The Water-Filling Game in Fading Multiple-Access Channels

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***Abstract*—A game-theoretic framework is developed to design and analyze the resource allocation algorithms in fading multiple-access channels (MACs), where the users are assumed to be selfish, rational, and limited by average power constraints. The maximum sum-rate point on the boundary of the MAC capacity region is shown to be the unique Nash equilibrium of the corresponding water-filling game. This result sheds a new light on the opportunistic communication principle. The base station is then introduced as a player interested in maximizing a weighted sum of the individual rates. A Stackelberg formulation is proposed in which the base station is the designated game leader. In this setup, the base station announces first its strategy defined as the decoding order of the different users, in the successive cancellation receiver, as a function of the channel state. In the second stage, the users compete conditioned on this particular decoding strategy. This formulation is shown to be able to achieve all the corner points of the capacity region, in addition to the maximum sum-rate point. On the negative side, it is shown that there does not exist a base station strategy in this formulation that achieves the rest of the boundary points. To overcome this limitation, a repeated game approach, which achieves the capacity region of the fading MAC, is presented. Finally, the study is extended to vector channels highlighting interesting differences between this scenario and the scalar channel case.**

***Index Terms*—Fading, multiple access, Nash equilibrium, power control, resource allocation.**

I. Introduction

THE design and analysis of efficient resource allocation algorithms for wireless channels has received significant research interest for many years. In a pioneering work, Tse and Hanly have characterized the capacity region of the fading multiple-access channel (MAC) and the corresponding optimal power and rate allocation policies [1]. The centralized nature of these policies relies on the assumption that the multiple access users will implement the power and rate allocation schemes dictated by the base station. On the other hand, when the users are selfish, there maybe an incentive for some, or all, of them to deviate from the centralized policies if the policies are not compatible with their individual interests. This motivates our work here on the design and analysis of distributed strategies that approach the optimal performance and are compatible with the selfish nature of the users. Arguably, such algorithms are more desirable from a practical perspective, since there is no incentive for the users to deviate from the specified policies.

We adopt a game-theoretic framework where the users are typically modeled as rational and selfish players interested in maximizing the utilities they obtain from the network. The selfish behavior implies that individual users do not care about the overall system performance. Over the last ten years, game-theoretic tools have been used to design distributed resource allocation strategies in a variety of contexts. For example, Mackenzie *et al.* consider the collision channel [2], Yu *et al.* focus on the digital subscriber line setup [3], Etkin *et al.* investigate the power allocation game in the Gaussian interference channel [4], [5], and La *et al.* model the power control problem in Gaussian MACs as a cooperative game where the users are allowed to form coalitions [6]. Probably the scenario closest to our work is the design of distributed power control algorithms for the uplink of code division multiple access (CDMA) systems considered in e.g., [7]–[12]. These papers focus on **time-invariant** channels and construct utility functions that allow the users to reach a socially optimal equilibrium. These works, however, reach the **negative** conclusion that the selfish behavior entails a fundamental performance loss in the sense that the achievable utilities at the equilibria points,[1] if they exist, are usually inefficient as compared with the centralized policy [7], [11]. The central contribution of this paper is showing how to overcome this negative conclusion in fading channels by exploiting the time varying nature of fading, modeling the base station as an additional player with the appropriate decoding strategy, and resorting to a repeated game formulation if needed.

We start with a static Nash formulation which only models the multiple access users as players interested in maximizing its achievable rate subject to an average power constraint. In this formulation, the base station is not allowed to explicitly influence the decision-making process of the multiple access players. It is assumed that every player treats the signals of other users as Gaussian noise, with the appropriate variance. This worst case assumption achieves our objective of eliminating the influence of the base station on the game, since the signal can be decoded by the receiver with any reasonable decoder. Remarkably, we show that the unique Nash equilibrium of this game is the sum-rate optimal point on the boundary of the capacity region [1]. Hoping to achieve other boundary points of the capacity region, we then introduce the base station as a player interested in maximizing a weighted sum of the individual rates.

[1]The rigorous definition of equilibria points will be given in the sequel.

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By allowing the base station to adopt the appropriate successive cancellation decoding strategy, which is announced at the first stage of the game, we transform our game into a Stackelberg formulation [13]. Here, we establish the ability of this approach to achieve all the corner points of the capacity region in addition to the sum-rate optimal point. The key idea is for the base station to use a successive cancellation decoding strategy while altering the decoding order as a function of the channel state. The final step, that allows for achieving all points on the boundary of the capacity region, is to use a dynamic game approach. In this setup, the base station can use the decoding order as a *punishment* tool forcing the multiple access users to adopt the optimal power control policies. We then extend our results to vector channels where different conclusions (as compared with the scalar case) are drawn. It is worth noting that our approach is purely information theoretic, and hence, we do not introduce other elements such as pricing mechanisms [7] into the problem. In particular, we limit the payoff functions to depend only on the achievable rate(s), and define the multiple access user strategy as a power allocation policy and the base station strategy as a decoding algorithm.

The rest of the paper is organized as follows. In Section II, we present the system model and review, briefly, known results on the capacity of fading MACs. Section III includes our results on the water-filling game for scalar fading channels. In particular, we devote Section III-A to the Nash formulation, Section III-B to the Stackelberg formulation, and Section III-C to the dynamic game scenario. Section IV highlights some interesting structural differences between scalar and vector channels. Finally, we close with some concluding remarks in Section V.

II. Background

We consider a discrete-time flat fading MAC with N users and one base station. The signal received by the base station at time n is[2]

$$y(n)=\sum_{i=1}^{N}\sqrt{h_i(n)}x_i(n)+z(n) \tag{1}$$

where $x_i(n)$ and $h_i(n)$ are the transmitted signal and fading channel gain of the ith user at time n. Similar to [1], we assume the fading process to be jointly stationary and ergodic. We further assume that the stationary distribution has a continuous density and is bounded. User i has an average power constraint $\bar{P}_i$ and $z(n)$ is a sample of a zero-mean white Gaussian noise process with variance σ^2. The capacity region of this channel depends on the fading process characteristics and the availability of the channel state information (CSI).

If the channel gains are assumed to be fixed and known *a priori* (i.e., time-invariant channel) then we are reduced to the Gaussian MAC where the capacity region is well known [14]. For the two-user case, this region $\mathcal{G}_g$ is given by

$$\begin{aligned}R_1 &\le \frac{1}{2}\log_2\left(1+\frac{h_1\bar{P}_1}{\sigma^2}\right)\\ R_2 &\le \frac{1}{2}\log_2\left(1+\frac{h_2\bar{P}_2}{\sigma^2}\right)\\ R_1+R_2 &\le \frac{1}{2}\log_2\left(1+\frac{h_1\bar{P}_1+h_2\bar{P}_2}{\sigma^2}\right).\end{aligned} \tag{2}$$

[2]In this paper, we use lower case letters for scalars, bold face lower case letters for vectors, and bold face upper case letters for matrices.

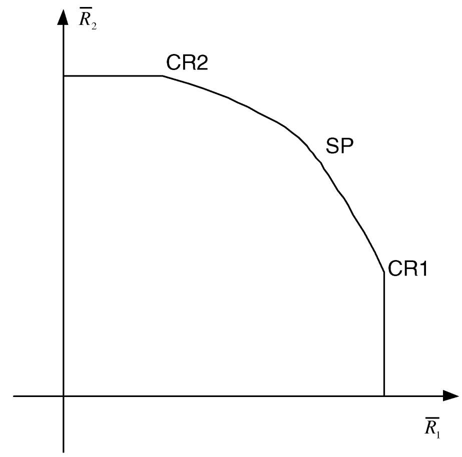

Fig. 1. The capacity region of the two-user fading MAC.

It is easy to see that the boundary of $\mathcal{G}_g$ is a pentagon. The two corner points are achieved by employing a successive cancellation decoding strategy at the base station and other boundary points can be achieved by appropriate time sharing between the two decoding strategies used at the corner points [14]. For time-varying channels with only receiver CSI, the capacity region is also known [15]. For the two-user case, the new capacity region can be interpreted as the average of the rate expressions in (2) with respect to the fading channel distribution.

Tse and Hanly [1] considered time-varying channels where the CSI is available *a priori* at all the transmitters and the receiver. In practice, this assumption can be realized by estimating the CSI at the base station and then broadcasting the information to the multiple access users. As pointed out in [1], this assumption is justified when the channel varies much slower than the data rate, resulting in a negligible cost for the estimation and feedback mechanism. Tse and Hanly characterized the capacity region $\mathcal{G}_c$ along with the corresponding centralized power and rate allocation policies $(\mathcal{P}_c, \mathcal{R}_c)$ for this scenario. It was also shown in [1] that the power and rate allocation policies are unique and each boundary point corresponds to the maximization of a weighted sum of the individual rates. All the boundary points are achieved by successive cancellation decoding, where the decoding order is determined by the rate award vector $\boldsymbol{\mu}$ [1].

The capacity region for the two-user case is shown in Fig. 1. The corner point CR_1 is achieved by using the following policy: user 1 water-fills over the background noise and user 2 water-fills over the sum of the interference from user 1 and the background noise. At the base station user 2 is decoded first followed by user 1. We denote the rate pair at this point as $(\bar{R}_{1,CR_1}, \bar{R}_{2,CR_1})$. At point CR_2, the roles of users 1 and 2 are reversed and we refer to the rate pair by $(\bar{R}_{1,CR_2}, \bar{R}_{2,CR_2})$. Another boundary point of particular interest is the maximum sum-rate point SP. Unlike the additive white Gaussian noise (AWGN) MAC, this point is unique and is achieved by a time-sharing policy where only one user is allowed to transmit at any fading state [1], [16]. This observation will prove instrumental in the development of the main result in Section III-A.

The centralized nature of the optimal power and rate allocation policies $(\mathcal{P}_c, \mathcal{R}_c)$ motivates our pursuit for distributed strategies that approach the capacity region of the fading MAC with selfish users who might have incentive to deviate from the centralized policy. Our assumption that the CSI is known everywhere implies that the games considered here are games with complete information [3], [5], [7]–[12]. Finally, we observe that the utility functions of the multiple access users in our games are concave in their own power control and the strategy spaces are convex. This allows for limiting our discussion to pure strategies without any loss of generality [13], [17].

III. The Water-Filling Game

For simplicity of presentation, we first consider in details the two-user scenario. Our arguments extend to the N-user channel as briefly outlined in Section III-D.

A. Nash Formulation

Here, we consider a static noncooperative game where the players are the multiple access users. In this game, the strategy of user i is the power control policy $\mathcal{P}_i$. The corresponding payoff function is defined as the average achievable rate $\bar{R}_i = \mathbb{E}_{\boldsymbol{h}}[\mathcal{R}_i]$ with $\boldsymbol{h} = [h_1, h_2]^T$. The goal of user i is to

$$\max_{\mathcal{P}_i} \bar{R}_i(\mathcal{P}_i, \mathcal{P}_{-i}) \text{ s.t. } \mathcal{P}_i \in \mathcal{F}_i \tag{3}$$

where $\mathcal{F}_i = \{\mathcal{P}_i : \mathbb{E}_{\boldsymbol{h}}[\mathcal{P}_i] \leq \bar{P}_i, \mathcal{P}_i(\boldsymbol{h}) \geq 0\}$ is the set of all feasible power control policies of user i, and $\mathcal{P}_{-i}$ represents the power control policy of the other user (in the more general case $\mathcal{P}_{-i}$ refers to the strategies of all users except user i). Since the base station is not a player of the game, we assume that each user will treat the signal of the other user as interference. This assumption is consistent with the lack of coordination among the users,[3] and allows the base station to use low-complexity single-user decoders. Given the power control policy $\mathcal{P}_2(h_1, h_2)$ of user 2, the payoff of user 1 is given by

$$\bar{R}_1 = \int\int \frac{1}{2} \log_2\left(1 + \frac{\mathcal{P}_1(h_1,h_2)h_1}{\sigma^2 + \mathcal{P}_2(h_1,h_2)h_2}\right) f(h_1,h_2)dh_1dh_2. \tag{4}$$

Here, $f(h_1, h_2)$ is the joint probability density function of the two fading coefficients. The payoff function of user 2 is defined similarly. As we can see that the payoff function of each user depends on the two power control policies $(\mathcal{P}_1, \mathcal{P}_2)$. Before proceeding further, we need the following definition from [17].

Definition 1: A Nash equilibrium is a policy pair $(\mathcal{P}_1^*, \mathcal{P}_2^*)$ such that

$$\begin{aligned} \bar{R}_1(\mathcal{P}_1^*, \mathcal{P}_2^*) &\geq \bar{R}_1(\mathcal{P}_1', \mathcal{P}_2^*), \quad \forall \mathcal{P}_1' \in \mathcal{F}_1 \\ \bar{R}_2(\mathcal{P}_1^*, \mathcal{P}_2^*) &\geq \bar{R}_2(\mathcal{P}_1^*, \mathcal{P}_2'), \quad \forall \mathcal{P}_2' \in \mathcal{F}_2. \end{aligned} \tag{5}$$

This definition means that at the Nash equilibrium, no user can benefit by unilaterally deviating. Given a fixed power control policy of user 2, the optimal strategy $\mathcal{P}_1(h_1, h_2)$ of user 1 is the solution to the optimization problem shown in (6) at the bottom of the page.

[3]Even if there is a coordination among them, the coordination is not trustful, since the users are selfish.

The solution to this optimization problem is the well-known water-filling power allocation, i.e.,

$$\mathcal{P}_1(h_1,h_2) = \left(\lambda_1 - \frac{\sigma^2}{h_1} - \frac{\mathcal{P}_2(h_1,h_2)h_2}{h_1}\right)^+ \tag{7}$$

in which $(x)^+ = \max\{x, 0\}$ and λ_1 is the power level that satisfies

$$\int\int \left(\lambda_1 - \frac{\sigma^2}{h_1} - \frac{\mathcal{P}_2(h_1,h_2)h_2}{h_1}\right)^+ f(h_1,h_2)dh_1dh_2 = \bar{P}_1. \tag{8}$$

Similarly, the optimal policy of user 2, given a fixed policy for user 1, is given by

$$\mathcal{P}_2(h_1,h_2) = \left(\lambda_2 - \frac{\sigma^2}{h_2} - \frac{\mathcal{P}_1(h_1,h_2)h_1}{h_2}\right)^+. \tag{9}$$

From these expressions, one can see that the optimal policy of each user depends largely on its *guess* of the other user's policy. Based on this guess, each user will determine its policy and adjusts its water-filling level to maximize its own average rate. At the Nash equilibrium, the water-filling pair (λ_1, λ_2) satisfies the two average power constraints with equality. Now we are ready to prove our first result.

Theorem 1: The maximum sum-rate point SP of the capacity region $\mathcal{G}_c$ is the unique Nash equilibrium of our water-filling game.

Proof: At first, let us show the existence of only time-sharing equilibria using similar argument as [18], [19]. Suppose there exists a non-time-sharing equilibrium with the corresponding water-level pair (λ_1, λ_2). Then for some channel realizations h_1, h_2, we have $\mathcal{P}_1(h_1, h_2) > 0$, $\mathcal{P}_2(h_1, h_2) > 0$, and

$$\begin{aligned} \frac{\sigma^2}{h_1} + \frac{\mathcal{P}_2(h_1,h_2)h_2}{h_1} + \mathcal{P}_1(h_1,h_2) &= \lambda_1 \\ \frac{\sigma^2}{h_2} + \frac{\mathcal{P}_1(h_1,h_2)h_1}{h_2} + \mathcal{P}_2(h_1,h_2) &= \lambda_2. \end{aligned} \tag{10}$$

From these two equations, we get

$$\lambda_1 = \lambda_2 \frac{h_2}{h_1}. \tag{11}$$

Since λ_1, λ_2 are constants, and the fading coefficients are characterized by a continuous distribution, (11) is satisfied with a zero probability. This implies the existence of only time-sharing Nash equilibria.

Under the time-sharing equilibrium, when $\mathcal{P}_1(h_1, h_2) > 0$, the sum of the background noise and the interference from user 1 should be larger than the water level of user 2. Thus, when user

$$\begin{aligned} \bar{R}_1 = \max_{\mathcal{P}_1(h_1,h_2)} &\int\int \frac{1}{2}\log_2\left(1 + \frac{\mathcal{P}_1(h_1,h_2)h_1}{\sigma^2 + \mathcal{P}_2(h_1,h_2)h_2}\right) f(h_1,h_2)dh_1dh_2 \\ \text{s.t.} \quad &\int\int \mathcal{P}_1(h_1,h_2)f(h_1,h_2)dh_1dh_2 \leq \bar{P}_1, \qquad \mathcal{P}_1(h_1,h_2) \geq 0. \end{aligned} \tag{6}$$

1 transmits, the channel conditions should satisfy the following inequality:

$$\mathcal{P}_1(h_1,h_2)\frac{h_1}{h_2}+\frac{\sigma^2}{h_2}=\left(\lambda_1-\frac{\sigma^2}{h_1}\right)\frac{h_1}{h_2}+\frac{\sigma^2}{h_2}=\frac{\lambda_1 h_1}{h_2}\geq\lambda_2.$$

Similarly, when user 2 transmits, the channel conditions should satisfy the following condition:

$$\frac{\lambda_2 h_2}{h_1}\geq\lambda_1.$$

The water-filling levels can now be obtained by solving the following two equations:

$$\iint_{\lambda_1 h_1\geq\lambda_2 h_2}\left(\lambda_1-\frac{\sigma^2}{h_1}\right)^+ f(h_1,h_2)dh_1dh_2=\bar{P}_1$$
$$\iint_{\lambda_1 h_1\leq\lambda_2 h_2}\left(\lambda_2-\frac{\sigma^2}{h_2}\right)^+ f(h_1,h_2)dh_1dh_2=\bar{P}_2. \quad (12)$$

Let $(\bar{\lambda}_1,\bar{\lambda}_2)$ be the solution to (12). The corresponding power control policies are unique and given by

$$\mathcal{P}_1(h_1,h_2)=\left(\bar{\lambda}_1-\frac{\sigma^2}{h_1}\right)^+,\quad \text{when } \bar{\lambda}_1 h_1\geq\bar{\lambda}_2 h_2 \quad (13)$$
$$\mathcal{P}_2(h_1,h_2)=\left(\bar{\lambda}_2-\frac{\sigma^2}{h_2}\right)^+,\quad \text{when } \bar{\lambda}_1 h_1\leq\bar{\lambda}_2 h_2 \quad (14)$$

with $\mathcal{P}_1(h_1,h_2)=0$ and $\mathcal{P}_2(h_1,h_2)=0$ in other cases.

It was shown in [1] that the centralized policy corresponding to the point SP is time sharing with the same power allocation levels as (13), (14). Finally, the fact that the solution to (12) is unique [1] implies that the only Nash equilibrium of the distributed power control game is the maximum sum-rate point of the capacity region (i.e., SP). □

Theorem 1 establishes the remarkable fact that the selfish behavior of the users will lead them to **jointly** optimize the sum-rate of the channel. In fact, this result provides a new interpretation of the opportunistic communication principle [16]. At any particular instance, the user with the strongest channel sees a relatively weak interference from the other user, and hence, decides to transmit with a high power level. On the other hand, the other user sees a strong interference in addition to a weak channel, and hence, decides to conserve the power for later usage. This way, they reach the *opportunistic* time-sharing equilibrium distributively. The underlying idea is that the selfishness of the different users will *balance-out* at the maximum sum-rate point. In this work, we only focus on the case where the channel state information is known by all the users. One possible justification is that the base station can estimate $\boldsymbol{h}$ and then broadcast it to all the users in the network. Developing models that only depend on local channel state information is an interesting avenue of future work. Theorem 1 contrasts the negative conclusions drawn in earlier works on the efficiency of game-theoretic approaches in CDMA uplink power control (e.g., [7]–[12]). The enabling vehicle behind this result is the time varying nature of the fading channel. With this temporal variations, the CSI (available at all transmitter) acts like a common randomness that allows the users to reach a more efficient equilibrium based on a selfish rationale. This is yet another manifestation of the positive impact that fading, if properly exploited, can have on certain aspects of wireless systems. We wish to stress the fact that, in our formulation, each user needs only to know the average power constraint of the other user (i.e., no need for knowing the instantaneous interference level). The two users, then, can compute the water levels (λ_1,λ_2) **off-line** based on the statistics of the channel, and will adjust their own instantaneous power level according to these water levels and the CSI. The users do not need to adjust their instantaneous rate, instead, they can use a codebook that spans several fading blocks with rate set to be the average rate achievable in these blocks. This approach relies on the assumption that the two users are rational and each user trusts that the other user is rational.

B. Stackelberg Formulation

In the previous section, we have shown that the only boundary point achievable by our Nash game is the maximum sum-rate point. One can attribute this limitation to the assumption that every user (player) will treat the other user's signal as noise. While this assumption does not entail a loss at the *time-sharing* point SP, it does not allow for achieving other boundary points. Such points require the base station to employ a more sophisticated decoding rule. In [1], it was shown that successive cancellation decoding, with the appropriate ordering, is sufficient to achieve all the boundary points. This observation motivates a game-theoretic formulation where the base station is introduced as an additional player. The base station strategy corresponds to a particular choice of the decoding order, as detailed next.

We wish to stress that, unlike the centralized scenario [1], the base station in our formulation does not dictate the power level and rate of the individual users.[4] Still, it is reasonable to assume that the roles of the base station and multiple access users are not totally symmetric. Therefore, we do not model the base station as an *ordinary* player in our game but rather appeal to the bi-level programming notion [20]. Bi-level programming is typically used in modeling a decision making process where there is a hierarchical relationship between the decision makers. In our context, bi-level programming corresponds to a Stackelberg game [17], [20], where the leader announces its strategy first and then the remaining players react according to a specific equilibrium concept among them. Here, we designate the base station as the game leader, and hence, it will announce its decoding strategy in the high-level game. This way, the base station can rely on the rational and selfish nature of the multiple access players to *influence* their behavior in the second stage (i.e., low level game).

In this work, we consider a class of successive decoding strategies parameterized by the decoding order as a function of the fading gains (h_1,h_2). More precisely, the base station divides the whole possible space of (h_1,h_2) into two subsets D_1, D_1^c. When $(h_1,h_2)\in D_1$, the base station will decode user 1's information first whereas $(h_1,h_2)\in D_1^c$ implies decoding user 2's signal first. After the base station announces its strategy, i.e., D_1, the multiple access users play the low-level game

[4]Even if the base station dictates the power and rate of individual user, each user will have an incentive to deviate from it if the dictated policy is not compatible with its own utility, as argued before.

using the Nash equilibrium concept. The strategy space of user i is still $\mathcal{F}_i$, and the payoff function of user i is defined as the supremum of the achievable rate. Here supremum refers to the fact that in the rate expressions to follow we always assume the users to be decoded successfully (which is a critical assumption in the successive decoding approach). We will show later that, at the Nash equilibrium this condition indeed holds. Hence, the supremum corresponds exactly to the achieved payoff. With a slight abuse of notation, the payoff function of each user is written as (15) at the bottom of the page. Here $I_{\{\cdot\}}$ is the indicator function. In order to achieve the average rate in (15), for a given base-station strategy D_1, each user will use two codebooks. The low-rate codebook is multiplexed across the fading states in which the user is decoded first and the high-rate codebook is multiplexed across the other fading states. The payoff function of the base station is defined as

$$\mu_1 \bar{R}_1(D_1, \mathcal{P}_1, \mathcal{P}_2) + \mu_2 \bar{R}_2(D_1, \mathcal{P}_1, \mathcal{P}_2). \tag{16}$$

This payoff function has a natural economic interpretation as the revenue of the base station where μ_i can be viewed as the payment that user i owes per unit rate. The value of μ_i can be decided using an auction process [21], where each user submits its proposed payment μ_i to the base station in order to maximize its own utility. In this work, we do not consider this auction process and assume that $\boldsymbol{\mu} = [\mu_1, \mu_2]^T$ is given.

We first study the properties of the low level game. The Nash equilibrium under a fixed base station strategy D_1 is a power control pair $(\mathcal{P}_1^*, \mathcal{P}_2^*)$ that satisfies

$$\begin{aligned} &\bar{R}_1(D_1, \mathcal{P}_1^*, \mathcal{P}_2^*) \geq \bar{R}_1(D_1, \mathcal{P}_1', \mathcal{P}_2^*), \quad \forall \mathcal{P}_1' \in \mathcal{F}_1 \\ &\bar{R}_2(D_1, \mathcal{P}_1^*, \mathcal{P}_2^*) \geq \bar{R}_2(D_1, \mathcal{P}_1^*, \mathcal{P}_2'), \quad \forall \mathcal{P}_2' \in \mathcal{F}_2. \end{aligned}$$

For any given power control policy $\mathcal{P}_2$, the optimal power control policy of user 1 is the solution to the optimization problem shown in (17), also at the bottom of the page. The optimal power control policy of user 2 is also the solution to a similar optimization problem for any power control policy of user 1. For a given D_1, the solution set for this low level game is written as

$$\begin{aligned} S(D_1) = \{&(\mathcal{P}_1, \mathcal{P}_2) \\ &: (\mathcal{P}_1, \mathcal{P}_2) \text{ is a Nash equilibrium of the low-level game}\}. \end{aligned}$$

The following result characterizes the pure-strategy Nash equilibria of our low-level game. The algorithm developed in the proof is reminiscent of the iterative algorithm in [1], [3].

Theorem 2: For any strategy D_1 of the base station, there exist Nash equilibria for the low-level power/rate control game.

Proof: At the Nash equilibrium, no user can benefit by deviating unilaterally. Suppose $\mathcal{P}_2(h_1, h_2)$ is given, user 1's strategy is the solution to (17), which is still the water-filling solution

$$\mathcal{P}_1(h_1, h_2) = \left(\lambda_1 - \frac{\sigma^2}{h_1} - \frac{\mathcal{P}_2(h_1, h_2) h_2 I_{\{(h_1, h_2) \in D_1\}}}{h_1}\right)^+ \tag{18}$$

where λ_1 is the power level chosen to satisfy the power constraint of user 1 with equality. For the same reason, if we fix $\mathcal{P}_1(h_1, h_2)$, the optimal response of user 2 is also water-filling over the sum of the interference from user 1 and the background noise, which is

$$\mathcal{P}_2(h_1, h_2) = \left(\lambda_2 - \frac{\sigma^2}{h_2} - \frac{\mathcal{P}_1(h_1, h_2) h_1 I_{\{(h_1, h_2) \in D_1^c\}}}{h_2}\right)^+. \tag{19}$$

The key of our proof is to establish the existence of a pair (λ_1, λ_2) that simultaneously satisfies the two power constraints with equality, and hence, constitutes a Nash equilibrium. If such (λ_1, λ_2) exists, we have solutions to the (18) and (19). One can easily check that if $(h_1, h_2) \in D_1$

$$\begin{aligned} \mathcal{P}_2(h_1, h_2) &= \left(\lambda_2 - \frac{\sigma^2}{h_2}\right)^+ \\ \mathcal{P}_1(h_1, h_2) &= \left(\lambda_1 - \frac{\sigma^2}{h_1} - \frac{\mathcal{P}_2(h_1, h_2) h_2}{h_1}\right)^+ \\ &= \left(\lambda_1 - \frac{\sigma^2}{h_1} - \left(\frac{\lambda_2 h_2}{h_1} - \frac{\sigma^2}{h_1}\right)^+\right)^+. \end{aligned} \tag{20}$$

Similarly, if $(h_1, h_2) \in D_1^c$

$$\begin{aligned} \mathcal{P}_1(h_1, h_2) &= \left(\lambda_1 - \frac{\sigma^2}{h_1}\right)^+ \\ \mathcal{P}_2(h_1, h_2) &= \left(\lambda_2 - \frac{\sigma^2}{h_2} - \frac{\mathcal{P}_1(h_1, h_2) h_1}{h_2}\right)^+ \\ &= \left(\lambda_2 - \frac{\sigma^2}{h_2} - \left(\frac{\lambda_1 h_1}{h_2} - \frac{\sigma^2}{h_2}\right)^+\right)^+. \end{aligned}$$

$$\begin{aligned} \bar{R}_1(D_1, \mathcal{P}_1, \mathcal{P}_2) &= \int\!\!\int \frac{1}{2} \log_2 \left(1 + \frac{\mathcal{P}_1(h_1, h_2) h_1}{\sigma^2 + \mathcal{P}_2(h_1, h_2) h_2 I_{\{(h_1, h_2) \in D_1\}}}\right) f(h_1, h_2) dh_1 dh_2, \\ \bar{R}_2(D_1, \mathcal{P}_1, \mathcal{P}_2) &= \int\!\!\int \frac{1}{2} \log_2 \left(1 + \frac{\mathcal{P}_2(h_1, h_2) h_2}{\sigma^2 + \mathcal{P}_1(h_1, h_2) h_1 I_{\{(h_1, h_2) \in D_1^c\}}}\right) f(h_1, h_2) dh_1 dh_2. \end{aligned} \tag{15}$$

$$\begin{aligned} \max_{\mathcal{P}_1(h_1, h_2)} \quad & \bar{R}_1(D_1, \mathcal{P}_1, \mathcal{P}_2) = \int\!\!\int \frac{1}{2} \log_2 \left(1 + \frac{\mathcal{P}_1(h_1, h_2) h_1}{\sigma^2 + \mathcal{P}_2(h_1, h_2) h_2 I_{\{(h_1, h_2) \in D_1\}}}\right) f(h_1, h_2) dh_1 dh_2 \\ \text{s.t.} \quad & \int\!\!\int \mathcal{P}_1(h_1, h_2) f(h_1, h_2) dh_1 dh_2 \leq \bar{P}_1, \quad \mathcal{P}_1(h_1, h_2) \geq 0. \end{aligned} \tag{17}$$

Thus, if the water-filling level pair (λ_1, λ_2) exists, it should be the solution to the following equation array:

$$\begin{aligned}&\iint_{D_1} \left(\lambda_1 - \frac{\sigma^2}{h_1} - \left(\frac{\lambda_2 h_2}{h_1} - \frac{\sigma^2}{h_1}\right)^+\right)^+ f(h_1,h_2)dh_1dh_2\\ &\qquad + \iint_{D_1^c} \left(\lambda_1 - \frac{\sigma^2}{h_1}\right)^+ f(h_1,h_2)dh_1dh_2 = \bar{P}_1\\ &\iint_{D_1^c} \left(\lambda_2 - \frac{\sigma^2}{h_2} - \left(\frac{\lambda_1 h_1}{h_2} - \frac{\sigma^2}{h_2}\right)^+\right)^+ f(h_1,h_2)dh_1dh_2\\ &\qquad + \iint_{D_1} \left(\lambda_2 - \frac{\sigma^2}{h_2}\right)^+ f(h_1,h_2)dh_1dh_2 = \bar{P}_2.\end{aligned} \tag{21}$$

Before proceeding further, we first observe the following. If there are two pairs (λ_1', λ_2') and (λ_1, λ_2), where $\lambda_1' > \lambda_1$, $\lambda_2' = \lambda_2$, then we have $\bar{P}_1(\lambda_1', \lambda_2') \geq \bar{P}_1(\lambda_1, \lambda_2)$, $\bar{P}_2(\lambda_1', \lambda_2') \leq \bar{P}_2(\lambda_1, \lambda_2)$.[5] One can easily verify this by observing that $\mathcal{P}_1(h_1, h_2)$ is a nondecreasing function of λ_1 and a nonincreasing function of λ_2. At the same time, $\mathcal{P}_2(h_1, h_2)$ is a nonincreasing function of λ_1 and a nondecreasing function of λ_2. Based on these observations, we have the following iterative method to solve (21). Set $\lambda_1(1) = 0$, $\lambda_2(1) = 0$, then fix λ_2 and increase λ_1 until $\bar{P}_1(\lambda_1, \lambda_2(1)) = \bar{P}_1$. This can be done by solving the following equation:

$$\begin{aligned}&\iint_{D_1^c} \left(\lambda_1 - \frac{\sigma^2}{h_1} - \left(\frac{\lambda_2(1) h_2}{h_1} - \frac{\sigma^2}{h_1}\right)^+\right)^+ f(h_1,h_2)dh_1dh_2\\ &\qquad + \iint_{D_1} \left(\lambda_1 - \frac{\sigma^2}{h_1}\right)^+ f(h_1,h_2)dh_1dh_2 = \bar{P}_1.\end{aligned}$$

Let $\lambda_1(2)$ represent the solution to this equation. At this time, we will have $\bar{P}_2(\lambda_1(2), \lambda_2(1)) \leq \bar{P}_2$. Then we can increase $\lambda_2(1)$ to $\lambda_2(2)$ such that $\bar{P}_2(\lambda_1(2), \lambda_2(2)) = \bar{P}_2$. After this step, $\bar{P}_1(\lambda_1(2), \lambda_2(2)) \leq \bar{P}_1$, thus, we can increase λ_1 again. Through this process, we can get nondecreasing sequences $\lambda_1(n)$, $\lambda_2(n)$ and $\bar{P}_1(\lambda_1(n), \lambda_2(n)) \rightarrow \bar{P}_1$, $\bar{P}_2(\lambda_1(n), \lambda_2(n)) \rightarrow \bar{P}_2$. Since $\bar{P}_1$, $\bar{P}_2$ are limited, $\lambda_1(n)$, $\lambda_2(n)$ are nondecreasing sequences with upper bounds. Then there exists constants λ_1^*, λ_2^* such that

$$\lim_{n\to\infty} \lambda_1(n) = \lambda_1^*, \qquad \bar{P}_1(\lambda_1^*, \lambda_2^*) = \bar{P}_1. \tag{22}$$
$$\lim_{n\to\infty} \lambda_2(n) = \lambda_2^*, \qquad \bar{P}_2(\lambda_1^*, \lambda_2^*) = \bar{P}_2. \tag{23}$$

This pair $(\lambda_1^*, \lambda_2^*)$ is therefore a Nash equilibrium of our power allocation game. □

[5]Here $\bar{P}_i(\lambda_1, \lambda_2)$ refers to the average power of user i when the users do water-filling according to the water levels (λ_1, λ_2).

Theorem 2 only establishes the existence of a Nash equilibrium, but it tells nothing about the uniqueness of this equilibrium. To prove uniqueness, one is typically forced to find a contraction mapping whose fixed point is the Nash equilibrium. In [3], [5], the authors apply this method to the interference game and find that uniqueness requires very restrictive conditions. Fortunately, we are able to prove uniqueness in our setup by using the concept of **admissible** Nash equilibrium ([17, Definition 3.3]).

Definition 2: A Nash equilibrium strategy pair $(\mathcal{P}_1^*, \mathcal{P}_2^*)$ is said to be admissible if there exists no other Nash equilibrium strategy pair $(\mathcal{P}_1', \mathcal{P}_2')$ such that $\bar{R}_1(D_1, \mathcal{P}_1', \mathcal{P}_2') \geq \bar{R}_1(D_1, \mathcal{P}_1^*, \mathcal{P}_2^*)$, $\bar{R}_2(D_1, \mathcal{P}_1', \mathcal{P}_2') \geq \bar{R}_2(D_1, \mathcal{P}_1^*, \mathcal{P}_2^*)$, and at least one of these equalities is strict.

Intuitively, this notion allows for eliminating Nash equilibria which are Pareto dominated by other equilibrium points. One would expect the rationality of the players to led them to choose one of the admissible Nash equilibria to operate, since these points are preferred for all the users. This approach allows for modifying the solution set for our low-level game to only include admissible Nash equilibria as shown in the equation at the bottom of the page.

The following result establishes the existence of a single admissible Nash equilibrium in this set (for any choice of D_1).

Theorem 3: For any strategy D_1 of the base station, there exists a **single** admissible Nash equilibrium for the low-level power game (i.e., for any D_1, $S^*(D_1)$ is a singleton).

Proof: First consider the case

$$D_1 = \{(h_1, h_2) : \bar{\lambda}_1 h_1 \leq \bar{\lambda}_2 h_2\}$$

where $(\bar{\lambda}_1, \bar{\lambda}_2)$ as given in the proof of Theorem 1. Based on the proof of Theorem 2, the power control policy of users will be water-filling. Let $(\lambda_1^*, \lambda_2^*)$ be the water-filling level of these two users at the equilibrium given this D_1. Then it is easy to check that $\lambda_1^* = \bar{\lambda}_1, \lambda_2^* = \bar{\lambda}_2$, hence, the optimal solution is time-sharing and the Nash equilibrium is unique. For other D_1, we establish uniqueness of the admissible Nash equilibrium by contradiction.

We let $(\lambda_1^*, \lambda_2^*)$ and (λ_1', λ_2') be the two pairs of water-levels corresponding to equilibria. Then, by definition, the two average power constraints are satisfied with equality with these two pairs of water-levels, that is, $\bar{P}_1(\lambda_1^*, \lambda_2^*) = \bar{P}_1$, $\bar{P}_2(\lambda_1^*, \lambda_2^*) = \bar{P}_2$, $\bar{P}_1(\lambda_1', \lambda_2') = \bar{P}_1$, $\bar{P}_2(\lambda_1', \lambda_2') = \bar{P}_2$. Noting that we **are not** at a time sharing point, we claim the following.

1) If $\lambda_1^* = \lambda_1'$, we have $\lambda_2^* = \lambda_2'$. If not, we will have $\bar{P}_1(\lambda_1', \lambda_2') > \bar{P}_1$, $\bar{P}_2(\lambda_1', \lambda_2') < \bar{P}_2$ when $\lambda_2^* > \lambda_2'$ and $\bar{P}_1(\lambda_1', \lambda_2') < \bar{P}_1$, $\bar{P}_2(\lambda_1', \lambda_2') > \bar{P}_2$ when $\lambda_2^* < \lambda_2'$. Thus, we come to a contradiction.
2) If $\lambda_1^* < \lambda_1'$, we have $\lambda_2^* < \lambda_2'$. If not, we will have $\bar{P}_1(\lambda_1', \lambda_2') > \bar{P}_1$, $\bar{P}_2(\lambda_1', \lambda_2') < \bar{P}_2$ when $\lambda_2^* \geq \lambda_2'$. Thus, we come to a contradiction.

$$S^*(D_1) = \{(\mathcal{P}_1, \mathcal{P}_2) : (\mathcal{P}_1, \mathcal{P}_2) \text{ is an admissible Nash equilibrium of the low-level game}\}.$$

3) If $\lambda_1^* > \lambda_1'$, we have $\lambda_2^* > \lambda_2'$. If not, we will have $\bar{P}_1(\lambda_1', \lambda_2') < \bar{P}_1$, $\bar{P}_2(\lambda_1', \lambda_2') > \bar{P}_2$ when $\lambda_2^* \leq \lambda_2'$. Thus, we come to a contradiction.

The two water-level pairs, therefore, have a strict order. We can define the relationship $<$ for the water-level pairs and say $(\lambda_1^*, \lambda_2^*) < (\lambda_1', \lambda_2')$, if $\lambda_1^* < \lambda_1'$ and $\lambda_2^* < \lambda_2'$. Suppose $(\lambda_1^*, \lambda_2^*) < (\lambda_1', \lambda_2')$, we claim that $\bar{R}_1(D_1, \mathcal{P}_1^*, \mathcal{P}_2^*) > \bar{R}_1(D_1, \mathcal{P}_1', \mathcal{P}_2')$ and $\bar{R}_2(D_1, \mathcal{P}_1^*, \mathcal{P}_2^*) > \bar{R}_2(D_1, \mathcal{P}_1', \mathcal{P}_2')$. Without loss of generality, we only need to prove the first part. To show this, we can see that the sum of the interference from user 2 and the background noise is $N_1'(\lambda_2) = \sigma^2$, if $(h_1, h_2) \in D_1^c$, and $N_1'(\lambda_2) = \sigma^2 + (\lambda_2 h_2 - \sigma^2)^+$ if $(h_1, h_2) \in D_1$.

Since our solution is not time sharing, we can see that $N_1'(\lambda_2)$ is a increasing function of λ_2. Thus, $\lambda_2^* < \lambda_2'$ implies that $\bar{R}_1(D_1, \mathcal{P}_1^*, \mathcal{P}_2^*) > \bar{R}_1(D_1, \mathcal{P}_1', \mathcal{P}_2')$ and our claim is true.

This claim means that the achievable utility pairs also have strict order, i.e., the smaller the water-filling pair, the larger the utility pair. With this strict order relationship among the achievable utilities at the Nash equilibria, the unique admissible Nash equilibrium is achieved with the minimum water-level pair. This completes the proof. □

An *explicit* approach for achieving the unique admissible equilibrium in our game is for all the users to follow the iterative algorithm used in the proof of Theorem 2 and agree off-line on the **convention** of starting the iteration with $\boldsymbol{\lambda}(1) = \mathbf{0}$. To see this, suppose that the users start with other $\boldsymbol{\lambda}^*(1) \geq \mathbf{0}$, then after one round of iteration, we will have $\boldsymbol{\lambda}^*(2) \geq \boldsymbol{\lambda}(2)$ by the proof of Theorem 2. Based on the same argument, we will have $\boldsymbol{\lambda}^*(n) \geq \boldsymbol{\lambda}(n), \forall n$. Thus, $\lim_{n\to\infty} \boldsymbol{\lambda}^*(n) \geq \lim_{n\to\infty} \boldsymbol{\lambda}(n)$, which means that the final water-levels when the users begin at $\mathbf{0}$ are smaller than the final water-levels if the users begin at any other $\boldsymbol{\lambda}^*(1)$. Hence, if the users start with $\boldsymbol{\lambda}(1) = \mathbf{0}$, the operating point will converge to the admissible Nash equilibrium. This agreement, that is to start the off-line iterative process with $\boldsymbol{\lambda}(1) = \mathbf{0}$, is clearly in the best interest of the two users since each of them achieves the largest throughput among all possible equilibria, and hence, is consistent with the selfish behavior assumption.

Now, we turn our attention to characterizing efficient base station strategies. In the following, we use $\mathcal{P}_{iD_1}$ to refer to the unique power control policy of each user, under strategy D_1, at the admissible Nash equilibrium. Here, we borrow the following definition from [17].

Definition 3: A strategy D_1^* is called a Stackelberg equilibrium strategy for a given (μ_1, μ_2), if with strategy D_1^*, the base station achieves

$$R^* = \sup_{D_1} \{\mu_1 \bar{R}_1(D_1, \mathcal{P}_{1D_1}, \mathcal{P}_{2D_1}) + \mu_2 \bar{R}_2(D_1, \mathcal{P}_{1D_1}, \mathcal{P}_{2D_1})\}.$$

Moreover, for any $\epsilon > 0$, a strategy $D_{1,\epsilon}^*$ is called an ϵ-Stackelberg strategy if

$$\mu_1 \bar{R}_1(D_{1,\epsilon}^*, \mathcal{P}_{1D_{1,\epsilon}^*}, \mathcal{P}_{2D_{1,\epsilon}^*}) + \mu_2 \bar{R}_2(D_{1,\epsilon}^*, \mathcal{P}_{1D_{1,\epsilon}^*}, \mathcal{P}_{2D_{1,\epsilon}^*}) \geq R^* - \epsilon. \quad (24)$$

Corollary 1: For every pair (μ_1, μ_2), $0 \leq \mu_1 < \infty$, $0 \leq \mu_2 < \infty$, an ϵ-Stackelberg strategy exists.

Proof: Based on Property 4.2 of [17], the only thing we need to prove is that R^* is bounded. Define R_i^o as the average rate the ith user can get when the other user is absent, then

$$\begin{aligned} R^* &= \mu_1 \bar{R}_1(D_1^*, \mathcal{P}_{1D_1^*}, \mathcal{P}_{2D_1^*}) + \mu_2 \bar{R}_2(D_1^*, \mathcal{P}_{1D_1^*}, \mathcal{P}_{2D_1^*}) \\ &\leq \mu_1 R_1^o + \mu_2 R_2^o. \end{aligned} \quad (25)$$

This completes the proof. □

Combining Theorem 3 and Corollary 1, we see that the proposed Stackelberg game setup has a very desirable structure. For any given vector $\boldsymbol{\mu}$, the existence of a base station policy which achieves a utility within an ϵ–difference from the optimal one is guaranteed; and for every rational multiple access user, the optimal policy in the low level game is unique. Therefore, the users will have no difficulty in deciding the power and rate levels in a distributed way. The following result characterizes the achievable performance of the proposed Stackelberg game.

Theorem 4: Let

$$\mathcal{G}_s = \bigcup_{D_1} \{(\bar{R}_1(D_1, \mathcal{P}_{1D_1}, \mathcal{P}_{2D_1}), \bar{R}_2(D_1, \mathcal{P}_{1D_1}, \mathcal{P}_{2D_1}))\}.$$

Then, $\mathcal{G}_s$ includes the three boundary points CR_1, CR_2, SP of the capacity region $\mathcal{G}_c$. However, $\mathcal{G}_s$ does not include any other boundary points of $\mathcal{G}_c$.

Proof: It is easy to verify that CR_1 can be achieved by setting $D_1 = \phi$, which means that the base station will always decode user 2's signal first. The corresponding policy for user 1 is to water-fill over the background noise, while the optimal policy for user 2 is also water-filling but over the sum of the interference from user 1 and the background noise. This is exactly the same as the centralized policy that achieves the boundary point CR_1. Similarly, CR_2 can be achieved by setting $D_1^c = \phi$, and SP can be achieved by setting

$$D_1 = \{(h_1, h_2) : \bar{\lambda}_1 h_1 \leq \bar{\lambda}_2 h_2\}$$

where $(\bar{\lambda}_1, \bar{\lambda}_2)$ as given in the proof of Theorem 1.

We prove our second claim by contradiction. Now, suppose that $\mathcal{G}_s$ includes another boundary point $(\bar{R}_{1b}, \bar{R}_{2b})$. Without loss of generality, suppose that at this point $\mu_1 > \mu_2$, the corresponding optimal central policy is $\mathcal{P}_b, \mathcal{R}_b$, the partition region that achieves this point is given by D_b, and the corresponding admissible power control pair is $\mathcal{P}_{1D_b}, \mathcal{P}_{2D_b}$. It was shown in [1] that the power control policy that achieves any boundary point is unique. Thus, if the partition D_b achieves this point, at any fading state (h_1, h_2), we have $\mathcal{P}_{1D_b}(h_1, h_2) = \mathcal{P}_{1,b}(h_1, h_2)$ and $\mathcal{P}_{2D_b}(h_1, h_2) = \mathcal{P}_{2,b}(h_1, h_2)$.

Then at any fading state, the capacity region pentagons formed by these two policies are the same, which is also shown on Fig. 2.

For every fading state, the optimal rate control policy $\mathcal{R}_b$ corresponds to the corner point $X1$. While for the distributed power control, when $(h_1, h_2) \in D_b$, the operating point is $X2$, and when $(h_1, h_2) \in D_b^c$, the operating point is $X1$. Thus

$$\begin{aligned} &\bar{R}_1(D_b, \mathcal{P}_{1D_b}, \mathcal{P}_{2D_b}) \\ &= \mathbb{E}_{\{\boldsymbol{h} \in D_b\}}[R_{1,X1}(\boldsymbol{h})] + \mathbb{E}_{\{\boldsymbol{h} \in D_b^c\}}[R_{1,X2}(\boldsymbol{h})] \\ &< \mathbb{E}_{\{\boldsymbol{h} \in D_b\}}[R_{1,X1}(\boldsymbol{h})] + \mathbb{E}_{\{\boldsymbol{h} \in D_b^c\}}[R_{1,X1}(\boldsymbol{h})] = \bar{R}_{1b} \end{aligned} \quad (26)$$

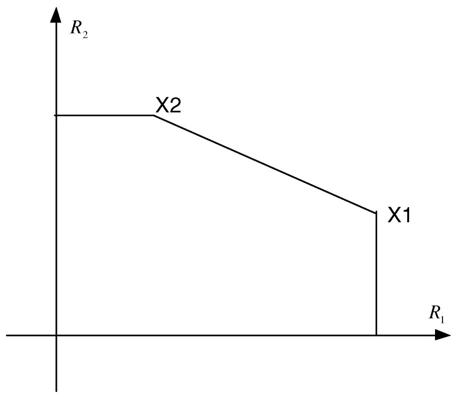

Fig. 2. The capacity region of the Gaussian MAC with fixed channel gains (h_1, h_2).

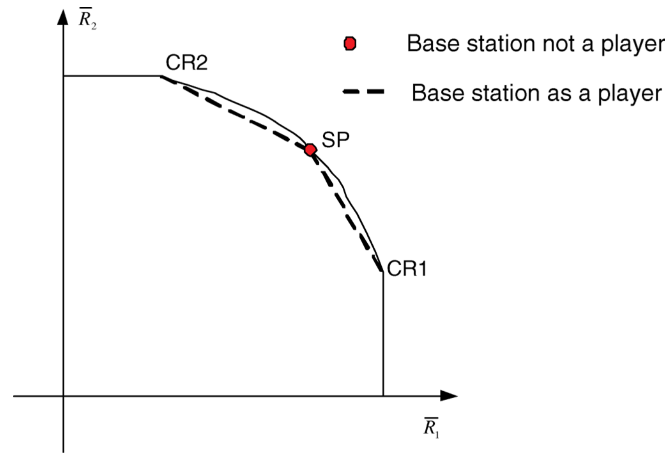

Fig. 3. The equilibria points of the Stackelberg power game.

which is a contradiction. This shows the nonexistence of D that achieves any other boundary point of the capacity region $\mathcal{G}_c$. □

Theorem 4 shows that the introduction of the base station as a leader of the game enlarges the achievable rate region (as compared with the Nash game discussed earlier) but this approach falls short of achieving the whole capacity region. Fig. 3 compares the capacity region with the Stackelberg achievable rate region assuming the following simple base station strategy: when $h_1 \leq \alpha h_2$ the base station decodes user 1 first and when $h_1 \geq \alpha h_2$ the base station decodes user 2 first. Under this strategy, the rates at the Nash-equilibrium are as shown in (27)–(28) at the bottom of the page, where λ_1, λ_2 are the solutions to the following equations:

$$\begin{aligned}\int_0^\infty \int_{\frac{\sigma^2}{\lambda_1}+\frac{(\lambda_2 h_2-\sigma^2)^+}{\lambda_1}}^{\alpha h_2} &\left(\lambda_1 - \frac{\sigma^2+(\lambda_2 h_2-\sigma^2)^+}{h_1}\right) f(h_1,h_2)dh_1dh_2 \\ &+\int_0^\infty \int_{\max\{\alpha h_2, \frac{\sigma^2}{\lambda_1}\}}^\infty \left(\lambda_1 - \frac{\sigma^2}{h_1}\right) f(h_1,h_2)dh_1dh_2 = \bar{P}_1 \\ \int_0^\infty \int_{\frac{\sigma^2}{\lambda_2}+\frac{(\lambda_1 h_1-\sigma^2)^+}{\lambda_2}}^{\frac{h_1}{\alpha}} &\left(\lambda_2 - \frac{\sigma^2+(\lambda_1 h_1-\sigma^2)^+}{h_2}\right) f(h_1,h_2)dh_1dh_2 \\ &+\int_0^\infty \int_{\max\{\frac{h_1}{\alpha}, \frac{\sigma^2}{\lambda_2}\}}^\infty \left(\lambda_2 - \frac{\sigma^2}{h_2}\right) f(h_1,h_2)dh_1dh_2 = \bar{P}_2. \end{aligned} \tag{29}$$

It is easy to verify that CR_1 is achieved by setting $\alpha = 0$, CR_2 is achieved by setting $\alpha = \infty$, and SP is achieved by setting $\alpha = \bar{\lambda}_2/\bar{\lambda}_1$, where $\bar{\lambda}_1, \bar{\lambda}_2$ are the water-filling levels given in the proof of Theorem 1. One can also prove the following statement.

Corollary 2: For the base station that adopts the simple region partition strategy, there always exists a Stackelberg equilibrium solution for any pair (μ_1, μ_2), if the support sets of the channel gains are compact and do not include 0.

Proof: Since $\mathcal{H}_1 = \{h_1\}$ and $\mathcal{H}_2 = \{h_2\}$ are compact, and $\min_{h_1 \in \mathcal{H}_1}(h_1) > 0$, $\min_{h_2 \in \mathcal{H}_2}(h_2) > 0$, then

$$\alpha \in \left[\min_{h_1\in\mathcal{H}_1}(h_1)/\max_{h_2\in\mathcal{H}_2}(h_2),\ \max_{h_1\in\mathcal{H}_1}(h_1)/\min_{h_2\in\mathcal{H}_2}(h_2)\right]$$

is a compact set. And for every α, we have proved in Theorem 3, $S^*(\alpha)$ is a singleton, thus based on [17], for any pair (μ_1, μ_2), there exists a Stackelberg equilibrium solution. □

C. Repeated Game Formulation

The inability of our Stackelberg game to achieve all the boundary points of the capacity region can be attributed to the structural difference between our successive decoding strategy and the optimal decoding strategy characterized in [1]. In particular, the optimal decoding strategy will always decode user 1 first (i.e., for all channel states) if $\mu_1 < \mu_2$, whereas in our formulation the decoding order is a function of the channel state. Unfortunately, if we adopt any *static* decoding order, the game will always settle at one of the corner points of the

$$\begin{aligned}\bar{R}_1(\alpha) = &\int_0^\infty \int_{\frac{\sigma^2}{\lambda_1}+\frac{(\lambda_2 h_2-\sigma^2)^+}{\lambda_1}}^{\alpha h_2} \frac{1}{2}\log_2\left(1+\frac{\lambda_1 h_1-\sigma^2-(\lambda_2 h_2-\sigma^2)^+}{\sigma^2+(\lambda_2 h_2-\sigma^2)^+}\right) f(h_1,h_2)dh_1dh_2 \\ &+\int_0^\infty \int_{\max\{\alpha h_2, \frac{\sigma^2}{\lambda_1}\}}^\infty \frac{1}{2}\log_2\left(1+\frac{\lambda_1 h_1-\sigma^2}{\sigma^2}\right) f(h_1,h_2)dh_1dh_2,\end{aligned} \tag{27}$$

$$\begin{aligned}\bar{R}_2(\alpha) = &\int_0^\infty \int_{\frac{\sigma^2}{\lambda_2}+\frac{(\lambda_1 h_1-\sigma^2)^+}{\lambda_2}}^{\frac{h_1}{\alpha}} \frac{1}{2}\log_2\left(1+\frac{\lambda_2 h_2-\sigma^2-(\lambda_1 h_1-\sigma^2)^+}{\sigma^2+(\lambda_1 h_1-\sigma^2)^+}\right) f(h_1,h_2)dh_1dh_2 \\ &+\int_0^\infty \int_{\max\{\frac{h_1}{\alpha}, \frac{\sigma^2}{\lambda_2}\}}^\infty \frac{1}{2}\log_2\left(1+\frac{\lambda_2 h_2-\sigma^2}{\sigma^2}\right) f(h_1,h_2)dh_1dh_2\end{aligned} \tag{28}$$

capacity region as argued in the previous section. To solve this problem, we pursue our last resort of replacing the static game formulation with a dynamic one.

The static formulation assumes that the players interact with each other only once. This assumption models the case where the topology of the network changes quickly. In a more slowly varying environment, a dynamic game formulation seems to be more appropriate. Specifically, we call a game where the players interact for $T > 1$ instances a dynamic game.[6] An example of a dynamic game is the repeated game where the same static game is played many times. Obviously, the users can play this game by repeating the same static strategy [13]. But, the advantage of the repeated game framework is that the players can do better than just repeating the same static strategy. The idea is that, since the players will interact with each other many times, they can learn each other's strategies, which may allow them to cooperate to obtain higher payoffs for both of them. In this case, the players can start cooperating and if one player deviates from the cooperation phase, the other players will adjust their strategies to punish the deviating player. The punishment *threat* is credible only if the deviating player achieves a lower payoff under punishment as compared with the cooperating phase. Under these circumstances, the users will have no desire to deviate from the cooperation phase, thus all the users can achieve higher utilities as compared with the static scenario.

In our repeated game setup, the players are the multiple access users and the base station. The strategy of the multiple access user is the power control policy. The strategy of the base station is the decoding order. The users choose their strategy simultaneously at each stage. The utility of each player can be defined as the time average, which assumes completely patient users, or as a discounted sum of the payoff achieved in each stage, where the discount factor $0 < \delta < 1$ models the level of patience: the larger δ is the more patient the player is. In the proof of the following theorem, we use a generalized version of a result due to Aumann and Shapley [13], [22] and define the payoff of the repeated game as the time-average of payoff at each stage. We later discuss briefly the discounted version of our game.

Theorem 5: As $T \to \infty$, all the boundary points of the capacity region are achievable under the repeated game setup. Moreover, the corresponding equilibria are subgame perfect.

Proof: In order to prove our claims, we need to construct a subgame perfect strategy that achieves every boundary point. Consider the following strategy: The base station announces its rate award vector $\boldsymbol{\mu}$, then the game proceeds in the following way.

1) $t = 1$, each user uses the optimal centralized control policy $\mathcal{P}_c$ and rate control policy $\mathcal{R}_c$ that maximize $\sum \mu_i \bar{R}_i$. Under this point, each user gets a rate $\bar{R}_i$.
2) If user 1 deviates from the centralized control policy at stage $t = t_d$, then the base station and user 2 punish user 1 by moving to the corner point CR_2 for T_1 periods (i.e., the base station decodes user 1 first for T_1 stages, and user 2 does water-filling over background noise during this period). The parameter T_1 is chosen such that

$$\bar{R}_{1,CR_1} + \sum_{i=2}^{T_1} \bar{R}_{1,CR_2} < \sum_{i=1}^{T_1} \bar{R}_1. \tag{30}$$

After T_1 periods, regardless of whether there are any deviations during the punishment period, the players return to the cooperative phase. If user 2 deviates, the base station and user 1 can also punish it for T_2 phases, which can be chosen in a similar way, by moving to the corner point CR_1.

The conditions on T_i ensures that any gain obtained from deviating is removed at the punishment phase, so no sequence of a finite or infinite number of deviations can increase user i's payoff. Moreover, although it is costly for the base station to carry out the punishment, any finite number of such losses are costless in the long run. This proves the subgame perfection of the strategy. □

In the previous proof, we assume that the users do not care about the loss they might incur by carrying out the punishment for the deviating user. If the users do care about this finite loss, as postulated by the overtaking criterion studied in [23], the users may deviate from the punishment period. In this case, we can modify our strategy to the following exponentially punishment strategy (inspired by Proposition 4.1 in [23]). The basic idea is to punish the user who deviates from the punishment period even more severely, and hence, ensuring that the punishment will be carried out. The first deviator is punished for T period. If during this punishment period, a punishing user (we call it the second deviator) deviates from the punishment for the first deviator, then the first punishment is terminated and all the remaining users punish the second deviator for T^2 period. If a punishing user (the third deviator) deviates from the punishment for the second deviator, then second punishment is terminated, and all the users punish the third deviator for T^3 period, and so on. Using a similar argument as [23], if T is large enough, no user will deviate. We note that the base station, as a user, can be punished by the multiple access users by adopting power allocation policies that correspond to a point outside the capacity region (resulting in a zero-utility for the base station).

When the users are not completely patient, the discounted version of our game becomes more appropriate. To prove our claim in this case, we need the following Fudenberg–Maskin theorem [13], [24]. Let $(v_1^*, \ldots, v_n^*)$ be the minimax utility vector, in which v_i^* is the maximum utility user i can get when all the other users are punishing it. Let $(v_1, \ldots, v_n)$ be any feasible utility vector, and $V^* = \{(v_1, \ldots, v_n) : v_i > v_i^*, \forall i\}$. Fudenberg–Maskin theorem states that if V^* has a nonempty interior, then for any $(v_1, \ldots, v_n) \in V^*$, there exists $\underline{\delta} \in (0,1)$ such that for all $\delta \in (\underline{\delta}, 1)$, there exist a subgame-perfect equilibrium of the infinitely repeated game with discount factor δ in which player i's average payoff is v_i. Now for the fading MAC under consideration, one can easily check that the minimax utility of user 1 is $\bar{R}_{1,CR2}$, the minimax utility of user

[6]We note that every game stage is assumed long enough to justify invoking the ergodic assumption within every stage.

2 is $\bar{R}_{2,CR1}$, the minimax utility of the base station is zero, any boundary points strictly Pareto dominates this minimax vector, and hence, the interior of the set that strictly Pareto dominates this minimax vector is nonempty. Applying the Fudenberg–Maskin theorem, we prove the discounted version of Theorem 5.

D. Arbitrary Number of Users

In this subsection, we briefly outline the extension to the $N \geq 3$ user channel.

In the Nash formulation, every user treats the signals from other users as noise. The optimal power control policy of each user is to water-fill over the sum of the interference and the background noise, i.e.,

$$\mathcal{P}_i(\boldsymbol{h}) = \left(\lambda_i - \frac{\sigma^2 + \sum_{j=1, j\neq i}^{N} h_j \mathcal{P}_j(\boldsymbol{h})}{h_i} \right)^+ . \tag{31}$$

Each user will adjust its water-level depending on the levels of the other users. At the Nash equilibrium points the water-levels $\lambda_i, \ i = 1, \ldots, N$ satisfy all power constraints with equality. In order to show that the only Nash equilibrium of this game is the maximum sum-rate point, we generalize the proof of Theorem 1. In particular, we show that at the equilibrium only one user will transmit at any fading state, then it is easy to verify that the power control policy of each user at the equilibrium is exactly the same as the corresponding central policy for the point SP. Without loss of generality, suppose that users 1 to M are transmitting simultaneously at certain fading states, then for each transmitting user, we have

$$\mathcal{P}_i + \frac{\sigma^2 + \sum_{j=1, j\neq i}^{M} h_j \mathcal{P}_j}{h_i} = \lambda_i, \quad i = 1, \ldots, M. \tag{32}$$

These conditions imply that $\lambda_i h_i = \lambda_j h_j, \forall i, j = 1, \ldots, M$. With continuous probability density functions, this happens with probability zero. Then with probability one, at any fading state, only one user will transmit. If user i transmits, the sum of background noise and the signal of user i should be larger than the water level of user j, and hence, h_i should satisfy

$$\left(\lambda_i - \frac{\sigma^2}{h_i} \right) \frac{h_i}{h_j} + \frac{\sigma^2}{h_j} = \lambda_i \frac{h_i}{h_j} \geq \lambda_j, \quad \forall j \neq i. \tag{33}$$

In the Stackelberg formulation, the receiver still uses the decoding order as its strategy. Given an arbitrary partition, using a similar argument to Theorem 2, it can be shown that $\mathcal{P}_i(\boldsymbol{h})$ is an nondecreasing function of λ_i, and is a nonincreasing function of $\lambda_j, j \neq i$. Hence, following the footsteps of Theorem 2, we can show that there always exist Nash equilibria in the lower level game. Unfortunately, the arguments used in the proof of Theorem 3 do not guarantee the uniqueness of the admissible Nash equilibrium with $N \geq 3$. But the ϵ-Stackelberg equilibria set still includes all the corner points and maximum sum rate point and does not include any other boundary points. To achieve any corner point, the base station can just fix its decoding order and to achieve the maximum sum rate point, the base station can choose the region as the time-sharing region. Further, the admissible power control policies of the multiple access users at these corner points and the maximum sum-rate point are unique. We can use a similar contradiction argument to Theorem 4 to show that we cannot achieve any other boundary point. In the repeated game formulation, the base station and multiple access users can punish the deviating user by operating at a nonfavorable corner point for this user, for an appropriate number of rounds. Therefore, the users will have no incentive to deviate which implies the achievability of any boundary point on the capacity region.

IV. Vector Channels

Thus far, we have presented our results for the scalar channel where the base station is only equipped with one receive antenna. In this section, we extend our study to the vector MAC, where the base station is equipped with N_r receive antennas. Our goal is to see if our previous conclusions carry through or not. Again to simplify the presentation, we focus on the two-user scenario. The signal received at any time n is given by

$$\boldsymbol{y}(n) = \sum_{i=1}^{2} \boldsymbol{h}_i(n) x_i(n) + \boldsymbol{z}(n) \tag{34}$$

where $\boldsymbol{h}_i(n) = [\sqrt{h_{1i}}, \sqrt{h_{2i}}, \ldots, \sqrt{h_{N_r i}}]^T$ is the $N_r \times 1$ fading vector from user i to the N_r receive antennas. As before, we assume that the fading processes have a joint continuous distribution with a bounded density. $\boldsymbol{z}(n)$ is the Gaussian noise vector at the N_r receive antenna with correlation matrix $\mathbb{E}[\boldsymbol{z}\boldsymbol{z}^T] = \sigma^2 \boldsymbol{I}_{Nr}$.

Similar to the scalar channel case, we first consider the static Nash formulation where the players of the game are the multiple access users and the base station employs single-user decoders. The strategy space of user i is still

$$\mathcal{F}_i = \{\mathcal{P}_i : \mathbb{E}_{\boldsymbol{H}}[\mathcal{P}_i] \leq \bar{P}_i, \mathcal{P}_i(\boldsymbol{H}) \geq 0\}$$

with $\boldsymbol{H} = [\boldsymbol{h}_1, \boldsymbol{h}_2]$. The payoff function of user i is still the average achievable rate $\bar{R}_i = \mathbb{E}_{\boldsymbol{H}}[\mathcal{R}_i]$. It is easy to see that for any power control strategy $\mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)$ of user 2, the optimal power control policy of user 1 is the solution to the following optimization problem:

$$\begin{aligned}
\max_{\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)} \bar{R}_1 = &\mathbb{E}_{\boldsymbol{H}} \left[\frac{1}{2} \log_2 \left(\det \left(\sigma^2 \boldsymbol{I}_{N_r} + \mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2) \boldsymbol{h}_1 \boldsymbol{h}_1^T \right.\right.\right. \\
&\left.\left. + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2) \boldsymbol{h}_2 \boldsymbol{h}_2^T \right) \right) \\
&\left. - \frac{1}{2} \log_2 \left(\det \left(\sigma^2 \boldsymbol{I}_{N_r} + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2) \boldsymbol{h}_2 \boldsymbol{h}_2^T \right) \right) \right] \\
\text{s.t.} \quad &\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2) \in \mathcal{F}_1.
\end{aligned} \tag{35}$$

Given any power control strategy $\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)$ of user 1, the optimal power control strategy of user 2 is a solution to a similar problem. The difference between the vector and scalar channels is highlighted in the following result.

Theorem 6: There exists a unique Nash equilibrium for the power allocation game in the vector MAC. At this equilibrium, the power control policy of each user is the same as the central policy that achieves the maximum sum-rate point SP. The achieved rates, however, are strictly smaller than the rates corresponding to SP.

Proof: Given the power control policy $\mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)$, it is easy to see that

$$\mathbb{E}_{\boldsymbol{H}}\left[\frac{1}{2}\log_2\left(\det\left(\sigma^2\boldsymbol{I}_{N_r} + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_2\boldsymbol{h}_2^T\right)\right)\right]$$

is a constant, thus the solution to the optimization problem (35) is the same as the solution to the following optimization problem:

$$\begin{aligned}\max_{\mathcal{P}_1(\boldsymbol{h}_1,\boldsymbol{h}_2)} f(\mathcal{P}_1) = \mathbb{E}_{\boldsymbol{H}}\Bigg[\frac{1}{2}\log_2\Bigg(\det\Big(&\sigma^2\boldsymbol{I}_{N_r} + \mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_1\boldsymbol{h}_1^T \\ &+ \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_2\boldsymbol{h}_2^T\Big)\Bigg)\Bigg] \\ \text{s.t.} \quad \mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2) \in \mathcal{F}_1.\end{aligned} \tag{36}$$

Since $\sigma^2\boldsymbol{I}_{N_r} + \mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_1\boldsymbol{h}_1^T + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_2\boldsymbol{h}_2^T$ is positive definite, and the $\log_2(\det(\cdot))$ function is concave in the set of positive-definite matrices, then the objective function is concave in the set of power allocation policies. The constraint set is convex. It is also of users' interest to meet the equality in the power constraint, that is, $\mathbb{E}_{\boldsymbol{H}}\{\mathcal{P}_i\} = \bar{P}_i$. Hence, there exists a constant γ_1, such that the solution to (35) is the same as the solution to the following optimization problem:

$$\begin{aligned}&\max_{\mathcal{P}_1(\boldsymbol{h}_1,\boldsymbol{h}_2)} L_1(\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2), \gamma_1) \\ &\quad = \mathbb{E}_{\boldsymbol{H}}\Bigg[\frac{1}{2}\log_2\Bigg(\det\Big(\sigma^2\boldsymbol{I}_{N_r} + \mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_1\boldsymbol{h}_1^T \\ &\qquad + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_2\boldsymbol{h}_2^T\Big)\Bigg)\Bigg] \\ &\qquad - \gamma_1\mathbb{E}_{\boldsymbol{H}}[\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)].\end{aligned} \tag{37}$$

The Karush–Kuhn–Tucker (KKT) necessary and sufficient conditions of this optimization problem is

$$\frac{\partial L_1}{\partial \mathcal{P}_1} = \boldsymbol{h}_1^T\left(\sigma^2\boldsymbol{I}_{N_r} + \mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_1\boldsymbol{h}_1^T + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_2\boldsymbol{h}_2^T\right)^{-1} \boldsymbol{h}_1 - \gamma_1 = 0. \tag{38}$$

Using the matrix inversion lemma [25]

$$(\boldsymbol{A} + \boldsymbol{x}\boldsymbol{x}^t)^{-1} = \boldsymbol{A}^{-1} - \frac{\boldsymbol{A}^{-1}\boldsymbol{x}\boldsymbol{x}^t\boldsymbol{A}^{-1}}{1 + \boldsymbol{x}^t\boldsymbol{A}^{-1}\boldsymbol{x}} \tag{39}$$

we come to

$$\frac{\partial L_1}{\partial \mathcal{P}_1} = \frac{\boldsymbol{h}_1^T\left(\sigma^2\boldsymbol{I}_{N_r} + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_2\boldsymbol{h}_2^T\right)^{-1}\boldsymbol{h}_1}{1 + \boldsymbol{h}_1^T\left(\sigma^2\boldsymbol{I}_{N_r} + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_2\boldsymbol{h}_2^T\right)^{-1}\boldsymbol{h}_1\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)} - \gamma_1 = 0. \tag{40}$$

Taking the constraint $\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2) \geq 0$ into consideration, we get

$$\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2) = \left(\lambda_1 - \frac{1}{\boldsymbol{h}_1^T\left(\sigma^2\boldsymbol{I}_{N_r} + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_2\boldsymbol{h}_2^T\right)^{-1}\boldsymbol{h}_1}\right)^+ \tag{41}$$

where $\lambda_1 = \frac{1}{\gamma_1}$ is a constant that satisfies the average power constraint of user 1 with equality. Similarly, given $\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)$, we get the following optimality condition:

$$\boldsymbol{h}_2^T\left(\sigma^2\boldsymbol{I}_{N_r} + \mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_1\boldsymbol{h}_1^T + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_2\boldsymbol{h}_2^T\right)^{-1}\boldsymbol{h}_2 - \gamma_2 = 0. \tag{42}$$

The optimal policy of user 2 is therefore

$$\mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2) = \left(\lambda_2 - \frac{1}{\boldsymbol{h}_2^T\left(\sigma^2\boldsymbol{I}_{N_r} + \mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_1\boldsymbol{h}_1^T\right)^{-1}\boldsymbol{h}_2}\right)^+ \tag{43}$$

where λ_2 is the constant that satisfies the average power constraint of user 2 with equality. Applying the results of [26] to the fading MAC with N_r receive antennas, we know that (38) and (42) are exactly the optimality conditions for the following optimization problem:

$$\begin{aligned}&\max_{\mathcal{P}_1,\mathcal{P}_2} \quad \bar{R}_{sum}(\mathcal{P}_1, \mathcal{P}_2) \\ &\quad = \mathbb{E}_{\boldsymbol{H}}[\mathcal{R}_1 + \mathcal{R}_2] \\ &\quad = \mathbb{E}_{\boldsymbol{H}}\Bigg[\frac{1}{2}\log_2\Bigg(\det\Big(\boldsymbol{I}_{N_r} \\ &\qquad + \frac{\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_1\boldsymbol{h}_1^T + \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2)\boldsymbol{h}_2\boldsymbol{h}_2^T}{\sigma^2}\Big)\Bigg)\Bigg] \\ &\quad \text{s.t.} \quad \mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2) \in \mathcal{F}_1, \mathcal{P}_2(\boldsymbol{h}_1, \boldsymbol{h}_2) \in \mathcal{F}_2.\end{aligned} \tag{44}$$

One can easily verify that the optimization problem (44) will maximize the sum-rate at the base station. This means that the optimal policy of each user aiming to maximize its **own** rate while treating the signal of the other user as interference is exactly the same as the power control policy that maximizes the sum-rate at the base station. A similar observation has been made in the vector Gaussian MAC in [27].

Therefore, we can apply the following iterative process to get the power control policy at the Nash equilibrium point. Starting at $\mathcal{P}_1 = 0$, $\mathcal{P}_2 = 0$, each user takes a turn to water-fill over the combined interference and the background noise. At each step, the objective function of (44) increases. But with limited average power at the users, the objective function (44) has an upper bound. Thus, this process will converge, which means that the Nash equilibrium exists. At the convergence point, the optimality conditions (38) and (42) hold, which means that the power control policy of each user at the Nash equilibrium is the same as the optimal policy that maximizes the sum-rate at the base station. The uniqueness of the power control policy that maximizes the sum-rate [26] implies the uniqueness of the Nash equilibrium point. This proves our first two claims.

From [26], we know that the optimal central control policy is not time-sharing. Hence, in some channel fading states, the transmission power of both users will be larger than zero. In these cases, the capacity region pentagon is shown in Fig. 4. We can easily see that the central rate control policy will always operate on one of the boundary points (the line between $X1$ and $X2$), but the distributed scheme will always choose the point

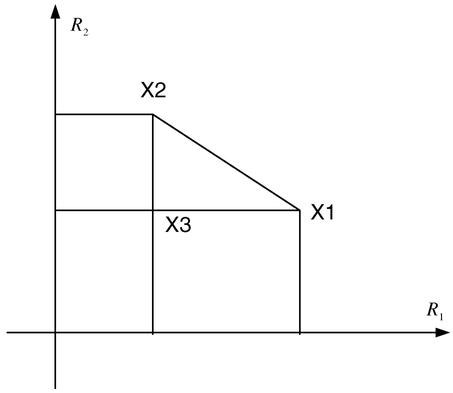

Fig. 4. The capacity region pentagon for fixed channel gains.

$X3$. We have either $\mathbb{E}_{\boldsymbol{H}}[R_{1N}] < \mathbb{E}_{\boldsymbol{H}}[R_{1,\text{sum}}]$ or $\mathbb{E}_{\boldsymbol{H}}[R_{2N}] < \mathbb{E}_{\boldsymbol{H}}[R_{2,\text{sum}}]$. This completes the proof. □

Theorem 6 contrasts the scalar scenario, where the Nash equilibrium rate is the same as the maximum sum-rate. The reason is that in the scalar MAC, the strategy that maximizes the sum-rate is time-sharing. In the vector case, on the other hand, we have $\min(N, N_r)$ degrees of freedom, and hence, more than one user are allowed to transmit at any fading state. The central control policy will choose to operate at one of the boundary points, but because of the interference, the multiple access users will distributively choose a point that is strictly inside the capacity region at the Nash equilibrium point.

Our Stackelberg game can also be extended to the vector MAC. Similar to the scalar case, the base station partitions the space of $(\boldsymbol{h}_1, \boldsymbol{h}_2)$ into two region D_1, D_1^c, and decodes user 1 first in D_1 and decode user 2 first in the region D_1^c. The following results do not depend on the specific choice of D_1. The strategy space of user i is still $\mathcal{F}_i$, and the payoff function of each user is still the supremum of achievable average rate.

Theorem 7: For any base station strategy D_1, there exists a unique admissible Nash equilibrium for the low-level game. The Stackelberg game achieves the two corner points of the capacity region but does not achieve the maximum sum-rate point.

Proof: The proof of the existence of a unique admissible Nash equilibrium for the low-level game under any base station strategy D_1 follows essentially the same lines as the proofs of Theorems 2 and 3. The only additional requirement is to prove that $\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)$ is a nondecreasing function of λ_1 and a nonincreasing function of λ_2.

Based on the proof of Theorem 6, we know that the optimal power control policy of user 1 is

$$\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2) = \left(\lambda_1 - \frac{\sigma^2}{\| \boldsymbol{h}_1 \|^2}\right)^+, \quad \text{if } (\boldsymbol{h}_1, \boldsymbol{h}_2) \in D_1^c$$
$$\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2) = \left(\lambda_1 - \frac{1}{f_1(\lambda_2)}\right)^+, \quad \text{if } (\boldsymbol{h}_1, \boldsymbol{h}_2) \in D_1 \quad (45)$$

in which

$$f_1(\lambda_2) = \boldsymbol{h}_1^T\left(\sigma^2 \boldsymbol{I}_{N_r} + \boldsymbol{h}_2\boldsymbol{h}_2^T\left(\lambda_2 - \frac{\sigma^2}{\| \boldsymbol{h}_2 \|^2}\right)^+\right)^{-1}\boldsymbol{h}_1.$$

It is easy to verify that $\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)$ is a nondecreasing function of λ_1. To show that $\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)$ is a nonincreasing function of λ_2, we only need to show that $f(\lambda_2)$ is a nonincreasing function of λ_2.

Using the matrix inversion lemma (39), we have

$$\begin{aligned} &\boldsymbol{h}_1^T\left(\sigma^2 \boldsymbol{I}_{N_r} + \boldsymbol{h}_2\boldsymbol{h}_2^T(\lambda_2 - \frac{\sigma^2}{\| \boldsymbol{h}_2 \|^2})^+\right)^{-1}\boldsymbol{h}_1 \\ &= \boldsymbol{h}_1^T\left(\frac{\boldsymbol{I}_{N_r}}{\sigma^2} - \frac{\left(\lambda_2 - \frac{\sigma^2}{\|\boldsymbol{h}_2\|^2}\right)^+ \boldsymbol{h}_2\boldsymbol{h}_2^T}{\sigma^4 + \sigma^2\left(\| \boldsymbol{h}_2 \|^2 \lambda_2 - \sigma^2\right)^+}\right)\boldsymbol{h}_1 \\ &= \frac{\| \boldsymbol{h}_1 \|^2}{\sigma^2} - | \boldsymbol{h}_2^T\boldsymbol{h}_1 |^2 \, g(\lambda_2) \end{aligned} \quad (46)$$

in which

$$g(\lambda_2) = \frac{\left(\lambda_2 - \frac{\sigma^2}{\|\boldsymbol{h}_2\|^2}\right)^+}{\sigma^4 + \sigma^2\left(\| \boldsymbol{h}_2 \|^2 \lambda_2 - \sigma^2\right)^+}. \quad (47)$$

It is easy to verify that $g(\lambda_2)$ is a nondecreasing function of λ_2, thus, we come to the conclusion that $\mathcal{P}_1(\boldsymbol{h}_1, \boldsymbol{h}_2)$ is a nondecreasing function of λ_1 and a nonincreasing function of λ_2. To achieve the corner points, the base station can just set D_1 to be the whole set, in one case, and the empty set in the other case.

We prove the nonexistence of a base station strategy that achieves the sum-rate point by contradiction. Suppose that a partition D_1 achieves the sum-rate point. Since the unique power control policy that achieves the maximum sum-rate point is to water-fill over the sum of the interference and the background noise for **both users**, then in the region D_1, user 1 should stop sending. Because in this region, the optimal distributed power control policy of user 2 is to water-fill only over the background noise. Similarly, in the region D_1^c, user 2 should also stop sending. Then we come to a time-sharing solution, which cannot achieve the maximum sum-rate point and we have our contradiction. □

Finally, if the users have the opportunity to interact many times then any boundary point of the capacity region of the vector MAC can be achieved as a subgame perfect equilibrium. Moreover, the users can use the same strategies developed in Theorem 5 to achieve these boundary points.

V. Conclusion

This paper has developed a game-theoretic framework for distributed resource allocation in fading MACs. In our first result, we showed that the opportunistic communications principle can be obtained as the unique Nash equilibrium of a water-filling game. By introducing the base station as a player, we were able to achieve all the corner points of the capacity region, in addition to the maximum sum-rate point, distributively. In slow-varying environments, where the multiple access users can be assumed to interact many times, the repeated game formulation was shown to achieve all the boundary points of the capacity region. Finally, we elucidated the limitations of our game-theoretic framework in vector MACs.

An interesting avenue for future work is to further investigate the practical aspects of our framework. For example, a natural extension is to consider the case with partial and/or distorted channel state information by borrowing tools from game theory with incomplete information.

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