On the Sum-Rate Capacity of Poisson MISO Multiple Access Channels

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Abstract—In this paper, we analyze the sum-rate capacity of two-user Poisson multiple access channels (MAC) when the receiver is equipped with single antenna. We first characterize the sum-rate capacity of the *non-symmetric* Poisson MAC when each transmitter has a single antenna. While the sum-rate capacity of the *symmetric* Poisson MAC with single antenna at each transmitter has been characterized in the literature, the special property exploited in the existing method for the symmetric case does not hold for the non-symmetric channel anymore. We obtain the optimal input that achieves the sum-rate capacity by solving a non-convex optimization problem. We show that, for certain channel parameters, it is optimal for a single-user to transmit to achieve the sum-rate capacity. This is in sharp contrast to the Gaussian MAC, in which both users must transmit, either simultaneously or at different times, in order to achieve the sum-rate capacity. We then characterize the sum-rate capacity of the Poisson multiple input single output (MISO) MAC with multiple antennas at each transmitter and single antenna at the receiver. By converting a non-convex optimization problem with a large number of variables into a non-convex optimization problem with two variables, we show that the sum-rate capacity of the Poisson MISO MAC with multiple transmit antennas is equivalent to a properly constructed Poisson MAC with a single antenna at each transmitter.

Index Terms—Poisson channels, multiple input single output, multi-access channels, optimal power allocation, sum-rate capacity

I. INTRODUCTION

Poisson Channel, whereby the arrival of photons is recorded by photon-sensitive devices incorporated in the receivers [2], is often used to model free-space optical (FSO) and visible light communication (VLC). The point-to-point single-user Poisson channel has been investigated from various perspectives, including single antenna [3], multiple antennas [4], fading channels [5], [6], in continuous-time [3], [7]–[9] and discrete-time [10]–[12]. On the other hand, Poisson channels with multiple users are not that well understood. Among limited

existing work, [13], [14] focus on the Poisson broadcast channel, [15] studies the Poisson multiple-access channel (MAC), [16] considers the optimization of the capacity region of Poisson MAC with respect to different power constraints, and [17] investigates the Poisson channel with side information at the transmitter. Furthermore, [18], [19] discuss the discrete-time Poisson channel and show that sum-capacity achieving distributions of the Poisson MAC under peak amplitude constraints are discrete with a finite number of mass points. [20] discusses a discrete memoryless Poisson MAC with noiseless feedback.

Of particular relevance to our study is [15], which thoroughly investigates the continuous-time Poisson MAC with each user equipped with single antenna. [15] shows that the approximation of the complex continuous-time continuous-input discrete-output Poisson MAC by a discrete-time binary-input binary-output MAC does not result in a loss in terms of the capacity region. [15] determines the sum-rate capacity of the symmetric Poisson MAC, in which the channel gains and power constraints for all users are identical under the maximum power constraint. Furthermore, it characterizes the boundary points on the capacity region of the symmetric MAC under maximum power constraint and analyzes the maximum-throughput under peak-power and average power constraints.

In this paper, we first study the single antenna *non-symmetric* Poisson MAC, in which the channel gains and power constraints at the two users are not necessarily the same. We refer to such a channel as Poisson SISO-MAC. This scenario naturally arises in multiuser optical communications when the transmitters have different distances to the receiver or have different transmission powers. Unfortunately, the method used in [15] to characterize the sum-rate capacity for the *symmetric* case does not apply to the *non-symmetric* case anymore. In particular, the method in [15] exploits the property that the objective function involved is a Schur concave function for the *symmetric* Poisson MAC, which greatly simplifies the analysis. However, in the *non-symmetric* channel, the objective function is not symmetric, and hence is not Schur concave anymore. As a result, we resort to a different approach from the one used in [15] to study the sum-rate capacity. More specifically, we show that characterizing the sum-rate capacity is equivalent to solving a non-convex optimization problem. We show that there are at most four possible candidates for the optimal solution to this optimization problem with two candidate solutions corresponding to the cases when only one user transmits. We further show that, for some channel parameters, it is indeed optimal to allow only one user to transmit

in order to achieve the sum-rate capacity under the maximum power constraint. This is in sharp contrast to the Gaussian MAC with an average power constraint, in which it is always optimal for both users to transmit, either simultaneously or at different time, to achieve the sum-rate capacity. We also identify conditions under which it is optimal for both users to transmit in order to achieve the sum-rate capacity.

We then extend the study to Poisson MAC with multiple antennas at each transmitter and one antenna at the receiver. We refer to this as Poisson MISO-MAC. Similarly to the Poisson SISO-MAC, the complex continuous-time continuous-input discrete-output Poisson MAC can be converted to a discrete-time binary-input binary-output Poisson MAC. However, the resulting problem is much more challenging than that of the Poisson SISO-MAC. In particular, to characterize the sum-rate capacity, we need to solve a non-convex optimization problem with $2^{J_1} + 2^{J_2}$ variables, in which J_n is the number of antennas at user n. Despite this challenge, we show that the optimal value obtained from this optimization problem with a large number of variables is the same as that of an optimization problem with only 2 variables. Furthermore, this reduced dimension optimization problem is equivalent to a problem arising in the Poisson SISO-MAC with properly chosen parameters. As the result, characterizing the sum-rate capacity of the Poisson MISO-MAC is equivalent to characterizing the sum-rate capacity of a Poisson SISO-MAC. Hence, the techniques and asymptotic analysis developed in the SISO-MAC case can be used for the MISO-MAC case. There are two major steps in our proof. In the first step, we show that the original optimization problem with $2^{J_1} + 2^{J_2}$ variables can be converted to a non-convex optimization problem with $J_1 + J_2$ variables by showing and exploiting the fact that, at the optimality, if the antenna with a smaller duty cycle is on, then the antenna with a larger duty cycle is also on. In the second step, we show that the optimization problem with $J_1 + J_2$ variables obtained in step 1 can be converted to an optimization problem with only 2 variables. The key ingredient in this step is to show that, at the optimality, all antennas at each transmitter have to be simultaneously on or off.

A natural next step is to characterize the sum-rate capacity of the scenario when there are also multiple antennas at the receiver. Another natural extension is to characterize the set of boundary points of the full capacity region. As will be discussed in sequel, there are significant technical challenges in extending our methods to these cases. Hence, new techniques are required for these extensions. One possible direction to pursue is to explore the relationship between the derivative of mutual information and mean squared error as discussed in [21] and [22].

The remainder of the paper is organized as follows. Section II describes the model under consideration. Section III analyzes the Poisson SISO-MAC. Section IV analyzes the Poisson MISO-MAC. Section V discusses challenges for certain extensions. Numerical analysis is presented in Section VI and concluding remarks are presented in Section VII.
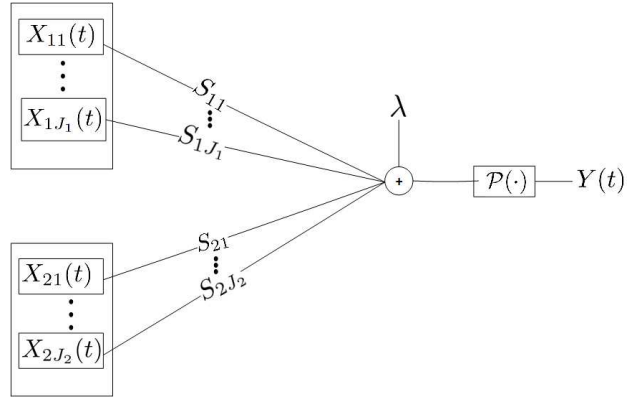

Fig. 1: The Poisson MISO-MAC model.

II. System Model

In this section, we introduce the model considered in this paper. As shown in the Fig. 1, we consider the continuous-time two-user Poisson MISO-MAC with two users communicating with a single antenna receiver. For transmitter n, it is equipped with J_n transmit antennas. Let $X_{nj}(t)$ be the input of the jth transmitter from n^{th} user and $Y(t)$ be the doubly-stochastic Poisson process observed at the receiver antenna. The input-output relationship can be described as:

$$Y(t) = \mathcal{P}\left(\sum_{n=1}^{2}\sum_{j=1}^{J_n} S_{nj}X_{nj}(t) + \lambda\right), \qquad (1)$$

in which S_{nj} is the channel response between the jth antenna of the nth user to the receiver, λ is the dark current at receiver antenna, and $\mathcal{P}(\cdot)$ is the nonlinear transformation converting the light strength to the doubly-stochastic Poisson process that records the timing and number of photon's arrivals. In particular, for any time interval $[t, t+\tau]$, the probability that there are k photons arriving at the receiver is

$$\Pr\{Y(t+\tau) - Y(t) = k\} = \frac{e^{-\Lambda}\Lambda^k}{k!}, \qquad (2)$$

where

$$\Lambda = \int_{t}^{t+\tau}\left[\sum_{n=1}^{2}\sum_{j=1}^{J_n} S_{nj}X_{nj}(t') + \lambda\right] dt'. \qquad (3)$$

We consider the peak power constraint, i.e., the transmitted signal $X_{nj}(t)$ must satisfy the following constraint:

$$0 \le X_{nj}(t) \le A_{nj}, \qquad (4)$$

where A_{nj} is the maximum power allowed by the jth antenna of the nth transmitter. We use μ_{nj} to denote the duty cycle of each transmitting antenna, i.e., μ_{nj} is the percentage of time at which the jth antenna of the nth user is on. We use $\boldsymbol{\mu}$ to denote the vector of all μ_{nj}s.

Throughout the paper, we use the following notation:

$$\varphi(x) \triangleq x\log(x), \qquad (5)$$

$$\zeta(x, y) \triangleq (x+y)\log(x+y) - y\log y, \qquad (6)$$

$$\alpha(x) \triangleq \frac{1}{x}\left(e^{-1}(1+x)^{\left(1+\frac{1}{x}\right)} - 1\right). \qquad (7)$$

It is easy to check that $0 < \alpha(x) < 1$ for $x \geq 0$.

Our goal is to characterize the sum-rate capacity of this Poisson MAC.

III. SISO-MAC ANALYSIS

In this section, we focus on the special case in which each transmitter has only one antenna, i.e., $J_1 = 1$ and $J_2 = 1$. Hence for the sake of convenience, we drop the subscript j in this section. The techniques developed in this section will be used in the more complicated setup considered in Section IV.

A. Optimality Conditions

It has been shown in [15] that the continuous-time continuous-input discrete-output Poisson MAC can be converted to a much simpler discrete-time binary-input binary-output MAC. In particular, the input waveform can be limited to be piecewise constant waveforms with two levels 0 or A_n for the nth transmitter. Let μ_n be the duty cycle of the nth transmitter (i.e., μ_n is the fraction of the time that transmitter n is on). It has been shown in [15] that the sum-rate capacity is given by

(P0): $C_{sum}^{SISO-MAC} = \max_{0 \leq \mu_1, \mu_2 \leq 1} I_{X_1, X_2; Y}(\mu_1, \mu_2),$ (8)

in which

$$I_{X_1, X_2; Y}(\mu_1, \mu_2) = (1 - \mu_1)(1 - \mu_2)\varphi(\lambda)$$
$$+ \mu_1(1 - \mu_2)\varphi(S_1 A_1 + \lambda) + (1 - \mu_1)\mu_2\varphi(S_2 A_2 + \lambda)$$
$$+ \mu_1 \mu_2 \varphi(S_1 A_1 + S_2 A_2 + \lambda) - \varphi(S_1 A_1 \mu_1 + S_2 A_2 \mu_2 + \lambda).$$

The optimization problem (8) has been solved by [15] for the symmetric case with $S_1 A_1 = S_2 A_2$. In particular, [15] shows that the objective function $I_{X_1, X_2; Y}(\mu_1, \mu_2)$ is a Schur concave function when $S_1 A_1 = S_2 A_2$. As the result, if $(\hat{\mu}_1, \hat{\mu}_2)$ is the optimal solution to (8) for the symmetric case, $\hat{\mu}_1$ must be equal to $\hat{\mu}_2$. Hence, the problem can be converted into a one-dimensional optimization problem, which can be solved easily.

However, the situation for the non-symmetric case is different. In particular, if $S_1 A_1 \neq S_2 A_2$, then $I_{X_1, X_2; Y}(\mu_1, \mu_2)$ is not a Schur concave function anymore. This can be observed from the fact that a Schur concave function must be a symmetric function (see page 258 of [23]), while $I_{X_1, X_2; Y}(\mu_1, \mu_2)$ is not a symmetric function when $S_1 A_1 \neq S_2 A_2$. Therefore, the techniques developed in [15] for the symmetric case cannot be extended to the non-symmetric case. Furthermore, for general values of $S_n A_n$ and λ, $I_{X_1, X_2; Y}(\mu_1, \mu_2)$ is not necessarily a concave function of (μ_1, μ_2), (see the proof in Appendix A). Hence, (8) is a non-convex optimization problem in general.

In the following, we solve this non-convex optimization problem. We start with the necessary KKT conditions (since the problem is not convex, these conditions are not sufficient for optimality). For convenience, we write $I_{X_1, X_2; Y} = I$ and hence the corresponding Lagrangian equation is given by:

$$\mathcal{L} = -I + \eta_1(\mu_1 - 1) - \eta_2 \mu_1 + \eta_3(\mu_2 - 1) - \eta_4 \mu_2. \quad (9)$$

The optimal solution $(\hat{\mu}_1, \hat{\mu}_2)$ must satisfy the following KKT constraints:

$$\frac{\partial I}{\partial \mu_1}\bigg|_{(\hat{\mu}_1, \hat{\mu}_2)} - \eta_1 + \eta_2 = 0,$$
$$\frac{\partial I}{\partial \mu_2}\bigg|_{(\hat{\mu}_1, \hat{\mu}_2)} - \eta_3 + \eta_4 = 0,$$
$$\eta_1(\hat{\mu}_1 - 1) = 0,$$
$$\eta_2 \hat{\mu}_1 = 0,$$
$$\eta_3(\hat{\mu}_2 - 1) = 0,$$
$$\eta_4 \hat{\mu}_2 = 0,$$

where

$$\frac{\partial I}{\partial \mu_1} = -(1 - \mu_2)\varphi(\lambda) + (1 - \mu_2)\varphi(S_1 A_1 + \lambda)$$
$$- \mu_2 \varphi(S_2 A_2 + \lambda) + \mu_2 \varphi(S_1 A_1 + S_2 A_2 + \lambda)$$
$$- S_1 A_1 \log(S_1 A_1 \mu_1 + S_2 A_2 \mu_2 + \lambda) - S_1 A_1, \quad (10)$$

and

$$\frac{\partial I}{\partial \mu_2} = -(1 - \mu_1)\varphi(\lambda) - \mu_1 \varphi(S_1 A_1 + \lambda)$$
$$+ (1 - \mu_1)\varphi(S_2 A_2 + \lambda) + \mu_1 \varphi(S_1 A_1 + S_2 A_2 + \lambda)$$
$$- S_2 A_2 \log(S_1 A_1 \mu_1 + S_2 A_2 \mu_2 + \lambda) - S_2 A_2. \quad (11)$$

In order to further analyze the above KKT conditions, we need to consider 16 cases corresponding to different combinations of active constraints (i.e., whether $\eta_i = 0$ or not for $i = 1, \cdots, 4$). For example, if $\eta_1 = 0, \eta_2 = 0, \eta_3 \neq 0, \eta_4 = 0$, then the above KKT conditions can be simplified to

$$\frac{\partial I}{\partial \mu_1}\bigg|_{(\hat{\mu}_1, \hat{\mu}_2)} = 0,$$
$$\frac{\partial I}{\partial \mu_2}\bigg|_{(\hat{\mu}_1, \hat{\mu}_2)} - \eta_3 = 0,$$
$$\eta_3(\hat{\mu}_2 - 1) = 0,$$

from which we obtain

$$\hat{\mu}_1 = \alpha\left(\frac{S_1 A_1}{S_2 A_2 + \lambda}\right),$$
$$\hat{\mu}_2 = 1. \quad (12)$$

Since $\max I(\hat{\mu}_1, 0) > \max I(\hat{\mu}_1, 1)$, (12) is clearly not an optimal solution.

Using similar arguments, we can show that among these 16 possible combinations, 13 constraint combinations result in non-optimal solutions. We are then left with the following three possible scenarios:

Scenario 1: $\eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \eta_4 = 0$**:**
The KKT conditions are simplified to

$$\frac{\partial I}{\partial \mu_1}\bigg|_{(\mu_1, \mu_2)} = 0, \quad (13)$$
$$\frac{\partial I}{\partial \mu_2}\bigg|_{(\mu_1, \mu_2)} = 0. \quad (14)$$

This scenario corresponds to the case where both users are active. From (10) and (11), we can see that both $\frac{\partial I}{\partial \mu_1}\big|_{(\mu_1, \mu_2)}$ and $\frac{\partial I}{\partial \mu_2}\big|_{(\mu_1, \mu_2)}$ are nonlinear functions of (μ_1, μ_2). Hence,

there can be multiple (μ_1, μ_2) pairs satisfying (13) and (14) simultaneously. However, we now show that there are in fact at most 2 possible (μ_1, μ_2) pairs that satisfy (13) and (14) simultaneously.

First, by (13)$\times S_2 A_2 = $ (14)$\times S_1 A_1$, we have

$$
\begin{aligned}
& S_2 A_2 (-(1-\mu_2)\varphi(\lambda) + (1-\mu_2)\varphi(S_1 A_1 + \lambda) \\
& -\mu_2\varphi(S_2 A_2 + \lambda)) + \mu_2\varphi(S_1 A_1 + S_2 A_2 + \lambda)) \\
& = S_1 A_1 (-(1-\mu_1)\varphi(\lambda) - \mu_1\varphi(S_1 A_1 + \lambda) \\
& -(1-\mu_1)\varphi(S_2 A_2 + \lambda) + \mu_1\varphi(S_1 A_1 + S_2 A_2 + \lambda)). \quad (15)
\end{aligned}
$$

Using (15), we can write μ_2 in terms of μ_1:

$$
\mu_2 = \frac{V}{W} + \frac{S_1 A_1}{S_2 A_2}\mu_1 \triangleq f(\mu_1), \quad (16)
$$

where

$$
\begin{aligned}
W & \triangleq \varphi(\lambda) - \varphi(S_1 A_1 + \lambda) - \varphi(S_2 A_2 + \lambda) \\
& \qquad + \varphi(S_1 A_1 + S_2 A_2 + \lambda), \quad (17) \\
V & \triangleq -\varphi(S_1 A_1 + \lambda) + \varphi(\lambda) - \frac{S_1 A_1}{S_2 A_2}\varphi(\lambda) \\
& \qquad + \frac{S_1 A_1}{S_2 A_2}\varphi(S_2 A_2 + \lambda). \quad (18)
\end{aligned}
$$

It is clear that $f(\mu_1)$ is a linear function of μ_1.

Using $\frac{\partial I}{\partial \mu_2} = 0$, we can write μ_2 in terms of μ_1:

$$
\begin{aligned}
\mu_2 & = \frac{1}{S_2 A_2} \left(\exp\left(\frac{1}{S_2 A_2}(-(1-\mu_1)\varphi(\lambda) \right.\right. \\
& -\mu_1\varphi(S_1 A_1 + \lambda) + (1-\mu_1)\varphi(S_2 A_2 + \lambda) \\
& \left.\left. + \mu_1\varphi(S_1 A_1 + S_2 A_2 + \lambda) - S_2 A_2) \right) \right) - \frac{S_1 A_1 \mu_1 + \lambda}{S_2 A_2} \\
& \triangleq g(\mu_1). \quad (19)
\end{aligned}
$$

It is easy to check that $g''(\mu_1) > 0$, and hence $g(\mu_1)$ is a strictly convex function of μ_1.

We have just converted (13) and (14) into equivalent forms:

$$
\mu_2 = f(\mu_1), \quad (20)
$$
$$
\mu_2 = g(\mu_1). \quad (21)
$$

Hence, (μ_1, μ_2) pairs where $f(\mu_1)$ and $g(\mu_1)$ intersect with each other satisfy (13) and (14) simultaneously. As $f(\mu_1)$ is a linear function of μ_1, while $g(\mu_1)$ is a strictly convex function of μ_1, they can have at most two intersecting points as shown in Fig. 2.

Therefore, there can be at most two pairs of (μ_1, μ_2) that satisfy both conditions simultaneously. Let these solutions be $(\tilde{\mu}_1, \tilde{\mu}_2)$ and $(\tilde{\mu}_1', \tilde{\mu}_2')$. We then need to check whether $(\tilde{\mu}_1, \tilde{\mu}_2)$ is in $[0,1] \times [0,1]$ or not. If yes, we keep it. If not, then for the presentation convenience, we replace it with $(0,0)$. We do the same for $(\tilde{\mu}_1', \tilde{\mu}_2')$.

Scenario 2: $\eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \eta_4 \neq 0$:

Solving the corresponding KKT conditions, we obtain

$$
\begin{aligned}
\bar{\mu}_1 & = \alpha(S_1 A_1/\lambda), \quad (22) \\
\bar{\mu}_2 & = 0.
\end{aligned}
$$

From the property of $\alpha(\cdot)$, we know that $0 \leq \bar{\mu}_1 \leq 1$, and hence $(\bar{\mu}_1, 0)$ is a valid input. This scenario corresponds to the case where only user 1 is active.
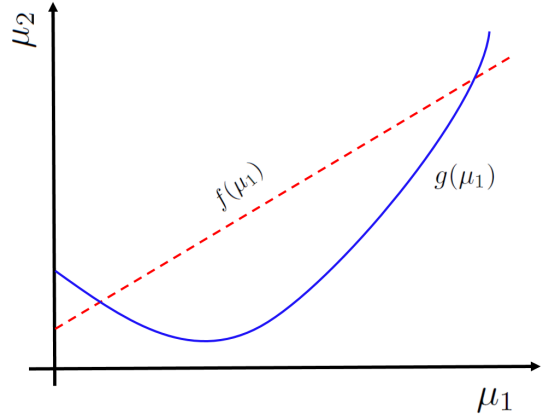

Fig. 2: $f(\mu_1)$ and $g(\mu_1)$ have at most two intersecting points.

Scenario 3: $\eta_1 = 0, \eta_2 \neq 0, \eta_3 = 0, \eta_4 = 0$:

Solving the corresponding KKT conditions, we obtain

$$
\begin{aligned}
\mu_1^* & = 0, \\
\mu_2^* & = \alpha(S_2 A_2/\lambda). \quad (23)
\end{aligned}
$$

Similarly, we have $0 \leq \mu_2^* \leq 1$, and hence $(0, \mu_2^*)$ is a valid input. This scenario corresponds to the case where only user 2 is active.

In summary, we have the following theorem.

Theorem 1. *The sum-rate capacity of the Poisson MAC is* $C_{sum}^{SISO-MAC} = I_{X_1 X_2; Y}(\hat{\mu}_1, \hat{\mu}_2)$ *with* $(\hat{\mu}_1, \hat{\mu}_2)$ *given by*

$$
(\hat{\mu}_1, \hat{\mu}_2) = \qquad (24)
$$
$$
\begin{cases}
(0, \mu_2^*) & \\
\quad if \quad I(0, \mu_2^*) \geq \max\left(I(\bar{\mu}_1, 0), I(\tilde{\mu}_1, \tilde{\mu}_2), I(\tilde{\mu}_1', \tilde{\mu}_2')\right) \\
(\bar{\mu}_1, 0) & \\
\quad if \quad I(\bar{\mu}_1, 0) \geq \max\left(I(0, \mu_2^*), I(\tilde{\mu}_1, \tilde{\mu}_2), I(\tilde{\mu}_1', \tilde{\mu}_2')\right) \\
(\tilde{\mu}_1', \tilde{\mu}_2') & \\
\quad if \quad I(\tilde{\mu}_1', \tilde{\mu}_2') \geq \max\left(I(0, \mu_2^*), I(\bar{\mu}_1, 0), I(\tilde{\mu}_1, \tilde{\mu}_2)\right) \\
(\tilde{\mu}_1, \tilde{\mu}_2) & otherwise
\end{cases}
$$

It is interesting to note that unlike the Gaussian MAC with an average power constraint, it can be optimal to allow only one user to transmit in order to achieve the sum-rate capacity in the Poisson MAC with a maximum power constraint. For example, when $S_1 A_1 = 5, S_2 A_2 = 50, \lambda = 0.5$, there is no solution for (20) and (21) in the desired range of $0 \leq \mu_1 \leq 1$ and $0 \leq \mu_2 \leq 1$, because (20) and (21) do not intersect (as shown in Fig. 3). Hence, for such a set of parameters, it is optimal to allow only one user (in this case, user 2) to transmit to achieve the sum-rate capacity.

On the other hand, there are scenarios under which it is optimal for both users to transmit. For example, when $S_1 A_1 = 10, S_2 A_2 = 15, \lambda = 0.5$, it is easy to check that it is optimal for both users to transmit in order to achieve the sum-rate capacity. Please refer to Fig. 7 in Section VI for an illustration of optimal operating regions for varying values of $S_1 A_1$ and $S_2 A_2$.
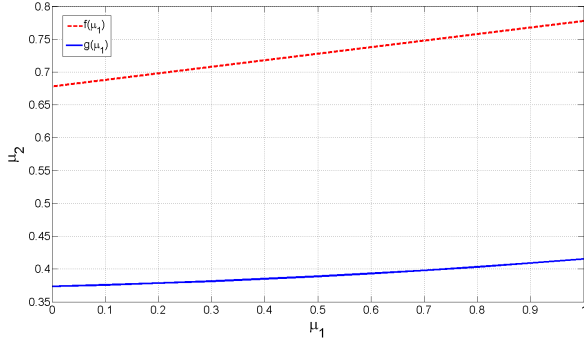
Fig. 3: $f(\mu_1)$ and $g(\mu_1)$ have no intersection in $0 \leq \mu_1 \leq 1$ and $0 \leq \mu_2 \leq 1$, when $S_1 A_1 = 5$, $S_2 A_2 = 50$, and $\lambda = 0.5$.

Motivated by these observations, we further analyze (24) to characterize conditions under which it is optimal to either allow one user or two users to transmit.

B. Single-User or Two-User Transmission?

In this subsection, we present conditions under which it is optimal for a single-user to transmit and conditions under which it is optimal for two-user transmission.

We first focus on the optimality of single-user transmission. As discussed above, the solution for two-user transmission is characterized by the intersections of (20) and (21). The following simple proposition characterize the conditions under which (20) and (21) do not have an intersection in the desired region $[0, 1] \times [0, 1]$ and hence two-user transmission is not optimal.

Proposition 2. *If $g(0) < f(0)$ and $g(1) < f(1)$, then single-user transmission is optimal to achieve the sum-rate capacity.*

Proof. It suffices to argue that two-user transmission is not optimal under the assumption of the proposition. This happens when (20) and (21) do not intersect. Therefore we prove that $g(\mu_1)$ does not intersect with $f(\mu_1)$ when $g(0) < f(0)$ and $g(1) < f(1)$ for $\mu_1 \in [0, 1]$. For any $\mu_1 \in [0, 1]$, we have

$$
\begin{aligned}
f(\mu_1) &\overset{(a)}{=} (1 - \mu_1) f(0) + \mu_1 f(1) \\
&\overset{(b)}{>} (1 - \mu_1) g(0) + \mu_1 g(1) \\
&\overset{(c)}{>} g(\mu_1).
\end{aligned}
$$

Here, (a) follows from the linearity of $f(\cdot)$, (b) follows from the assumption $g(0) < f(0)$ and $g(1) < f(1)$, and (c) follows from the strict convexity of $g(\cdot)$. $\square$

Proposition 2 gives sufficient conditions for single-user transmission to be optimal. Even if the conditions in Proposition 2 are not satisfied, which implies there are solutions for two-user transmission, it is still possible for single-user transmission to be optimal if the corresponding rate achieved is the largest. In the following, we will show that if one of the $S_i A_i$s is sufficiently large, then it is optimal for one user to transmit. As the roles of users are symmetric, we restrict our analysis to $S_2 A_2 \to \infty$ as an example. We show that as

$S_2 A_2 \to \infty$, the conditions in Proposition 2 are satisfied and hence $f(\mu_1)$ and $g(\mu_1)$ do not intersect.

Lemma 3. *The functions $f(\mu_1)$ and $g(\mu_1)$ have the following properties:*

$$
\lim_{S_2 A_2 \to \infty} f(\mu_1) = \lim_{S_2 A_2 \to \infty} f(0) = \lim_{S_2 A_2 \to \infty} f(1) = 1
$$

and

$$
\lim_{S_2 A_2 \to \infty} g(\mu_1) = \lim_{S_2 A_2 \to \infty} g(1) = \lim_{S_2 A_2 \to \infty} g(0) = \frac{1}{e}.
$$

Therefore, (20) and (21) do not intersect as $S_2 A_2 \to \infty$.
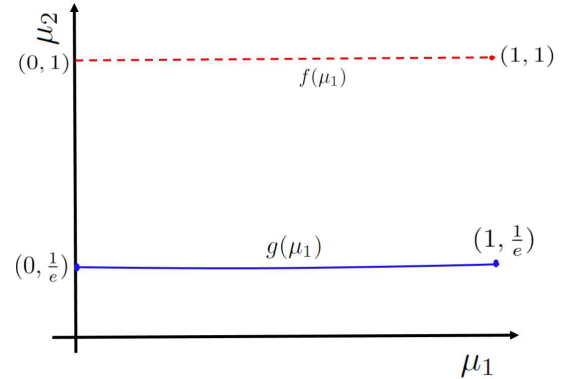
Proof. Please refer to Appendix B. $\square$

Fig. 4: μ_2 vs. μ_1 as $S_2 A_2 \to \infty$

Lemma 3 is illustrated in Fig. 4. As discussed in Scenario 1 of Section III-A, (20) and (21) do not intersect in our region of interest, and hence we replace $(\tilde{\mu}_1', \tilde{\mu}_2')$ and $(\tilde{\mu}_1, \tilde{\mu}_2)$ by $(0, 0)$ when $S_2 A_2 >> S_1 A_1$. This implies that either $(0, \mu_2^*)$ or $(\bar{\mu}_1, 0)$ is the possible solutions. With $S_2 A_2 >> S_1 A_1$, solving (24) yields $(0, \mu_2^*)$ as the optimal solution, i.e., only user 2 transmitting achieves the sum-rate capacity.

Now, we discuss the conditions under which it is optimal for both users to transmit. In particular, the following proposition characterizes conditions under which single-user transmission is not optimal.

Proposition 4. *User 1 transmitting alone is not optimal if*

$$
\alpha(S_1 A_1 / \lambda) > \gamma_1
$$
$$
\triangleq \frac{\left(1 - \frac{S_2 A_2}{S_1 A_1}\right) \varphi(\lambda) - \varphi(S_2 A_2 + \lambda) + \frac{S_2 A_2}{S_1 A_1} \varphi(S_1 A_1 + \lambda)}{W},
$$

in which W is defined in (17). Similarly, user 2 transmitting alone is not optimal if

$$
\alpha(S_2 A_2 / \lambda) > \gamma_2
$$
$$
\triangleq \frac{\left(1 - \frac{S_1 A_1}{S_2 A_2}\right) \varphi(\lambda) - \varphi(S_1 A_1 + \lambda) + \frac{S_1 A_1}{S_2 A_2} \varphi(S_2 A_2 + \lambda)}{W}.
$$

Furthermore, if both conditions above are satisfied, it is optimal for both users to be active.

Proof. To prove this proposition, we will find out conditions under which a single-user transmission can be eliminated as a candidate for optimality. To eliminate $(0, \mu_2^*)$, which is

obtained in (23), as a candidate for the optimal solution, we check whether $\frac{\partial I}{\partial \mu_1}\big|_{(\mu_1=0,\mu_2^*)}$ is larger than 0 or not. If it is larger than 0, then we know that $(0,\mu_2^*)$ cannot be the optimal solution. By replacing the value of $\log(S_2 A_2 \mu_2^* + \lambda)$ from the $\frac{\partial I}{\partial \mu_2}\big|_{\mu_1=0,\mu_2^*} = 0$ in $\frac{\partial I}{\partial \mu_1}\big|_{(\mu_1=0,\mu_2^*)} = 0$, we have

$$
\begin{aligned}
\frac{\partial I}{\partial \mu_1}\Big|_{(\mu_1=0,\mu_2^*)} =\ & -(1-\mu_2^*)\varphi(\lambda) + (1-\mu_2^*)\varphi(S_1 A_1 + \lambda) \\
& -\mu_2^*\varphi(S_2 A_2 + \lambda) + \mu_2^*\varphi(S_1 A_1 + S_2 A_2 + \lambda) \\
& -\frac{S_1 A_1}{S_2 A_2}\varphi(S_2 A_2 + \lambda) + \frac{S_1 A_1}{S_2 A_2}\varphi(\lambda).
\end{aligned}
\tag{25}
$$

Hence the condition for $\frac{\partial I}{\partial \mu_1}\big|_{(\mu_1=0,\mu_2^*)} > 0$ to hold true is $\mu_2^* > \gamma_2$, where $\gamma_2 = \frac{r_2}{W}$ with $r_2 = (1-\frac{S_1 A_1}{S_2 A_2})\varphi(\lambda) - \varphi(S_1 A_1 + \lambda) + \frac{S_1 A_1}{S_2 A_2}\varphi(S_2 A_2 + \lambda)$. Therefore, the case $(0,\mu_2^*)$ is not optimal if $\mu_2^* > \gamma_2$.

Following similar arguments, we can conclude that $(\bar{\mu}_1, 0)$, which is obtained in (22), is not optimal when $\bar{\mu}_1 > \gamma_1$, in which $\gamma_1 = \frac{r_1}{W}$ with $r_1 = (1 - \frac{S_2 A_2}{S_1 A_1})\varphi(\lambda) - \varphi(S_2 A_2 + \lambda) + \frac{S_2 A_2}{S_1 A_1}\varphi(S_1 A_1 + \lambda)$.

If $\bar{\mu}_1 > \gamma_1$ and $\mu_2^* > \gamma_2$, two-user transmission is the optimal solution. $\square$

C. Special Case: Symmetric Channel

In this section we show that the results obtained in Section III-A can recover the results obtained in [15] for the symmetric case. We show this using the following three steps.

Step 1: Among the four possible solutions in (24), we first rule out $(0,\mu_2^*)$ and $(\bar{\mu}_1, 0)$. It is easy to check that, when $S_1 A_1 = S_2 A_2$, $\gamma_1 = 0$ and $\gamma_2 = 0$. Hence, as discussed in Section III-B, $(0,\mu_2^*)$ is not optimal, as we clearly have $\mu_2^* > \gamma_2 = 0$. Similarly, $(\bar{\mu}_1, 0)$ is not optimal, as $\bar{\mu}_1 > \gamma_1 = 0$. Hence, scenario 2 and scenario 3 cannot be optimal, and we are left with only scenario 1.

Step 2: We show that, if (μ_1, μ_2) is a solution to (20) and (21) of scenario 1, then μ_1 must be equal to μ_2. This can be easily seen by setting $S_1 A_1 = S_2 A_2$ in (16), which yields $\mu_1 = \mu_2$.

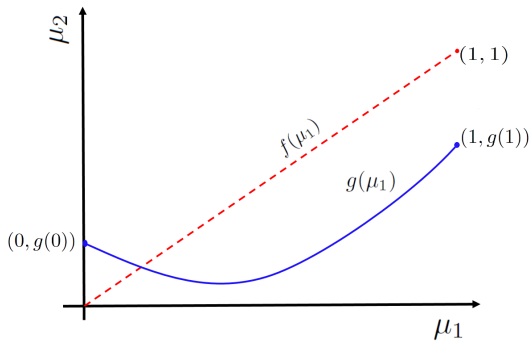**Step 3:** We show that there is a unique pair (μ_1, μ_2) that satisfies (20) and (21) of scenario 1. To prove the uniqueness of the solution, as illustrated in Fig. 5, we show that $g(0) >$

Fig. 5: $f(\mu_1)$ and $g(\mu_1)$ has a single intersecting point in the region of interest when $S_1 A_1 = S_2 A_2$.

$0 = f(0)$ and $g(1) < 1 = f(1)$. Since $g(\cdot)$ is a strictly convex function while $\mu_2 = \mu_1$ is a linear function, $f(\mu_1)$ and $g(\mu_1)$ have a single intersecting point in the range $0 \le \mu_1 \le 1$.

Lemma 5. *If $S_1 A_1 = S_2 A_2$, $g(1) < 1$ and $g(0) > 0$.*

Proof. Please refer to Appendix C. $\square$

Hence it can be concluded that if $S_1 A_1 = S_2 A_2$, then there is a unique solution to the problem and at optimality $\hat{\mu}_1 = \hat{\mu}_2$. This result is consistent with the one shown in [15].

IV. MISO-MAC ANALYSIS

In this section, we extend the analysis to the case when the transmitters are equipped with more than one antennas.

A. Sum-rate Capacity of MISO-MAC

Similarly to the single-antenna case studied in Section III, the continuous-time continuous-input discrete-output Poisson MAC can be converted to discrete-time binary-input binary-output MAC. In particular, the input waveform of each antenna can be limited to be piecewise constant waveforms with two levels 0 or A_{nj} for the jth antenna of the nth transmitter, which can be established by following analysis in [15] and [24]. At the end of Section IV-A, we also provide another intuitive way to understand the optimality of two-level signaling. Depending on the on-off states of each antenna of user n, there are 2^{J_n} states at user n. In the following, we use $i_n \in [1, \cdots, 2^{J_n}]$ to index each of these 2^{J_n} states at user n. We will use $P_n(i_n)$ to denote the probability that user n lies in state i_n and $\mathbf{p}_n \triangleq [P_n(1), \cdots, P_n(2^{J_n})]$ to denote the vector of probabilities of states at user n. We will use the binary variable $b_{nj}(i_n)$ to indicate whether the jth antenna of the nth user is on or off at state i_n, i.e., $b_{nj}(i_n) = 1$ if the jth antenna of the nth user is on for state i_n and is 0 otherwise. The sum-rate achievable using $(\mathbf{p}_1, \mathbf{p}_2)$ is given by

$$
\begin{aligned}
I_{\mathbf{X}_N;Y}(\mathbf{p}_1, \mathbf{p}_2) = \\
\sum_{i_1=1}^{2^{J_1}} \sum_{i_2=1}^{2^{J_2}} & \left[P_1(i_1) P_2(i_2) \zeta\left(\sum_{n=1}^{2} \sum_{j=1}^{J_n} S_{nj} A_{nj} b_{nj}(i_n), \lambda \right) \right] \\
& -\zeta\left(\sum_{n=1}^{2} \sum_{j=1}^{J_n} S_{nj} A_{nj} \mu_{nj}, \lambda \right).
\end{aligned}
\tag{26}
$$

It is easy to see that

$$
\mu_{nj} = \sum_{i_n=1}^{2^{J_n}} P_n(i_n) b_{nj}(i_n).
\tag{27}
$$

Fig 6 (a) shows 4 possible states for user 2 with 2 antennas.

To characterize the sum-rate capacity, we need to solve the following optimization problem:

$$
\textbf{(P1):}\ C_{sum}^{MISO-MAC} = \max_{\mathbf{p}_1,\mathbf{p}_2} I_{\mathbf{X}_N;Y}(\mathbf{p}_1, \mathbf{p}_2),
\tag{28}
$$

$$
\text{s.t}\quad 0 \le P_n(i_n) \le 1,\ i_n = 1, \cdots, 2^{J_n}, n = 1, 2,
\tag{29}
$$

$$
\sum_{i_n=1}^{2^{J_n}} P_n(i_n) = 1, n = 1, 2.
\tag{30}
$$

Problem (P1) is a complex non-convex optimization problem with a large number of variables. In particular, the number of variables $2^{J_1} + 2^{J_2}$ increases exponentially with the number of antennas. The main result of this section is the following theorem.

Theorem 6. *Solving problem (P1) is equivalent to solving the following problem*

$$\textbf{(P2):} \quad C_{sum}^{MISO-MAC} = \max_{0 \leq \mu_1, \mu_2 \leq 1} I(\mu_1, \mu_2), \quad (31)$$

with

$$
\begin{aligned}
&I(\mu_1, \mu_2) \\
&= (1-\mu_1)(1-\mu_2)\varphi(\lambda) + \mu_1(1-\mu_2)\varphi(B_1 + \lambda) \\
&\quad + (1-\mu_1)\mu_2\varphi(B_2 + \lambda) + \mu_1\mu_2\varphi(B_1 + B_2 + \lambda) \\
&\quad - \varphi(B_1\mu_1 + B_2\mu_2 + \lambda),
\end{aligned}
\quad (32)
$$

where

$$B_n \triangleq \sum_{j=1}^{J_n} S_{nj} A_{nj}. \quad (33)$$

Remark 1. *Compared with (P1), there are only 2 variables in (P2). Although (P2) is still a non-convex optimization problem, it has the same form as the problem (P0) solved in Section III-A and hence all techniques and results there (e.g., the analysis on whether single-user transmission is optimal or not) can be applied here. Intuitively, this theorem says that the sum capacity of a MISO-MAC (with channel gains (S_{n1}, S_{n2}) and power constraints (A_{n1}, A_{n2}) for each transmitter n) is the same as the sum capacity of a SISO-MAC (with channel gain $S_{n1}A_{n1} + S_{n2}A_{n2}$ and power constraint 1 for each transmitter n).*

The proof of Theorem 6 has the following two major steps.

In Step-1, we prove the following proposition that simplifies the optimization problem from $(\mathbf{p}_1, \mathbf{p}_2)$ to $\boldsymbol{\mu}$.

Proposition 7. *At the optimality, for each user, if the antenna with a smaller duty cycle is on then all antennas with a larger duty cycle must also be on.*

This proposition shows that, at the optimality, instead of being a function of $(\mathbf{p}_1, \mathbf{p}_2)$, the objective function can be simplified to a function of $\boldsymbol{\mu}$. As the result, the dimension of the problem is reduced from $2^{J_1} + 2^{J_2}$ to $J_1 + J_2$. The central issue addressed here is that, for a given $\boldsymbol{\mu}$, there are infinite number of combinations of $(\mathbf{p}_1, \mathbf{p}_2)$ that satisfy (27). The main idea is to show that, for any user n, if the antenna with a smaller duty cycle is on, then the antenna with a larger duty cycle is also on at the optimality. As the result, at the optimality, the value of $(\mathbf{p}_1, \mathbf{p}_2)$ is determined by $\boldsymbol{\mu}$. Detailed proof of this proposition can be found in Section IV-B. For the example shown in Figure 6, assuming $\mu_{21} \geq \mu_{22}$, there are four initial states shown in Fig. 6 (a): only the antenna with the larger duty cycle is on, both of the antennas are on, only the antenna with the smaller duty cycle is on and none of the antennas is on. We argue that a state with only the antenna having smaller duty cycle to be on is not optimal. Hence, at the optimality, we have the scenario shown in Fig. 6 (b).

In Step-2, we show the following proposition that characterizes the optimal value of $\boldsymbol{\mu}$.

Proposition 8. *At the optimality, for each user, all antennas must have the same duty cycle and they must be on or off simultaneously.*

This proposition shows that, at the optimality, the antennas of each user must have the same duty cycle (i.e., $\mu_{n1} = \cdots = \mu_{nJ_n} \triangleq \mu_n$) and are aligned. Hence, the dimension of the problem is further reduced from $J_1 + J_2$ to 2. The main idea of this step is to show that, at the optimality, all antennas of user n are either simultaneously on or off. Hence, from receiver's perspective, transmitter n can be viewed as a single antenna with power constraint 1 and channel gain $\sum_{j=1}^{J_n} S_{nj}A_{nj}$. Step 2 is illustrated Fig. 6 (b) and Fig. 6 (c). The proof can be found in Section IV-C.

In order to prove Theorem 6, Propositions 7 and 8 need to be proved. In the following subsections, we prove these propositions in detail. For the MISO case, the results can be proved in the continuous time domain as follows. For any codeword for transmitter 1 that sends $x_{1j}(t)$ at antenna j for $t \in [0, T]$. Let $x_{\text{tot}} \triangleq \sum_{j=1}^{J_1} S_{1j}x_{1j}$. Now consider a different codeword that sends on the jth antenna

$$x_{1j}' = \frac{A_{1j}}{\sum_{j=1}^{J_1} S_{1j}A_{1j}} x_{\text{tot}}.$$

One can easily see that x_{1j}' satisfies the peak power constraint of the jth antenna, and that the new codeword induces the same output distribution as the original one. Hence there is no loss of optimality in letting all transmit antennas send the same waveform. Hence, the MISO-MAC can be converted to a SISO-MAC. Since in SISO-MAC, letting each antenna to send two-level signal is optimal, this also shows that letting each antenna sending two-level signal is optimal in the multiple transmit antenna case. We note that, however, the proofs presented below are more general and can be used to handle more sophisticated scenarios when the receiver also has multiple antennas, as discussed in Section V, or when we have average power constraint at each transmit antenna.

B. Proof of Proposition 7

In this subsection, we prove Proposition 7 by characterizing the optimal value of $(\mathbf{p}_1, \mathbf{p}_2)$ for any given $\boldsymbol{\mu}$. Hence, in this subsection, $\boldsymbol{\mu}$ is fixed. More specifically, we show that, at the optimality in the MISO-MAC, if the antenna with the smaller duty cycle is on, then the other antenna should also be on.

From (26), it is clear that to optimize over $(\mathbf{p}_1, \mathbf{p}_2)$ for a given $\boldsymbol{\mu}$, we only need to focus on

We focus on finding the optimal values of $\mathbf{p}_2$ first. To facilitate the understanding, we list the labeling of states of user 2 and the corresponding values of b_{2j}s in Table I.

Using the definition of ζ function in (6), we can easily check that

$$
\begin{aligned}
d(i_1, 1) &< \min\{d(i_1, 2), d(i_1, 3)\} \\
&\leq \max\{d(i_1, 2), d(i_1, 3)\} < d(i_1, 4),
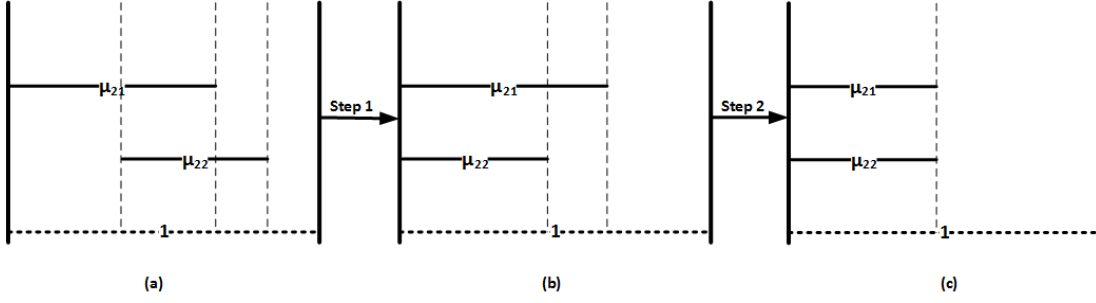\end{aligned}
$$

Fig. 6: (a) Four possible states for user 2; (b) Step 1 shows, at the optimality, the timing of antennas being on is aligned; (c) Step 2 shows, at the optimality, the duty cycles of both antennas are the same and are aligned.

$$
\begin{aligned}
\mathcal{H} &\triangleq \sum_{i_1=1}^{2^{J_1}} \sum_{i_2=1}^{2^{J_2}} P_1(i_1) P_2(i_2) \zeta \left(\sum_{n=1}^{2} \sum_{j=1}^{J_n} S_{nj} A_{nj} b_{nj}(i_n), \lambda \right) \\
&= \sum_{i_1=1}^{2^{J_1}} P_1(i_1) \underbrace{\sum_{i_2=1}^{2^{J_2}} P_2(i_2) \underbrace{\zeta \left(\sum_{j=1}^{J_1} S_{1j} A_{1j} b_{1j}(i_1) + \sum_{j=1}^{J_2} S_{2j} A_{2j} b_{2j}(i_2), \lambda \right)}_{d(i_1,i_2)}}_{\mathcal{H}_1(i_1)}.
\end{aligned}
\tag{36}
$$

	b_{21}	b_{22}
$(i_1, 1)$	0	0
$(i_1, 2)$	0	1
$(i_1, 3)$	1	0
$(i_1, 4)$	1	1

TABLE I: The states of user 2 and the corresponding values of b_{2j}s.

which is simultaneously true for any value of i_1. As $\mathcal{H}_1(i_1)$ is simply a linear combination of $d(i_1, i_2)$s, where $\mathcal{H}_1(i_1)$ and $d(\cdot, \cdot)$ are defined in (36), hence, for any given μ, maximizing $\mathcal{H}_1(i_1)$ is a linear programming problem, for which we have the following (assuming $\mu_{21} \geq \mu_{22}$, the other case being similar):

1) As $d(i_1, 4)$ is the largest, $P_2(4)$ should be as large as possible. Therefore, we assign $P_2(4) = \mu_{22}$.
2) As μ_{22} has been all used, we should set $P_2(2) = 0$.
3) As $d(i_1, 3) > d(i_1, 1)$, we assign the remaining part of μ_{21} to state $(i_1, 3)$ and hence $P_2(3) = \mu_{21} - \mu_{22}$.
4) For the last state related to the term $d(i_1, 1)$, allot the remaining probability. Hence $P_2(1) = 1 - \mu_{21}$.

This assignment implies that if the antenna with a smaller duty cycle is on, the antenna with a larger duty cycle should also be on. This is illustrated in Fig. 6 (b). Note that the above arguments are true for all i_1s, and hence this assignment maximizes $\mathcal{H}_1(i_1)$ for all i_1 simultaneously. Furthermore, this assignment is independent of $\mathbf{p}_1$.

Notice that the above discussion for $J_2 = 2$, can be extended for $J_2 > 2$.

Similarly, by writing

$$
\mathcal{H} = \sum_{i_2=1}^{2^{J_2}} P_2(i_2) \sum_{i_1=1}^{2^{J_1}} P_1(i_1)
$$
$$
\zeta \left(\sum_{j=1}^{J_1} S_{1j} A_{1j} b_{1j}(i_1) + \sum_{j=1}^{J_2} S_{2j} A_{2j} b_{2j}(i_2), \lambda \right),
$$

then following the same procedure as above, we can calculate the optimal values of $\mathbf{p}_1$.

As the result, we know that (26) can be simplified to a function μ depending on the relationships between the values of μ_{nj}s. For example, in the case of two transmitter antennas, we have four symmetric cases, i.e. $(\mu_{11} \geq \mu_{12}, \mu_{21} \geq \mu_{22})$, $(\mu_{11} \leq \mu_{12}, \mu_{21} \geq \mu_{22})$, $(\mu_{11} \geq \mu_{12}, \mu_{21} \leq \mu_{22})$ and $(\mu_{11} \leq \mu_{12}, \mu_{21} \leq \mu_{22})$. For the case of $(\mu_{11} \geq \mu_{12}, \mu_{21} \geq \mu_{22})$, $I_{\mathbf{X}_N;Y}$ can be simplified to

$$
\begin{aligned}
&I(\mu_{11} - \mu_{12}, \mu_{12}, \mu_{21} - \mu_{22}, \mu_{22}) = \\
&(1 - \mu_{11})((1 - \mu_{21})\varphi(\lambda) + (\mu_{21} - \mu_{22})\varphi(S_{21}A_{21} + \lambda) \\
&+ \mu_{22}\varphi(B_2 + \lambda)) + (\mu_{11} - \mu_{12})((1 - \mu_{21})\varphi(S_{11}A_{11} + \lambda) \\
&+ (\mu_{21} - \mu_{22})\varphi(S_{21}A_{21} + S_{11}A_{11} + \lambda) \\
&+ \mu_{22}\varphi(B_2 + S_{11}A_{11} + \lambda)) + \mu_{12}((1 - \mu_{21})\varphi(B_1 + \lambda) \\
&+ (\mu_{21} - \mu_{22})\varphi(S_{21}A_{21} + B_1 + \lambda) \\
&+ \mu_{22}\varphi(B_2 + B_1 + \lambda)) - \varphi \left(\sum_{n=1}^{2} \sum_{j=1}^{J_n} S_{nj} A_{nj} \mu_{nj} + \lambda \right).
\end{aligned}
\tag{37}
$$

As the result, the objective function is simplified to characterizing

$$
C_{sum}^{MISO-MAC} = \max \left(C_{\mu_{11} \geq \mu_{12}, \mu_{21} \geq \mu_{22}}, C_{\mu_{11} \leq \mu_{12}, \mu_{21} \geq \mu_{22}}, \right.
$$
$$
\left. C_{\mu_{11} \geq \mu_{12}, \mu_{21} \leq \mu_{22}}, C_{\mu_{11} \leq \mu_{12}, \mu_{21} \leq \mu_{22}} \right),
\tag{38}
$$

in which

(P3): $C_{\mu_{11}\geq\mu_{12},\mu_{21}\geq\mu_{22}} =$

$$\max I(\mu_{11} - \mu_{12}, \mu_{12}, \mu_{21} - \mu_{22}, \mu_{22}), \quad (39)$$

$$\text{s.t.} \quad 0 \leq \mu_{12} \leq \mu_{11} \leq 1, (40)$$

$$0 \leq \mu_{22} \leq \mu_{21} \leq 1. (41)$$

Other terms in (38) are defined in a similar manner. Due to symmetry, in the following, we only provide details on how to solve (P3).

C. Proof of Proposition 8

In this subsection, we prove Proposition 8 by solving (P3). For the ease of calculation, we define $q_1 = \mu_{11} - \mu_{12}$, $q_2 = \mu_{12}$, $q_3 = \mu_{21} - \mu_{22}$ and $q_4 = \mu_{22}$ and let $\mathbf{q} = [q_1, q_2, q_3, q_4]$. Then (37) can be re-written as

$$I(\mathbf{q}) = (1 - (q_1 + q_2))((1 - (q_3 + q_4))\varphi(\lambda)$$
$$+ q_3\varphi(S_{21}A_{21} + \lambda) + q_4\varphi(B_2 + \lambda))$$
$$+ q_1((1 - (q_3 + q_4))\varphi(S_{11}A_{11} + \lambda)$$
$$+ q_3\varphi(S_{21}A_{21} + S_{11}A_{11} + \lambda)$$
$$+ q_4\varphi(B_2 + S_{11}A_{11} + \lambda)) + q_2((1 - (q_3 + q_4)\varphi(B_1 + \lambda)$$
$$+ q_3\varphi(S_{21}A_{21} + B_1 + \lambda) + q_4\varphi(B_2 + B_1 + \lambda))$$
$$- \varphi(S_{11}A_{11}q_1 + B_1q_2 + S_{21}A_{21}q_3 + B_2q_4 + \lambda). \quad (42)$$

Correspondingly, (P3) is equivalent to

(P4): $C_{\mu_{11}\geq\mu_{12},\mu_{21}\geq\mu_{22}} = \max I(\mathbf{q}) \quad (43)$

$$\text{s.t.} \quad q_k \geq 0, \ k = 1, \cdots, 4, \quad (44)$$

$$q_1 + q_2 \leq 1, \quad (45)$$

$$q_3 + q_4 \leq 1. \quad (46)$$

Similarly to (8), the objective function (43) is not a convex function in general. We use the KKT conditions as necessary conditions to characterize the set of possible candidates for the optimal solution. In the following, we consider only the constraint (44) explicitly. We check (45) and (46) after obtaining the solution.

The Langrangian equation for (P4) with constraint (44) is given by

$$\mathcal{L} = -I - \sum_{k=1}^{4} \eta_k q_k.$$

The corresponding KKT conditions are:

$$\frac{\partial I}{\partial q_k} + \eta_k = 0, \qquad k = 1, \cdots, 4, \quad (47)$$

$$\eta_k q_k = 0, \qquad k = 1, \cdots, 4, \quad (48)$$

where

$$\frac{\partial I}{\partial q_1} = \zeta(S_{11}A_{11}, \lambda) + q_3(\zeta(S_{21}A_{21}, S_{11}A_{11} + \lambda)$$
$$- \zeta(S_{21}A_{21}, \lambda)) + q_4(\zeta(B_2, S_{11}A_{11} + \lambda) - \zeta(B_2, \lambda))$$
$$- S_{11}A_{11}(\log(S_{11}A_{11}q_1 + B_1q_2 + S_{21}A_{21}q_3 + B_2q_4 + \lambda) + 1),$$

$$\frac{\partial I}{\partial q_2} = \zeta(B_1, \lambda) + q_3(\zeta(S_{21}A_{21}, B_1 + \lambda)$$
$$- \zeta(S_{21}A_{21}, \lambda)) + q_4(\zeta(B_2, B_1 + \lambda) - \zeta(B_2, \lambda))$$
$$- B_1(\log(S_{11}A_{11}q_1 + B_1q_2 + S_{21}A_{21}q_3 + B_2q_4 + \lambda) + 1),$$

$$\frac{\partial I}{\partial q_3} = (1 - (q_1 + q_2))\zeta(S_{21}A_{21}, \lambda)$$
$$+ q_1\zeta(S_{21}A_{21}, S_{11}A_{11} + \lambda) + q_2\zeta(S_{21}A_{21}, B_1 + \lambda)$$
$$- S_{21}A_{21}(\log(S_{11}A_{11}q_1 + B_1q_2 + S_{21}A_{21}q_3 + B_2q_4 + \lambda) + 1),$$

and

$$\frac{\partial I}{\partial q_4} = (1 - (q_1 + q_2))\zeta(B_2, \lambda)$$
$$+ q_1\zeta(B_2, S_{11}A_{11} + \lambda) + q_2\zeta(B_2, B_1 + \lambda)$$
$$- B_2(\log(S_{11}A_{11}q_1 + B_1q_2 + S_{21}A_{21}q_3 + B_2q_4 + \lambda) + 1).$$

Now in order to find the set of optimal solutions, we may solve the above KKT conditions (47) and (48). As we can see from the expressions of $\frac{\partial I}{\partial q_k}$s, we need to solve a set of nonlinear equations, which are in general difficult to solve and may have infinite number of solutions. Nevertheless, by exploring the structure of problem, we obtain the following result.

Proposition 9. *There are only three possible cases for the solution to Problem (P4):*

1) $\mathbf{q} = (0, \alpha(B_1/\lambda), 0, 0)$, *which implies that both antennas of user 1 are active with the same duty cycle $\alpha(B_1/\lambda)$ while both antennas of user 2 are off.*

2) $\mathbf{q} = (0, 0, 0, \alpha(B_2/\lambda))$, *which implies that both antennas of user 1 are off while while both antennas of user 2 are active with the same duty cycle $\alpha(B_2/\lambda)$.*

3) $\mathbf{q} = (0, \mu_1, 0, \mu_2)$, *which implies that both antennas of user 1 are active with the same duty cycle μ_1 and both antennas of user 2 are active with the same duty cycle μ_2. Furthermore, there are only two possible pairs of (μ_1, μ_2) and can be obtained by solving (13) and (14) with S_1A_1 being replaced with B_1 and S_2A_2 being replaced with B_2.*

Proof. Please refer to Appendix D. □

Proposition 9 states that the solution to (P4) is the same as the solution to (P2), and hence Theorem 6 is proved.

V. DISCUSSIONS

In this section, we discuss a few interesting open directions along the lines of the results presented in this paper and potential challenges associated with these problems.

A. Boundary Points of the Capacity Region

After characterizing the sum-rate capacity, a natural next step is to characterize all boundary points on the capacity region. Towards this goal, we can follow the same approach developed in this paper to solve the following optimization problem to obtain any boundary rate pair (R_1, R_2) for a given $0 \le \gamma \le 1/2$

$$\max \quad \gamma R_1 + (1 - \gamma)R_2. \tag{49}$$

Here,

$$R_1 = I_{X_1;Y}, \quad R_2 = I_{X_2;Y|X_1}, \tag{50}$$

whose expression can be written out as functions of duty cycles of each antenna. Let $\mathcal{E}_1$ be the set of obtained rate pairs (R_1, R_2) by solving (49) with γ varying in $[0, 1/2]$. Similarly, we can obtain the set $\mathcal{E}_2$ by setting $R_1 = I_{X_1;Y|X_2}$ and $R_2 = I_{X_2;Y}$.

Let $\mathcal{E} = \mathcal{E}_1 \bigcup \mathcal{E}_2$. [15] shows that $\mathcal{E}$ is a description of the boundary points of the capacity region, if the set $\mathcal{R} \triangleq \{(R_1, R_2) : \exists (R_1^*, R_2^*) \in \mathcal{E} \text{ such that } 0 \le R_1 \le R_1^*, 0 \le R_2 \le R_2^*\}$ is a convex set. However, although one can verify the convexity of $\mathcal{R}$ numerically, it turns out to be difficult to analytically verify such convexity even for the symmetric case considered in [15] and further for the non-symmetric case. Such an open problem is interesting and solution to it requires more refined understanding of the structure of the set $\mathcal{R}$.

B. Poisson MIMO-MAC

Another extension is to consider Poisson MIMO-MAC in which the receiver is also equipped with multiple antennas. A natural idea is still to characterize the sum-rate capacity by following the two-step methodology developed in this paper. In particular, in our approach, the first step is to reduce the dimension of the problem by arguing that if the antenna with a smaller duty cycle is on, the other antennas with larger duty cycles must also be on. This step can still be carried out for Poisson MIMO-MAC by following the method presented in this paper. The second step is to characterize the optimal value of the duty cycles. In the case of the Poisson MISO-MAC, we do it by converting the set of KKT conditions related nonlinear equations, to a set of linear equations and a convex equation by substituting the common nonlinear term (i.e., the log term) from one equation to all the others (see, for example, the proof of Case-8 in Appendix D). This enables us to conclude that there are only a finite number of solutions to our problem. However, this procedure requires further exploration for the general MIMO case. In particular, as the nonlinear terms in KKT conditions have different forms, it is difficult to convert nonlinear equations to a set of linear equations so that it is not clear whether there are a finite number of solutions or an infinite number of solutions from the set of nonlinear equations related to KKT conditions. Therefore, the extension to the Poisson MIMO-MAC may require new treatments of the optimization objective function. One possible venue is to explore the relationship between the derivative of the mutual information and the mean squared error as discussed in [21] and [22]. The interesting work on single user MIMO Poisson channel [5], [24], [25] will be also be useful for our study.

VI. NUMERICAL ANALYSIS

In this section, we provide numerical examples to illustrate results obtained in this paper. As shown in the paper, the case of Poisson MISO-MAC can be converted to a Poisson SISO-MAC. Hence, in the following, we provide only example related to the SISO-MAC case.
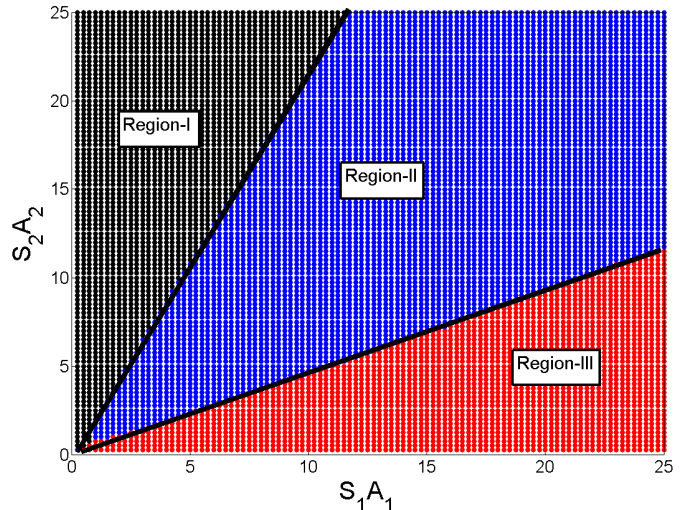

Fig. 7: Optimal operating schemes over the ranges of $S_1 A_1$ and $S_2 A_2$

Fig. 7 shows the optimal operating scenarios for different combinations of $S_1 A_1$ and $S_2 A_2$ when they range from 0 to 25. In generating this figure, we set $\lambda = 0.5$. In Region-I, it is optimal for user 2 to transmit alone. Region-II corresponds to the case in which it is optimal for both users to transmit. In Region-III, it is optimal for user 1 to transmit alone.
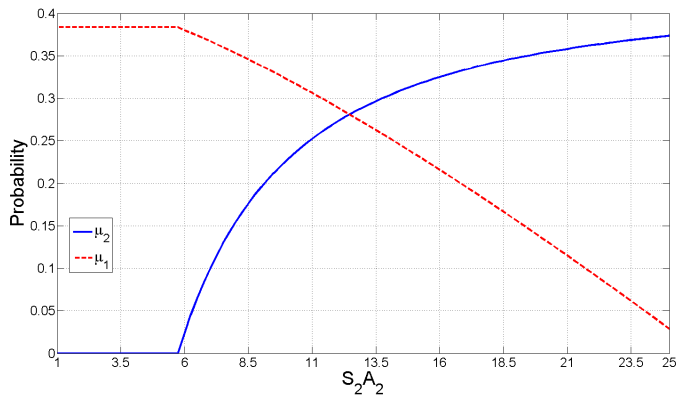
Fig. 8: $(\hat{\mu}_1, \hat{\mu}_2)$ vs. $S_2 A_2$

Fig. 8 illustrates the effect of increasing $S_2 A_2$ on the optimal value of $(\hat{\mu}_1, \hat{\mu}_2)$ when $S_1 A_1$ is constant. In this figure, $S_1 A_1 = 12.5$. We can see that when $S_2 A_2$ is small, the optimal value $\hat{\mu}_2$ is equal to 0, i.e., it is optimal for user 2 to stay silent. We also observe that once $S_2 A_2$ starts to increase and has noticeable value compared to $S_1 A_1$, $\hat{\mu}_1$ starts to decrease while $\hat{\mu}_2$ starts to increase. Furthermore, μ_1 and μ_2 intersect with each other, i.e. $\hat{\mu}_1 = \hat{\mu}_2$, when $S_1 A_1 = S_2 A_2$. This is consistent with the result obtained in [15] for the symmetric case.

VII. Conclusion

In this paper, by solving non-convex optimization problems, we have characterized the sum-rate capacity for both non-symmetric Poisson SISO-MAC and non-symmetric Poisson MISO-MAC. We have shown that under certain channel conditions, it is optimal for both users to be active and we have also established conditions under which it is optimal for only one user to be active.

There are a few open issues that are interesting to investigate. For example, one can aim to characterize the boundary points of the capacity region. One can also consider the scenario with multiple receiver antennas. These extensions require exploration of new treatment and understanding of the problem as we discuss in Section V. Furthermore, we can investigate the error exponents of Poisson MAC by extending the work in [26]. When the error exponent is considered, even for the non-symmetric Poisson MAC, single user transmit may not be optimal anymore.

Appendix A
Concavity of $I_{X_1,X_2;Y}(\mu_1, \mu_2)$

In this Appendix, we show that $I_{X_1,X_2;Y}(\mu_1, \mu_2)$ is not necessarily concave for general values of S_1A_1, S_2A_2 and λ. For $I_{X_1,X_2;Y}(\mu_1, \mu_2)$ to be concave, $\bigtriangledown^2 I$ needs to be negative semi-definite. For $\bigtriangledown^2 I$ to be negative semi-definite, there are two conditions to be satisfied [27]. The first condition is that its first order principle minor must be non-positive. As $\frac{\partial^2 I}{\partial \mu_1^2} = -\frac{S_1^2 A_1^2}{S_1 A_1 \mu_1 + S_2 A_2 \mu_2 + \lambda} < 0$, this condition holds. The second condition is that the determinant of the Hessian matrix must be non-negative. It is easy to check that

$$
| \bigtriangledown^2 I | = (\varphi(\lambda) - \varphi(S_1 A_1 + \lambda) - \varphi(S_2 A_2 + \lambda)
$$
$$
+ \varphi(S_1 A_1 + S_2 A_2 + \lambda)) \left(\frac{2 S_1 A_1 S_2 A_2}{S_1 A_1 \mu_1 + S_2 A_2 \mu_2 + \lambda} \right.
$$
$$
- (\varphi(\lambda) - \varphi(S_1 A_1 + \lambda) - \varphi(S_2 A_2 + \lambda)
$$
$$
+ \varphi(S_1 A_1 + S_2 A_2 + \lambda))).
$$

The two terms on the right hand side can be dealt separately. First, we show the following.

Lemma 10. $\varphi(\lambda) - \varphi(S_1 A_1 + \lambda) - \varphi(S_2 A_2 + \lambda) + \varphi(S_1 A_1 + S_2 A_2 + \lambda) > 0.$

Proof. Using the definition of φ, it is easy to see that φ' is a strictly increasing function. Let $a = \lambda, b = S_1 A_1 + \lambda, c = S_2 A_2 + \lambda$ and $d = S_1 A_1 + S_2 A_2 + \lambda$. then using the mean value theorem, we have:

$$
\exists x_1 \in (a, b) \quad \text{s.t. } \varphi'(x_1) = \frac{\varphi(b) - \varphi(a)}{b - a},
$$
$$
\exists x_2 \in (c, d) \quad \text{s.t. } \varphi'(x_2) = \frac{\varphi(d) - \varphi(c)}{d - c}.
$$

Without loss of generality we can assume that $S_1 A_1 < S_2 A_2$, then we will have $a < b < c < d$. As φ' is an increasing function and $x_1 < x_2$, we have $\varphi'(x_1) < \varphi'(x_2)$ and $b - a = d - c$, then:

$$
\frac{\varphi(b) - \varphi(a)}{b - a} < \frac{\varphi(d) - \varphi(c)}{d - c}
$$
$$
\varphi(b) - \varphi(a) < \varphi(d) - \varphi(c). \tag{51}
$$

Hence $\varphi(d) + \varphi(a) > \varphi(b) + \varphi(c)$. $\qquad\square$

As the first term is always greater than 0, for the function to be concave, the second term, $\frac{2 S_1 A_1 S_2 A_2}{S_1 A_1 \mu_1 + S_2 A_2 \mu_2 + \lambda} - (\varphi(\lambda) - \varphi(S_1 A_1 + \lambda) - \varphi(S_2 A_2 + \lambda) + \varphi(S_1 A_1 + S_2 A_2 + \lambda))$, must also be non-negative. This, however, is not true. For example, taking $\mu_1 = 0.9, \mu_2 = 0.7$ and setting $S_1 A_1 = 50, S_2 A_2 = 100$ and $\lambda = 0.5$, the second term results in the value of -6.2943. Hence, we can conclude that $I_{X_1,X_2;Y}(\mu_1, \mu_2)$ is not always concave.

Appendix B
Proof of Lemma 3

In this section we present the asymptotic analysis of (16) and (19). As the case $S_1 A_1 \to \infty$ is similar to analysis for $S_2 A_2 \to \infty$ due to symmetry, we restrict our analysis to $S_2 A_2 \to \infty$ in this section. We will show that as $S_2 A_2 \to \infty$, $f(\mu_1)$ and $g(\mu_1)$ do not intersect. Denoting $S_2 A_2$ as x, we calculate $\lim_{x \to \infty} g(0)$, $\lim_{x \to \infty} g(1)$ and $\lim_{x \to \infty} f(0)$ as $\lim_{x \to \infty} f(0) = \lim_{x \to \infty} f(1)$.

$$
\lim_{x \to \infty} g(0)
$$
$$
= \lim_{x \to \infty} \left(\frac{1}{x} \exp \left(\frac{1}{x} \left(-\varphi(\lambda) + \varphi(x + \lambda) - x \right) \right) + \frac{\lambda}{x} \right)
$$
$$
= \lim_{x \to \infty} \left(\frac{1}{xe} \exp \left(\log(\lambda)^{\frac{-\lambda}{x}} + \log(x + \lambda)^{\frac{x+\lambda}{x}} \right) \right)
$$
$$
= \lim_{x \to \infty} \left(\frac{1}{xe} \lambda^{\frac{-\lambda}{x}} (x + \lambda)^{\left(\frac{x+\lambda}{x} \right)} \right)
$$
$$
= \lim_{x \to \infty} \left(\frac{1}{e} \lambda^{\frac{-\lambda}{x}} \frac{1}{x} \frac{x^{\frac{\lambda}{x}}}{x^{\frac{\lambda}{x}}} (x + \lambda)^{\left(\frac{x+\lambda}{x} \right)} \right)
$$
$$
= \lim_{x \to \infty} \left(\frac{1}{e} \lambda^{\frac{-\lambda}{x}} x^{\frac{\lambda}{x}} \left(1 + \frac{\lambda}{x} \right)^{\left(1 + \frac{\lambda}{x} \right)} \right).
$$

As $\lim_{x \to \infty} \lambda^{\frac{-\lambda}{x}} = 1$, and

$$
\lim_{x \to \infty} x^{\frac{\lambda}{x}} = \lim_{x \to \infty} e^{\log \left(x^{\frac{\lambda}{x}} \right)} = \lim_{x \to \infty} e^{\frac{\lambda}{x} \log(x)} = 1,
$$

and

$$
\lim_{x \to \infty} \left(1 + \frac{\lambda}{x} \right)^{\left(1 + \frac{\lambda}{x} \right)} = 1.
$$

Hence, we obtain $\lim_{x \to \infty} g(0) = \frac{1}{e}$.

Similarly

$$\lim_{x\to\infty} g(1)$$

$$= \lim_{x\to\infty} \left(\frac{1}{x} \exp\left(\frac{1}{x} \left(-\varphi(S_1 A_1 + \lambda) \right. \right. \right.$$
$$\left. \left. + \varphi(S_1 A_1 + x + \lambda) - x) \right) + \frac{S_1 A_1 + \lambda}{x} \right)$$

$$= \lim_{x\to\infty} \left(\frac{1}{xe} \exp\left(\log(S_1 A_1 + \lambda)^{\frac{-(S_1 A_1 + \lambda)}{x}} \right. \right.$$
$$\left. \left. + \log(S_1 A_1 + x + \lambda)^{\frac{S_1 A_1 + x + \lambda}{x}} \right) \right)$$

$$= \lim_{x\to\infty} \left(\frac{1}{xe} (S_1 A_1 + \lambda)^{\frac{-(S_1 A_1 + \lambda)}{x}} \right.$$
$$\left. (S_1 A_1 + x + \lambda)^{\left(\frac{S_1 A_1 + x + \lambda}{x} \right)} \right)$$

$$= \lim_{x\to\infty} \left(\frac{1}{e} (S_1 A_1 + \lambda)^{\frac{-(S_1 A_1 + \lambda)}{x}} \frac{1}{x} \frac{x^{\frac{(S_1 A_1 + \lambda)}{x}}}{x^{\frac{(S_1 A_1 + \lambda)}{x}}} \right.$$
$$\left. (S_1 A_1 + x + \lambda)^{\left(\frac{S_1 A_1 + x + \lambda}{x} \right)} \right)$$

$$= \lim_{x\to\infty} \left(\frac{1}{e} (S_1 A_1 + \lambda)^{\frac{-(S_1 A_1 + \lambda)}{x}} x^{\frac{(S_1 A_1 + \lambda)}{x}} \right.$$
$$\left. \left(1 + \frac{S_1 A_1 + \lambda}{x} \right)^{\left(1 + \frac{S_1 A_1 + \lambda}{x} \right)} \right)$$

$$= \frac{1}{e}.$$

Now for the $f(\mu_1)$, we notice that $\lim_{x\to\infty} f(0) = \lim_{x\to\infty} f(1)$. Hence we calculate $\lim_{x\to\infty} f(0)$.

$$\lim_{x\to\infty} f(0)$$

$$= \lim_{x\to\infty} \frac{\zeta(S_1 A_1, \lambda) + \frac{S_1 A_1}{x} \zeta(x, \lambda)}{\zeta(x, S_1 A_1 + \lambda) - \zeta(x, \lambda)}$$

$$\overset{(a)}{=} \lim_{x\to\infty} \frac{\frac{-S_1 A_1}{x^2} \lambda \log\left(1 + \frac{x}{\lambda}\right) + \frac{S_1 A_1}{x + \lambda} + \frac{S_1 A_1 \lambda}{x(x + \lambda)}}{\log(1 + \frac{S_1 A_1}{x + \lambda})}$$

$$\overset{(b)}{=} \lim_{x\to\infty} \frac{-S_1 A_1 \lambda \log\left(1 + \frac{x}{\lambda}\right) + 2 S_1 A_1 x}{O}$$

$$= \lim_{x\to\infty} \frac{\frac{-S_1 A_1 \lambda}{2x} \log\left(1 + \frac{x}{\lambda}\right) + S_1 A_1}{R}$$

$$= 1,$$

where $O = (x + \lambda) \log\left(1 + \frac{S_1 A_1}{x + \lambda}\right) + \frac{x}{2} \log\left(1 + \frac{S_1 A_1}{x + \lambda}\right) - \frac{x S_1 A_1}{2(S_1 A_1 + x + \lambda)}$, $R = (x + \lambda) \log\left(1 + \frac{S_1 A_1}{x + \lambda}\right) + \frac{x}{2} \log\left(1 + \frac{S_1 A_1}{x + \lambda}\right) - \frac{x S_1 A_1}{2(S_1 A_1 + x + \lambda)}$, (a) follows from the L'hospital rule and (b) follows from multiplying by $\frac{x^2(x + \lambda)}{x^2(x + \lambda)}$ and L'hospital rule.

APPENDIX C
PROOF OF LEMMA- 5

In this section we will prove that when $S_1 A_1 = S_2 A_2$, we have $g(1) < 1$ and $g(0) > 0$.

Proof. Using $S_1 A_1 = S_2 A_2$, we will show that

$$g(1) - 1 < 0.$$

By plugging $\mu_1 = 1$ in (19), this is equivalent to show

$$\frac{1}{S_1 A_1} \exp\left(\frac{1}{S_1 A_1} \left(-\varphi(S_2 A_2 + \lambda) \right. \right.$$
$$\left. \left. + \varphi(S_1 A_1 + S_2 A_2 + \lambda) - S_1 A_1 \right) \right) - 2 - \frac{\lambda}{S_1 A_1} < 0,$$

$$\Leftrightarrow \frac{1}{S_1 A_1} \exp\left(-(1 + \frac{\lambda}{S_1 A_1}) \log(S_1 A_1 + \lambda) \right.$$
$$\left. + (2 + \frac{\lambda}{S_1 A_1}) \log(2 S_1 A_1 + \lambda) \right).e^{-1} < 2 + \frac{\lambda}{S_1 A_1},$$

which is equivalent to show

$$\exp\left(\log\left(\frac{(2 S_1 A_1 + \lambda)^{\left(2 + \frac{\lambda}{S_1 A_1}\right)}}{(S_1 A_1 + \lambda)^{\left(1 + \frac{\lambda}{S_1 A_1}\right)}} \right) \right) < (2 S_1 A_1 + \lambda)e,$$

$$\Leftrightarrow \frac{(2 S_1 A_1 + \lambda)^{\left(2 + \frac{\lambda}{S_1 A_1}\right)}}{(S_1 A_1 + \lambda)^{\left(1 + \frac{\lambda}{S_1 A_1}\right)}} < (2 S_1 A_1 + \lambda)e,$$

$$\Leftrightarrow \left(1 + \frac{S_1 A_1}{S_1 A_1 + \lambda} \right)^{\left(1 + \frac{\lambda}{S_1 A_1}\right)} < e,$$

$$\Leftrightarrow \left(1 + \frac{\lambda}{S_1 A_1} \right) \log\left(1 + \frac{S_1 A_1}{S_1 A_1 + \lambda} \right) < 1,$$

which is true as $\log(1 + x) < x$. Following the similar steps, $g(0) > 0$ can also be proved. $\square$

APPENDIX D
PROOF OF PROPOSITION 9

The proof strategy is to analyze different cases corresponding to whether η_ks are zero or not. By exploiting the structure of the problem, we will show that, except for three cases, all other cases are not optimal. It will be clear in the sequel, while some cases are easy to handle, it needs significant amount of work to rule out certain cases.

Case-1: $\eta_1 \neq 0, \eta_2 \neq 0, \eta_3 \neq 0, \eta_4 \neq 0 \Rightarrow$

$$\eta_k \neq 0 \to q_k = 0.$$

This implies that none of the users are active. It is clear that $I(0, 0, 0, 0)$ can not the optimal solution.

Case-2: $\eta_1 \neq 0, \eta_2 \neq 0, \eta_3 \neq 0, \eta_4 = 0 \Rightarrow$

$$\eta_1 \neq 0 \to q_1 = 0,$$
$$\eta_2 \neq 0 \to q_2 = 0,$$
$$\eta_3 \neq 0 \to q_3 = 0,$$
$$\frac{\partial I}{\partial q_4} = 0.$$

These equations imply that user 1 is inactive while at user 2 both transmitting antennas are active with a same duty cycle. These equations lead to

$$\zeta(B_2, \lambda) - B_2(\log(B_2 q_4 + \lambda) + 1) = 0,$$

from which we solve q_4:

$$\tilde{q}'_4 = \alpha(B_2/\lambda).$$

As $0 \leq \alpha(B_2/\lambda) \leq 1$, we obtain a feasible candidate $(0, 0, 0, \tilde{q}_4)$.

Case-3: $\eta_1 \neq 0, \eta_2 \neq 0, \eta_3 = 0, \eta_4 \neq 0 \Rightarrow$

$$
\begin{aligned}
\eta_1 \neq 0 \rightarrow q_1 &= 0, \\
\eta_2 \neq 0 \rightarrow q_2 &= 0, \\
\frac{\partial I}{\partial q_3} &= 0, \\
\eta_4 \neq 0 \rightarrow q_4 &= 0.
\end{aligned}
$$

These equations imply that user 1 is inactive and at user 2 only one antenna is active. As user 1 is inactive, the scenario is same as a single-user MISO Poisson channel. It is easy to check that, for a single-user MISO Poisson channel, the maximal rate achievable using only a single antenna is less than the maximal rate achievable when both antennas are active, which is Case-2 mentioned above. Hence, Case-3 cannot be the optimal solution.

Case-4: $\eta_1 \neq 0, \eta_2 \neq 0, \eta_3 = 0, \eta_4 = 0 \Rightarrow$

$$
\begin{aligned}
\eta_1 \neq 0 \rightarrow q_1 &= 0, \\
\eta_2 \neq 0 \rightarrow q_2 &= 0, \\
\frac{\partial I}{\partial q_3} &= 0, \\
\frac{\partial I}{\partial q_4} &= 0.
\end{aligned}
$$

This case refers to the scenario when user 1 is inactive and user 2 transmits with both antennas having different duty cycles. Plugging $q_1 = 0$ and $q_2 = 0$ into the last two equations leads to the following two equations:

$$
S_{21}A_{21} \log \frac{\left(1 + \alpha\left(\frac{S_{21}A_{21}}{\lambda}\right)\left(\frac{S_{21}A_{21}}{\lambda}\right)\right)}{1 + \frac{S_{21}A_{21}}{\lambda}q_3 + \frac{B_2}{\lambda}q_4} = 0,
$$

$$
B_2 \log \frac{\left(1 + \alpha\left(\frac{B_2}{\lambda}\right)\left(\frac{B_2}{\lambda}\right)\right)}{1 + \frac{S_{21}A_{21}}{\lambda}q_3 + \frac{B_2}{\lambda}q_4} = 0,
$$

which requires:

$$
\begin{aligned}
\frac{S_{21}A_{21}}{\lambda}q_3 + \frac{B_2}{\lambda}q_4 &= \alpha\left(\frac{S_{21}A_{21}}{\lambda}\right)\left(\frac{S_{21}A_{21}}{\lambda}\right), \\
\frac{S_{21}A_{21}}{\lambda}q_3 + \frac{B_2}{\lambda}q_4 &= \alpha\left(\frac{B_2}{\lambda}\right)\left(\frac{B_2}{\lambda}\right).
\end{aligned}
$$

It is easy to check that $z(x) \triangleq \alpha(x)x$ is a monotonically increasing function. As the result, there does not exist (q_3, q_4) that satisfies these two equations simultaneously as $S_{21}A_{21} < B_2$. Hence, Case-4 is not possible.

Case-5: $\eta_1 \neq 0, \eta_2 = 0, \eta_3 \neq 0, \eta_4 \neq 0 \Rightarrow$

$$
\begin{aligned}
\eta_1 \neq 0 \rightarrow q_1 &= 0, \\
\frac{\partial I}{\partial q_2} &= 0, \\
\eta_3 \neq 0 \rightarrow q_3 &= 0, \\
\eta_4 \neq 0 \rightarrow q_4 &= 0.
\end{aligned}
$$

These imply that user 2 is inactive and at user 1 both antennas are active with a same duty cycle. From these equations, we obtain

$$
\zeta(B_1, \lambda) - B_1(\log(B_1 q_2 + \lambda) + 1) = 0,
$$

from which we solve q_2:

$$
\tilde{q}_2 = \alpha(B_1/\lambda).
$$

Hence, the obtained feasible candidate for optimal solution from this case is $(0, \tilde{q}_2, 0, 0)$.

Case-6: $\eta_1 \neq 0, \eta_2 = 0, \eta_3 \neq 0, \eta_4 = 0 \Rightarrow$

$$
\begin{aligned}
\eta_1 \neq 0 \rightarrow q_1 &= 0, \\
\frac{\partial I}{\partial q_2} &= 0, \qquad (52) \\
\eta_3 \neq 0 \rightarrow q_3 &= 0, \\
\frac{\partial I}{\partial q_4} &= 0. \qquad (53)
\end{aligned}
$$

This case corresponds to the scenario when all of the antennas are active and both antennas at user 1 have same duty cycle and both antennas at user 2 have same duty cycle.

By plugging $q_1 = 0$ and $q_3 = 0$ into (52) and (53), these two equations have the same form as (13) and (14) (with $S_1 A_1$ replaced by B_1 and $S_2 A_2$ replaced by B_2 respectively). Hence, (52)-(53) can be solved in the same manner as (13)-(14). In particular, these two nonlinear equations can be converted into a linear equation and a convex equation, therefore we know that there can be only two such values of q_2 and q_4 that satisfy the equations simultaneously. Lets those values be $(0, \bar{q}_2, 0, \bar{q}_4)$ and $(0, \bar{q}_2', 0, \bar{q}_4')$. If the solutions lies outside the range of $(0, 1) \times (0, 1) \times (0, 1) \times (0, 1)$, we replace it with $(0, 0, 0, 0)$ for the sake of presentations convenience.

Case-7: $\eta_1 \neq 0, \eta_2 = 0, \eta_3 = 0, \eta_4 \neq 0 \Rightarrow$

$$
\begin{aligned}
\eta_1 \neq 0 \rightarrow q_1 &= 0, \\
\frac{\partial I}{\partial q_2} &= 0, \\
\frac{\partial I}{\partial q_3} &= 0, \\
\eta_4 \neq 0 \rightarrow q_2 &= 0.
\end{aligned}
$$

This case refers to the scenario when both of the antennas at user 1 are active with a same duty cycle but at user 2 only the antenna with the larger duty cycle is active. In Appendix E-A, we show that any sum-rate achieved in this case can also be achieved by the letting both antennas of each user to be simultaneously on or off, which is Case-6. Hence, Case-7 can be ruled out.

Case-8: $\eta_1 \neq 0, \eta_2 = 0, \eta_3 = 0, \eta_4 = 0 \Rightarrow$

$$
\begin{aligned}
\eta_1 \neq 0 \rightarrow q_1 &= 0, \\
\frac{\partial I}{\partial q_2} &= 0, \\
\frac{\partial I}{\partial q_3} &= 0, \\
\frac{\partial I}{\partial q_4} &= 0.
\end{aligned}
$$

This case corresponds to the scenario when both antennas at the user 2 are active and have different duty cycles but at user 1 both transmitting antennas have the same duty cycle.

Following a similar approach as how to obtain (15), we can combine $\frac{\partial I}{\partial q_3} = 0$ and $\frac{\partial I}{\partial q_4} = 0$ to obtain a linear equation in

terms of q_1 and q_2. By plugging $q_1 = 0$ to the obtained linear equation, we solve

$$q_2 = \frac{c_1}{c_1 + c_2}, \tag{54}$$

where $c_1 = h_1(\lambda), c_2 = -h_1(B_1 + \lambda)$ with

$$h_1(x) = \left(1 + \frac{S_{22}A_{22}}{S_{21}A_{21}}\right)\zeta(S_{21}A_{21}, x) - \zeta(B_2, x).$$

Now for q_2 to be feasible, we need $0 \le q_2 \le 1$, which requires c_1 and c_2 to have the same sign. To rule out this case, we need the following lemma.

Lemma 11. $h_1(x) < 0$ for $x > 0$.

Proof. Please see Appendix E-B. □

Using this lemma, we know $c_1 < 0$ and $c_2 > 0$, so $q_2 \notin [0, 1]$. Hence this case not a valid choice.
Case-9: $\eta_1 = 0, \eta_2 \ne 0, \eta_3 \ne 0, \eta_4 \ne 0 \Rightarrow$

$$\begin{aligned}
\frac{\partial I}{\partial q_1} &= 0, \\
\eta_2 \ne 0 \to q_2 &= 0, \\
\eta_3 \ne 0 \to q_3 &= 0, \\
\eta_4 \ne 0 \to q_4 &= 0.
\end{aligned}$$

In this case user 2 is inactive and at user 1 only the antenna with a larger duty cycle is active. As user 2 is inactive, the scenario is same as the single-user MISO Poisson channel. It is easy to check that, for a single-user MISO Poisson channel, the maximal rate achievable using only a single antenna is less than the maximal rate achievable when both antennas are active, which is Case-5 mentioned above. Hence, Case-9 cannot be the optimal solution.
Case-10: $\eta_1 = 0, \eta_2 \ne 0, \eta_3 \ne 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned}
\frac{\partial I}{\partial q_1} &= 0, \\
\eta_2 \ne 0 \to q_2 &= 0 \\
\eta_3 \ne 0 \to q_3 &= 0 \\
\frac{\partial I}{\partial q_4} &= 0.
\end{aligned}$$

This case refers to the scenario when both antennas at user 2 are active with a same duty cycle, while at user 1 only one antenna is active. This case can be ruled out using the same reason as Case-7.
Case-11: $\eta_1 = 0, \eta_2 \ne 0, \eta_3 = 0, \eta_4 \ne 0 \Rightarrow$

$$\begin{aligned}
\frac{\partial I}{\partial q_1} &= 0, \\
\eta_2 \ne 0 \to q_2 &= 0, \\
\frac{\partial I}{\partial q_3} &= 0, \\
\eta_4 \ne 0 \to q_4 &= 0.
\end{aligned}$$

In this case, only one antenna at both of the users are active. Following similar argument as that in Case-7, we know this

case cannot be the optimal solution.
Case-12: $\eta_1 = 0, \eta_2 \ne 0, \eta_3 = 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned}
\frac{\partial I}{\partial q_1} &= 0, \\
\eta_2 \ne 0 \to q_2 &= 0, \\
\frac{\partial I}{\partial q_3} &= 0, \\
\frac{\partial I}{\partial q_4} &= 0.
\end{aligned}$$

This case occurs when both antennas at user 2 are active and have different duty cycles while at user 1 only one antenna is active. Following the same steps in Case-8, we obtain

$$q_1 = \frac{c_1}{c_1 + c_3},$$

in which $c_1 = h_1(\lambda)$ and $c_3 = -h_1(S_{11}A_{11} + \lambda)$. Using Lemma 11, we know that $q_1 \notin [0, 1]$, hence we may conclude that Case-12 is not a valid case.
Case-13: $\eta_1 = 0, \eta_2 = 0, \eta_3 \ne 0, \eta_4 \ne 0 \Rightarrow$

$$\begin{aligned}
\frac{\partial I}{\partial q_1} &= 0, \\
\frac{\partial I}{\partial q_2} &= 0, \\
\eta_3 \ne 0 \to q_3 &= 0, \\
\eta_4 \ne 0 \to q_4 &= 0.
\end{aligned}$$

In this case user 2 is inactive while at user 1 both antennas transmit with different duty cycles. By plugging $q_3 = 0$ and $q_4 = 0$ into $\frac{\partial I}{\partial q_1} = 0$ and $\frac{\partial I}{\partial q_2} = 0$, we have that (q_1, q_2) must satisfy the following two equations simultaneously:

$$S_{11}A_{11} \log \frac{\left(1 + \alpha\left(\frac{S_{11}A_{11}}{\lambda}\right)\left(\frac{S_{11}A_{11}}{\lambda}\right)\right)}{1 + \frac{S_{11}A_{11}}{\lambda}q_1 + \frac{B_1}{\lambda}q_2} = 0,$$

$$B_1 \log \frac{\left(1 + \alpha\left(\frac{B_1}{\lambda}\right)\left(\frac{B_1}{\lambda}\right)\right)}{1 + \frac{S_{11}A_{11}}{\lambda}q_1 + \frac{B_1}{\lambda}q_2} = 0.$$

As mentioned in Case-4, $z(x) = \alpha(x)x$ is a monotonically increasing function. As $S_{11}A_{11} \ne B_1$, we may conclude that there does not exist such (q_1, q_2) pair and hence this case is not possible.
Case-14: $\eta_1 = 0, \eta_2 = 0, \eta_3 \ne 0, \eta_4 = 0 \Rightarrow$

$$\begin{aligned}
\frac{\partial I}{\partial q_1} &= 0, \\
\frac{\partial I}{\partial q_2} &= 0, \\
\eta_3 \ne 0 \to q_3 &= 0, \\
\frac{\partial I}{\partial q_4} &= 0.
\end{aligned}$$

This case corresponds to the scenario when at user 1 both antennas are active with different duty cycles and at user 2 both antennas have same duty cycle.

Following the same steps in Case-8, we obtain q_4 as shown in (55).

However, it is difficult to make any definitive conclusion about q_4 from this form. To rule out this case, we use the following lemma.

$$q_4 = \frac{\left(1 + \frac{S_{12}A_{12}}{S_{11}A_{11}}\right)\zeta(S_{11}A_{11},\lambda) - \zeta(B_1,\lambda)}{\zeta(B_2,B_1+\lambda) - \left(1 + \frac{S_{12}A_{12}}{S_{11}A_{11}}\right)\zeta(B_2,S_{11}A_{11}+\lambda) + \frac{S_{12}A_{12}}{S_{11}A_{11}}\zeta(B_2,\lambda)}. \tag{55}$$

Lemma 12.

$$(55) = \frac{c_4}{c_4 + c_5},$$

in which $c_4 = h_2(\lambda)$ and $c_5 = -h_2(B_2 + \lambda)$ with

$$h_2(x) = \left(1 + \frac{S_{12}A_{12}}{S_{11}A_{11}}\right)\zeta(S_{11}A_{11},x) - \zeta(B_1,x).$$

Proof. Please see Appendix E-C. $\square$

Similar to Lemma 11, we can show that $h_2(x) < 0$ when $x > 0$. As the result, $q_4 \notin [0,1]$. Hence, we know that Case-14 is not a valid choice.

Case-15: $\eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \eta_4 \neq 0 \Rightarrow$

$$\frac{\partial I}{\partial q_1} = 0,$$
$$\frac{\partial I}{\partial q_2} = 0,$$
$$\frac{\partial I}{\partial q_3} = 0,$$
$$\eta_4 \neq 0 \to q_4 = 0.$$

This case corresponds to the scenario when both antennas at user 1 are active with a different duty cycles while at user 2 only the antenna with the larger duty cycle is active. Following the same steps in Case-8, we obtain the value of q_3 as (56).

Similar to Case-14, it is difficult to directly make any conclusion about the value of q_3. Following similar steps as in Lemma 12, we can show that

$$(56) = \frac{c_4}{c_4 + c_6},$$

in which $c_4 = h_2(\lambda)$ and $c_6 = -h_2(S_{21}A_{21} + \lambda)$. Hence, similar to Case-14, we can conclude that $q_3 \notin [0,1]$, and hence this case is not a valid choice.

Case-16: $\eta_1 = 0, \eta_2 = 0, \eta_3 = 0, \eta_4 = 0 \Rightarrow$

$$\frac{\partial I}{\partial q_k} = 0, k = 1, \cdots, 4$$

This case refers to the scenario when both of the antennas of each user is active and have different duty cycles. Following similar argument in Appendix E-A, we can rule this case out.

In summary, we are left with only three candidates for the optimality, i.e. Case-2, Case-5 and Case-6. Case-2 corresponds to the scenario where only user 2 is active with both antennas are simultaneously on or off with duty cycle $\alpha(B_2/\lambda)$ and hence the optimal value of $\mathbf{q}$ is $(0,0,0,\alpha(B_2/\lambda))$. Case-5 is the scenario where only user 1 is active with both antennas are simultaneously on or off with duty cycle $\alpha(B_1/\lambda)$ and therefore $\mathbf{q} = (0, \alpha(B_1/\lambda), 0, 0)$. Case-6 is the scenario where both users are active with both antennas at user 1 are simultaneously on or off and both antennas at user 2 are also simultaneously on or off and hence $\mathbf{q} = (0, \mu_1, 0, \mu_2)$ where μ_1 and μ_2 are obtained by solving (13) and (14) with

S_1A_1 replaced by $B_1 = S_{11}A_{11} + S_{12}A_{12}$ and S_2A_2 replaced by $B_2 = S_{21}A_{21} + S_{22}A_{22}$. It is clear that results obtained for MISO-MAC are the same as a SISO-MAC with properly chosen parameter.

<div align="center">

APPENDIX E

PROOFS OF LEMMAS USED IN THE PROOF OF PROPOSITION 9

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A. Proof of Case-7

In this section we show that any sum-rate achievable for scheme A, where both of the antennas at user 1 are active with a same duty cycle but at user 2 only the antenna with the larger duty cycle is active, can also be achieved by scheme B, where both antennas of each user are simultaneously on or off.

Let p^* be the duty cycle used by both antennas of user 1 and x^* be the duty cycle used by the antenna with the larger duty cycle of user 2. Then the sum-rate achieved by scheme A is

$$\begin{aligned}
I_A &= (1-p^*)(1-x^*)\varphi(\lambda) + (1-p^*)x^*\varphi(S_{21}A_{21} + \lambda) \\
&+ p^*(1-x^*)\varphi(S_{11}A_{11} + S_{12}A_{12} + \lambda) \\
&+ p^*x^*\varphi(S_{21}A_{21} + S_{11}A_{11} + S_{12}A_{12} + \lambda) \\
&- \varphi((S_{11}A_{11} + S_{12}A_{12})p^* + S_{21}A_{21}x^* + \lambda).
\end{aligned}$$

Now consider scheme B, in which both antennas of user 1 to be simultaneously on-off with duty-cycle p^*, for user 2, we let both antennas to be simultaneously on or off with duty cycle x^* but with reduced amplitude. In particular, for antenna 1, it uses $\beta_1 A_{21}$. For antenna 2, it uses $\beta_2 A_{22}$. We select β_1 and β_2 such that $\beta_1 S_{21}A_{21} + \beta_2 S_{22}A_{22} = S_{21}A_{21}$. It is easy to check that there always exists $0 \leq \beta_1 \leq 1$ and $0 \leq \beta_2 \leq 1$ such that this relationship holds. Hence, scheme B is a valid scheme. For this scheme, the achievable sum-rate is

$$\begin{aligned}
I_B &= (1-p^*)(1-x^*)\varphi(\lambda) \\
&+ (1-p^*)x^*\varphi(\beta_1 S_{21}A_{21} + \beta_2 S_{22}A_{22} + \lambda) \\
&\quad p^*(1-x^*)\varphi(S_{11}A_{11} + S_{12}A_{12} + \lambda) \\
&+ p^*x^*\varphi(\beta_1 S_{21}A_{21} + \beta_2 S_{22}A_{22} + S_{11}A_{11} + S_{12}A_{12} + \lambda) \\
&- \varphi((S_{11}A_{11} + S_{12}A_{12})p^* + (\beta_1 S_{21}A_{21} + \beta_2 S_{22}A_{22})x^* + \lambda).
\end{aligned}$$

As $\beta_1 S_{21}A_{21} + \beta_2 S_{22}A_2 2 = S_{21}A_{21}$, we have $I_A = I_B$. Therefore, we can conclude that any sum-rate achievable by letting both of the antennas at user 1 to be active with a same duty cycle but letting only one antenna of user 2 to be active can also be achieved by letting both antennas of each user to be simultaneously on or off.

$$q_3 = \frac{\left(1 + \frac{S_{12}A_{12}}{S_{11}A_{11}}\right)\zeta(S_{11}A_{11}, \lambda) - \zeta(B_1, \lambda)}{\zeta(S_{21}A_{21}, B_1 + \lambda) - \left(1 + \frac{S_{12}A_{12}}{S_{11}A_{11}}\right)\zeta(S_{21}A_{21}, S_{11}A_{11} + \lambda) + \frac{S_{12}A_{12}}{S_{11}A_{11}}\zeta(S_{21}A_{21}, \lambda)}. \tag{56}$$

B. Property of Lemma 11

To lighten up the notation, we set $a = S_{22}A_{22}$ and $b = S_{21}A_{21}$. We have

$$\begin{aligned}
&h_1(x) \\
&= \left(1 + \frac{b}{a}\right)\zeta(a, x) - \zeta(a + b, x) \\
&= \zeta(a, x) + \frac{b}{a}\zeta(a, x) - \zeta(a + b, x) \\
&= (a + x)\log(a + x) - x\log x + \frac{b}{a}(a + x)\log(a + x) \\
&\quad - \frac{b}{a}x\log x - (a + b + x)\log(a + b + x) + x\log x \\
&= -(a + x)\log\left(\frac{a + b + x}{a + x}\right) - b\log\left(\frac{a + b + x}{a + x}\right) \\
&\quad + \frac{b}{a}x\log\left(\frac{a + x}{x}\right).
\end{aligned}$$

Using the fact that for $x > 0$,

$$\frac{x}{1 + x} < \ln(1 + x) < x,$$

we obtain

$$\begin{aligned}
&h_1(x) \\
&< \frac{1}{\ln(2)}\left(-(a + x)\frac{b}{a + b + x} - \frac{b^2}{a + b + x} + \frac{b}{a}x\frac{a}{x}\right) \\
&= 0.
\end{aligned}$$

C. Proof of Lemma 12

In order to come to a firm conclusion about the value of q_4, we will write the value of $I(\mathbf{q})$ in (42) in a different form. In (42), all the terms are written separated by q_1 and q_2 terms, we will now write $I(\mathbf{q})$ written separated by q_3 and q_4. Clearly $I(\mathbf{q})$ in (42) can be written as

$$\begin{aligned}
I(\mathbf{q}) &= (1 - (q_3 + q_4))\Big((1 - (q_1 + q_2))\varphi(\lambda) \\
&+ q_1\varphi(S_{11}A_{11} + \lambda) + q_2\varphi(B_1 + \lambda)\Big) \\
&+ q_3\Big((1 - (q_1 + q_2))\varphi(S_{21}A_{21} + \lambda) \\
&+ q_1\varphi(S_{11}A_{11} + S_{21}A_{21} + \lambda) \\
&+ q_2\varphi(B_1 + S_{21}A_{21} + \lambda)\Big) + q_4\Big((1 - (q_1 + q_2)\varphi(B_2 + \lambda) \\
&+ q_1\varphi(S_{11}A_{11} + B_2 + \lambda) + q_2\varphi(B_1 + B_2 + \lambda) \\
&- \varphi(S_{11}A_{11}q_1 + B_1q_2 + S_{21}A_{21}q_3 + B_2q_4 + \lambda). \tag{57}
\end{aligned}$$

Using this new form, then we have:

$$\begin{aligned}
\frac{\partial I}{\partial q_1} &= (1 - (q_3 + q_4))\zeta(S_{11}A_{11}, \lambda) \\
&+ q_3\zeta(S_{11}A_{11}, S_{21}A_{21} + \lambda) + q_4\zeta(S_{11}A_{11}, B_2 + \lambda) \\
&- S_{11}A_{11}\left(\log\left(S_{11}A_{11}q_1 + B_1q_2 + S_{21}A_{21}q_3 + B_2q_4 + \lambda\right) + 1\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial I}{\partial q_2} &= (1 - (q_3 + q_4))\zeta(B_1, \lambda) \\
&+ q_3\zeta(B_1, S_{21}A_{21} + \lambda) + q_4\zeta(B_1, B_2 + \lambda) \\
&- B_1\left(\log(S_{11}A_{11}q_1 + B_1q_2 + S_{21}A_{21}q_3 + B_2q_4 + \lambda) + 1\right),
\end{aligned}$$

$$\begin{aligned}
\frac{\partial I}{\partial q_3} &= \zeta(S_{21}A_{21}, \lambda) \\
&+ q_1\Big(\zeta(S_{11}A_{11}, S_{21}A_{21} + \lambda) - \zeta(S_{11}A_{11}, \lambda)\Big) \\
&+ q_2\Big(\zeta(B_1, S_{21}A_{21} + \lambda) - \zeta(B_1, \lambda)\Big) \\
&- S_{21}A_{21}(\log(S_{11}A_{11}q_1 + B_1q_2 + S_{21}A_{21}q_3 + B_2q_4 + \lambda) + 1),
\end{aligned}$$

and

$$\begin{aligned}
\frac{\partial I}{\partial q_4} &= \zeta(B_2, \lambda) + q_1\Big(\zeta(S_{11}A_{11}, B_2 + \lambda) - \zeta(S_{11}A_{11}, \lambda)\Big) \\
&+ q_2\Big(\zeta(B_1, B_2 + \lambda) - \zeta(B_1, \lambda)\Big) \\
&- B_2(\log(S_{11}A_{11}q_1 + B_1q_2 + S_{21}A_{21}q_3 + B_2q_4 + \lambda) + 1).
\end{aligned}$$

Recall that we need to solve

$$\begin{aligned}
\frac{\partial I}{\partial q_1} &= 0, \\
\frac{\partial I}{\partial q_2} &= 0, \\
q_3 &= 0, \\
\frac{\partial I}{\partial q_4} &= 0.
\end{aligned}$$

By combining $\frac{\partial I}{\partial q_1} = 0$ and $\frac{\partial I}{\partial q_2} = 0$, we can eliminate the term with $\log$ and obtain a linear equation in terms of q_3 and q_4. By plugging $q_3 = 0$ to the obtained linear equation, we obtain an alternative form of (55):

$$q_4 = \frac{c_4}{c_4 + c_5},$$

in which $c_4 = h_2(\lambda)$ and $c_5 = -h_2(B_2 + \lambda)$ with

$$h_2(x) = \left(1 + \frac{S_{12}A_{12}}{S_{11}A_{11}}\right)\zeta(S_{11}A_{11}, x) - \zeta(B_1, x).$$

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