetric cases, i.e., $(\mu_{11} \geq \mu_{12}, \mu_{21} \geq \mu_{22}), (\mu_{11} \leq \mu_{12}, \mu_{21} \geq \mu_{22}), (\mu_{11} \geq \mu_{12}, \mu_{21} \leq \mu_{22})$ and $(\mu_{11} \leq \mu_{12}, \mu_{21} \leq \mu_{22})$. For the case of $(\mu_{11} \geq \mu_{12}, \mu_{21} \geq \mu_{22})$, $I_{\mathbf{X}_N;Y}$ can be simplified to

$$
\begin{aligned}
& I(\mu_{11}-\mu_{12},\mu_{12},\mu_{21}-\mu_{22},\mu_{22}) \\
& =\sum_{m=1}^{2}\Bigg[(1-\mu_{11})((1-\mu_{21})\varphi(\lambda_m)+(\mu_{21}-\mu_{22})\varphi(b_{1m}+\lambda_m) \\
& \quad +\mu_{22}\varphi(b_{1m}+b_{2m}+\lambda_m))+ \\
& \quad +(\mu_{11}-\mu_{12})((1-\mu_{21})\varphi(a_{1m}+\lambda_m) \\
& \quad +(\mu_{21}-\mu_{22})\varphi(a_{1m}+b_{1m}+\lambda_m)+ \\
& \quad +\mu_{22}\varphi(a_{1m}+b_{1m}+b_{2m}+\lambda_m)) \\
& \quad +\mu_{12}((1-\mu_{21})\varphi(a_{1m}+a_{2m}+\lambda_m)+ \\
& \quad +(\mu_{21}-\mu_{22})\varphi(a_{1m}+a_{2m}+b_{1m}+\lambda_m)+ \\
& \quad +\mu_{22}\varphi(a_{1m}+a_{2m}+b_{1m}+b_{2m}+\lambda_m)) \\
& \quad -\varphi\left(\sum_{n=1}^{2}\sum_{j=1}^{J_n}S_{njm}A_{nj}\mu_{nj}+\lambda_m\right)\Bigg]. \qquad (36)
\end{aligned}
$$

As the result, the objective function is simplified to characterizing

$$
\begin{aligned}
C_{sum}^{MIMO-MAC} = \max\big(& C_{\mu_{11}\geq\mu_{12},\mu_{21}\geq\mu_{22}}, C_{\mu_{11}\leq\mu_{12},\mu_{21}\geq\mu_{22}}, \\
& C_{\mu_{11}\geq\mu_{12},\mu_{21}\leq\mu_{22}}, C_{\mu_{11}\leq\mu_{12},\mu_{21}\leq\mu_{22}}\big), \qquad (37)
\end{aligned}
$$

in which

$$
\textbf{(P3):}\ C_{\mu_{11}\geq\mu_{12},\mu_{21}\geq\mu_{22}} = \max I(\mu_{11}-\mu_{12},\mu_{12},\mu_{21}-\mu_{22},\mu_{22}), \qquad (38)
$$

$$
\text{s.t.}\quad 0\leq\mu_{12}\leq\mu_{11}\leq 1, \qquad (39)
$$

$$
0\leq\mu_{22}\leq\mu_{21}\leq 1. \qquad (40)
$$

Other terms in (37) are defined in a similar manner. Due to symmetry, we provide details only on how to solve (**P3**) in Step-2. □

APPENDIX C
PROOF OF PROPOSITION 6

Proof. For the ease of calculation, we define $q_1 = \mu_{11} - \mu_{12}$, $q_2 = \mu_{12}$, $q_3 = \mu_{21} - \mu_{22}$ and $q_4 = \mu_{22}$ and let $\mathbf{q} = [q_1, q_2, q_3, q_4]$. Then (36) can be re-written as

$$\begin{aligned} I(\mathbf{q}) \;=\; \sum_{m=1}^{2} \Bigg[& (1-(q_1+q_2))((1-(q_3+q_4))\varphi(\lambda_m) \\ & +q_3\varphi(b_{1m}+\lambda_m)+q_4\varphi(b_{1m}+b_{2m}+\lambda_m)) \\ & +q_1((1-(q_3+q_4))\varphi(a_{1m}+\lambda_m)+q_3\varphi(b_{1m}+a_{1m}+\lambda_m) \\ & +q_4\varphi(b_{1m}+b_{2m}+a_{1m}+\lambda_m)) \\ & +q_2((1-(q_3+q_4))\varphi(a_{1m}+a_{2m}+\lambda_m) \\ & +q_3\varphi(b_{1m}+a_{1m}+a_{2m}+\lambda_m) \\ & +q_4\varphi(b_{1m}+b_{2m}+a_{1m}+a_{2m}+\lambda_m)) \\ & -\varphi(a_{1m}q_1+(a_{1m}+a_{2m})q_2+b_{1m}q_3+(b_{1m}+b_{2m})q_4+\lambda_m)\Bigg]. \end{aligned} \tag{41}$$

Correspondingly, (**P3**) is equivalent to

$$\textbf{(P4):}\quad C_{\mu_{11}\geq\mu_{12},\mu_{21}\geq\mu_{22}} \;=\; \max I(\mathbf{q}) \tag{42}$$

$$\text{s.t.}\quad q_k \geq 0,\ k=1,\cdots,4, \tag{43}$$

$$q_1+q_2 \leq 1, \tag{44}$$

$$q_3+q_4 \leq 1. \tag{45}$$

Now in order to find the optimal solution for (**P4**), we show that any sum-rate achievable when both antennas at both transmitters 1 and 2 are active with different duty cycles (called scheme A), can be achieved by setting the duty cycles of antennas of the same transmitter to be the same (called scheme B) in a properly constructed weaker channel. This implies that, for the original channel, we can restrict to the case where the antennas of the same transmitter are simultaneously on or off without losing optimality.

For scheme A, let q_1 and q_2 be the duty cycles of each of the antennas at transmitter 1 and q_3 and q_4 be the duty cycle of each antenna at transmitter 2, respectively.

$$\begin{aligned} I_A(\mathbf{q}) \;=\; \sum_{m=1}^{2} \Bigg[& (1-(q_1+q_2))((1-(q_3+q_4))\varphi(\lambda_m) \\ & +q_3\varphi(b_{1m}+\lambda_m)+q_4\varphi(b_{1m}+b_{2m}+\lambda_m)) \\ & +q_1((1-(q_3+q_4))\varphi(a_{1m}+\lambda_m)+q_3\varphi(b_{1m}+a_{1m}+\lambda_m) \\ & +q_4\varphi(b_{1m}+b_{2m}+a_{1m}+\lambda_m)) \\ & +q_2((1-(q_3+q_4))\varphi(a_{1m}+a_{2m}+\lambda_m) \\ & +q_3\varphi(b_{1m}+a_{1m}+a_{2m}+\lambda_m) \\ & +q_4\varphi(b_{1m}+b_{2m}+a_{1m}+a_{2m}+\lambda_m)) \\ & -\varphi(a_{1m}q_1+(a_{1m}+a_{2m})q_2+b_{1m}q_3+(b_{1m}+b_{2m})q_4+\lambda_m)\Bigg]. \end{aligned} \tag{46}$$

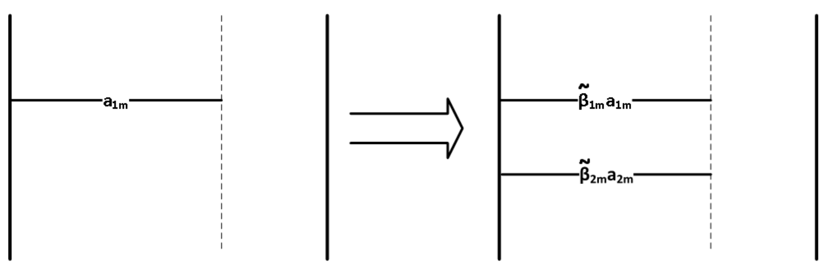

Fig. 4: Transformation from Scheme A to Scheme B.

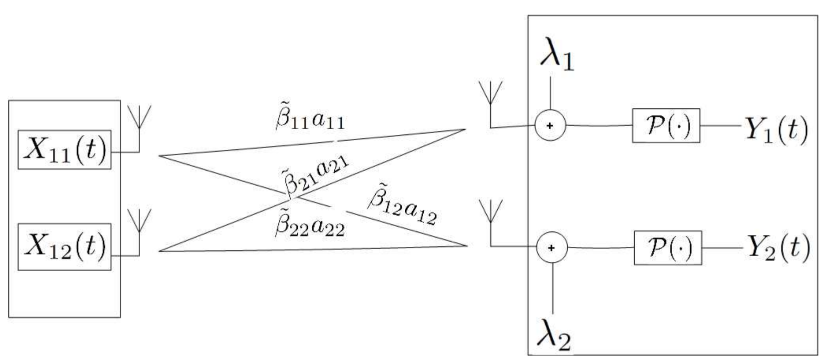

Fig. 5: Scheme B elaborated.

Now we show that $I_A(\mathbf{q})$ can be achieved in a weakened channel but with both antennas to be simultaneously on or off. In particular, in the weakened channel, we restrict the channel gains for the stage of Scheme A when only the stronger antenna is active. For transmitter 1, as shown in Fig. 5, we reduce each a_{jm} to $\tilde{\beta}_{jm}a_{jm}$ by reducing S_{1jm} to $\tilde{\beta}_{jm}S_{1jm}$ with $0 \leq \tilde{\beta}_{jm} \leq 1$. We choose the values of $\tilde{\beta}_{jm}$ such that

$$a_{11} = \tilde{\beta}_{11}a_{11} + \tilde{\beta}_{21}a_{21}, \tag{47}$$

$$a_{12} = \tilde{\beta}_{12}a_{12} + \tilde{\beta}_{22}a_{22}. \tag{48}$$

It is easy to check that we can always find $\tilde{\beta}_{jm}$s in $[0, 1]$ that satisfy (47) and (48).

Same can be done for transmitter 2 by choosing $b_{1m} = \bar{\beta}_{1m}b_{1m} + \bar{\beta}_{2m}b_{2m}$ by restricting the channel gain at each of the channels for transmitter 2. We know that this scheme is also feasible for $0 \leq \bar{\beta}_{1m}, \bar{\beta}_{2m} \leq 1$. Therefore, the sum-rate capacity for scheme B is:

$$\begin{aligned} I_B(\mathbf{q}) \;=\; \sum_{m=1}^{2} \Bigg[& (1-(q_1+q_2))((1-(q_3+q_4))\varphi(\lambda_m) \\ & +q_3\varphi(\bar{\beta}_{1m}b_{1m} + \bar{\beta}_{2m}b_{2m} + \lambda_m) + q_4\varphi(b_{1m} + b_{2m} + \lambda_m)) \\ & +q_1((1-(q_3+q_4))\varphi(\tilde{\beta}_{1m}a_{1m} + \tilde{\beta}_{2m}a_{2m} + \lambda_m) \\ & +q_3\varphi(\bar{\beta}_{1m}b_{1m} + \bar{\beta}_{2m}b_{2m} + \tilde{\beta}_{1m}a_{1m} + \tilde{\beta}_{2m}a_{2m} + \lambda_m) \\ & +q_4\varphi(b_{1m} + b_{2m} + \tilde{\beta}_{1m}a_{1m} + \tilde{\beta}_{2m}a_{2m} + \lambda_m)) \\ & +q_2((1-(q_3+q_4))\varphi(a_{1m} + a_{2m} + \lambda_m) \\ & +q_3\varphi(\bar{\beta}_{1m}b_{1m} + \bar{\beta}_{2m}b_{2m} + a_{1m} + a_{2m} + \lambda_m) \\ & +q_4\varphi(b_{1m} + b_{2m} + a_{1m} + a_{2m} + \lambda_m)) \\ & -\varphi(a_{1m}q_1 + (a_{1m} + a_{2m})q_2 + b_{1m}q_3 + (b_{1m} + b_{2m})q_4 + \lambda_m) \Bigg]. \end{aligned} \tag{49}$$

Clearly, we have $I_A = I_B$. Therefore, we conclude that any sum-rate achievable by both antennas at transmitter be active, at different duty cycles, can also be achieved by letting both antennas of each user to be simultaneously on or off. Note that scheme A represents a channel model with strong channel gains and scheme B represents a channel model with weak channel gains. Hence, we conclude that in order to achieve the sum-rate capacity, the duty cycles of antennas of the same transmitter should be simultaneously on or off. $\square$